A method for solving solvable conditions of restricted three-body problems.

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Abstract

In the classical mechanics, the problem of many body gravitation has always been a hot topic.But in addition to the early two-body gravity problem, three-body and more than three-body can not provide analytical solutions.Because in three-dimensional space, only 10 algebraic equations can be obtained after the vector is decomposed(Three equations can be obtained from centroid,three equations can be obtained from the law of conservation of momentum,three equations can be obtained from the law of conservation of angular momentum,and the last can be obtained from the law of conservation of energy).We can't get enough algebraic equations to relate the differential equations of three-body motion problems,so it can't be solved accurately.But not all cases are unsolved. By simplifying some conditions, we can still get the analytical solution of some three-body problems, which are called restricted three-body problems.In this paper, a theorem is introduced and proved, which can provide a new way to solve the solvable condition of restricted three body problem.

Introduction.-For three objects A, B and C of any initial position, velocity and mass,if the influence of volume and shape is ignored,through Newton's second law and the law of universal gravitation, we can get the differential equation of motion of object A is

$$m_A\vec{a}_A = Gm_A[\frac{m_B}{r_{AB}^3}(\vec{r}_B-\vec{r}_A)+\frac{m_C}{r_{AC}^3}(\vec{r}_C-\vec{r}_A)]$$

This is a vector, which can establish a proper rectangular coordinate system, and the projection is

$$a_A^x = G[\frac{m_B}{r_{AB}^3}(x_B-x_A)+\frac{m_C}{r_{AC}^3}(x_C-x_A)]$$

$$a_A^y = G[\frac{m_B}{r_{AB}^3}(y_B-y_A)+\frac{m_C}{r_{AC}^3}(y_C-y_A)]$$

$$a_A^z = G[\frac{m_B}{r_{AB}^3}(z_B-z_A)+\frac{m_C}{r_{AC}^3}(z_C-z_A)]$$

After the same operation for B and C, we can get 9 second order ordinary differential equations, 18 orders in total. That is to say, 18 algebraic equations are needed to correlate and solve them.In this part, there are four mechanical laws about motion state, corresponding to three vector equations and one scalar equation.

$$\sum m_i \vec{v}_i = \vec{C}_1$$

$$\sum m_i \vec{v}_i \times \vec{r}_i = \vec{C}_3$$

$$\sum m_i \vec{r}_i = \vec{C}_1 t + \vec{C}_2$$

$$\sum E_k + \sum E_p = C_4$$

According to the conventional method, there are only 10 algebraic equations after projection, which is obviously not enough. In 1843, Jacobi proved that if $6n-2$ algebraic equations are found in the many body problem, then the remaining two also can be found. According to his theory, the two-body problem is obviously solvable, and only 16 are needed for the three-body problem. For any one of the three objects, its energy and momentum are not conserved, so the algebraic equations of these physical quantities can only be listed with the form of a system. But for the angular momentum, if the angular momentum of each object is conserved, the number of algebraic equations will increase to 16. Based on this idea, the following theorem is introduced.

Theorem.-Three objects are regarded as particles in space.,if there is interaction between every two objects, the extension line of the resultant forces of each object will intersect with each other in one point.

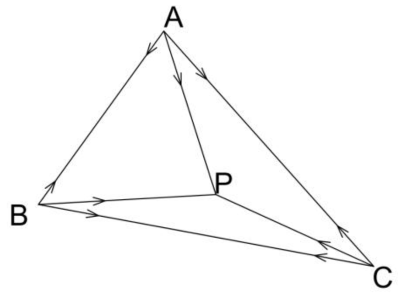

Figure 1 Diagram

Prove.-Since three objects are regarded as particles, plane $O-xy$ can be established. Because $\vec{F}_{AB}$ and $\vec{F}_{AC}$ are in plane $O-xy$,the resultant force of A $\vec{F}_A$ also in $O-xy$. The same is true for $\vec{F}_B$ and $\vec{F}_C$ .We know they can't be parallel with each other,so let the intersection of extension lines of $\vec{F}_A$ and $\vec{F}_B$ be the origin O of coordinate system.Let

$$F_{AB} = F_{BA} = F_1$$

$$F_{BC} = F_{CB} = F_2$$

$$F_{CA} = F_{AC} = F_3$$

and

$$r_{AB} = r_{BA} = r_1$$

$$r_{BC} = r_{CB} = r_2$$

$$r_{CA} = r_{AC} = r_3 .$$

Because the extension line of the resultant force passes through the origin,there is

$\vec{F}_A \times \vec{r}_A = 0$, $(F_A^x, F_A^y) \times (x_A, y_A) = 0$.

We can get

$$(F_{AB} \cdot \frac{x_B - x_A}{r_{AB}} + F_{AC} \cdot \frac{x_C - x_A}{r_{AC}}) \cdot y_A -$$
$$(F_{AB} \cdot \frac{y_B - y_A}{r_{AB}} + F_{AC} \cdot \frac{y_C - y_A}{r_{AC}}) \cdot x_A = 0$$

Simplification

$$\frac{F_1}{r_1} \cdot x_B y_A + \frac{F_3}{r_3} \cdot x_C y_A = \frac{F_1}{r_1} \cdot y_B x_A + \frac{F_3}{r_3} \cdot y_C x_A$$

(1)

Empathy

$\vec{F}_B \times \vec{r}_B = 0$, $(F_B^x, F_B^y) \times (x_B, y_B) = 0$

$$(F_{BA} \cdot \frac{x_A - x_B}{r_{BA}} + F_{BC} \cdot \frac{x_C - x_B}{r_{BC}}) \cdot y_B -$$
$$(F_{BA} \cdot \frac{y_B - y_A}{r_{BA}} + F_{BC} \cdot \frac{y_C - y_B}{r_{BC}}) \cdot x_B = 0$$

Simplification

$$\frac{F_1}{r_1} \cdot x_A y_B + \frac{F_2}{r_2} \cdot x_C y_B = \frac{F_1}{r_1} \cdot y_A x_B + \frac{F_2}{r_2} \cdot y_C x_B$$

(2)

(2)-(1),We can get

$$\frac{F_2}{r_2} \cdot x_B y_C - \frac{F_3}{r_3} \cdot x_C y_A = \frac{F_2}{r_2} \cdot y_B x_C - \frac{F_3}{r_3} \cdot y_C x_A$$

It means

$$\frac{F_2}{r_2} \cdot x_B y_C + \frac{F_3}{r_3} \cdot x_A y_C = \frac{F_2}{r_2} \cdot y_B x_C + \frac{F_3}{r_3} \cdot y_A x_C$$

(3)

Subtract $\frac{F_2}{r_2} x_c y_c$ and $\frac{F_3}{r_3} x_c y_c$ from both sides of the equation,we can get

$$(F_{CA} \cdot \frac{x_A - x_C}{r_{CA}} + F_{CB} \cdot \frac{x_B - x_C}{r_{CB}}) \cdot y_C -$$
$$(F_{CA} \cdot \frac{y_A - y_C}{r_{CA}} + F_{CB} \cdot \frac{y_B - y_C}{r_{CB}}) \cdot x_C = 0$$

So we got

$(F_C^x, F_C^y) \times (x_C, y_C) = 0$, $\vec{F}_C \times \vec{r}_C = 0$

That is to say, the resultant force of Object C also passes through the origin, that is, when three objects interact, the extension lines of the resultant force of each object intersect at one point, which may be called "force center" P.

It's worth noting that there's a simpler way to prove:

If it is known that point P is the intersection of A and B resultant extension lines, there is an equation

$$(\vec{F}_{AB} + \vec{F}_{AC}) \times \overrightarrow{AP} = 0$$

and

$$(\vec{F}_{BA} + \vec{F}_{BC}) \times \overrightarrow{BP} = 0$$

It means

$$\vec{F}_{AB} \times (\overrightarrow{AP} + \overrightarrow{PB}) + \vec{F}_{AC} \times \overrightarrow{AP} + \vec{F}_{BC} \times \overrightarrow{BP} = 0$$

(4)

Notice that

$$\vec{F}_{AB} \times \overrightarrow{AB} = 0$$

$$\vec{F}_{CA} \times \overrightarrow{AC} = 0$$

$$\vec{F}_{CB} \times \overrightarrow{BC} = 0$$

So equation **(4)** is equivalent to that

$$\vec{F}_{CA} \times \overrightarrow{CP} + \vec{F}_{CB} \times \overrightarrow{CP} = 0$$

This means that the extension line of the resultant force of Object C crosses point P.

The force center of three-body problem.-The Universal gravitation is also an interaction force, so the resultant force of each of the three objects only points to a point in space with the action of universal gravitation.

We all know $F_{ij} = F_{ji} = G\frac{m_i m_j}{r_{ij}^2}$,

put it into the equations,there is

$$m_A r_{BC}^3 x_A + m_B r_{CA}^3 x_B + m_C r_{AB}^3 x_C = 0$$

$$m_A r_{BC}^3 y_A + m_B r_{CA}^3 y_B + m_C r_{AB}^3 y_C = 0$$

So

$$m_A r_{BC}^3(\vec{x}_A + \vec{y}_A) + m_B r_{CA}^3(\vec{x}_B + \vec{y}_B) + m_C r_{AB}^3(\vec{x}_C + \vec{y}_C) = 0$$

We can get

$$m_A r_{BC}^3 \vec{r}_A + m_B r_{CA}^3 \vec{r}_B + m_C r_{AB}^3 \vec{r}_C = 0 \quad \textbf{(5)}$$

In other coordinate system ξ

$$\vec{r}_{\xi A} = \vec{r}_{\xi P} + \vec{r}_{PA}$$

$$\vec{r}_{\xi B} = \vec{r}_{\xi P} + \vec{r}_{PB}$$

$$\vec{r}_{\xi C} = \vec{r}_{\xi P} + \vec{r}_{PC}$$

Substituting into equation **(5)** and get

$$\vec{r}_{\xi P} = \frac{m_A r_{BC}^3 \vec{r}_{\xi A} + m_B r_{CA}^3 \vec{r}_{\xi B} + m_C r_{AB}^3 \vec{r}_{\xi C}}{m_A r_{BC}^3 + m_B r_{CA}^3 + m_C r_{AB}^3} \quad \textbf{(6)}$$

It's kind of like an expression for centroid.

Now we can get $\vec{v}_{\xi P} = \frac{d\vec{r}_{\xi P}}{dt}$,and $\vec{a}_{\xi P} = \frac{d\vec{v}_{\xi P}}{dt}$ .When $\vec{a}_{\xi P} \equiv 0$,it means that point P always moves in a straight line at a constant speed.We can use point P as the origin to establish the coordinate system,because the resultant force on each object always points to P ,conservation of angular momentum of each object to point P.

Project to three axes,for A , B and C ,there will be 9 equations of conservation of angular momentum. Now there are 16 algebraic equations, according to Jacoby, it can be solved.

Therefore, equation **(6)** can be used as a condition to solve the group of restricted three-body problems.

Derivation of the position vector of the force center P with the interaction force.-From **(1)** we can get

$$\frac{F_1}{r_1} \cdot x_B y_A - \frac{F_1}{r_1} \cdot y_B x_A = \frac{F_3}{r_3} \cdot x_A y_C - \frac{F_3}{r_3} \cdot x_C y_A$$

Which means

$$(\frac{F_{AB}}{r_{AB}} \cdot \vec{r}_B + \frac{F_{CA}}{r_{CA}} \cdot \vec{r}_C) \times \vec{r}_A = 0$$

Empathy

$$(\frac{F_{AB}}{r_{AB}} \cdot \vec{r}_A + \frac{F_{BC}}{r_{BC}} \cdot \vec{r}_C) \times \vec{r}_B = 0$$

$$(\frac{F_{BC}}{r_{BC}} \cdot \vec{r}_B + \frac{F_{AB}}{r_{AB}} \cdot \vec{r}_A) \times \vec{r}_C = 0$$

Notice that

$$\vec{r}_A \times \vec{r}_A = 0, \vec{r}_B \times \vec{r}_B = 0, \vec{r}_C \times \vec{r}_C = 0$$

we can get

$$(\frac{F_{AB}\cdot F_{BC}}{r_{AB}\cdot r_{BC}}\cdot\vec{r}_B+\frac{F_{CA}\cdot F_{BC}}{r_{CA}\cdot r_{BC}}\cdot\vec{r}_C+\frac{F_{CA}\cdot F_{AB}}{r_{CA}\cdot r_{AB}}\vec{r}_A)\times\vec{r}_A=0$$

$$(\frac{F_{AB}\cdot F_{BC}}{r_{AB}\cdot r_{BC}}\cdot\vec{r}_B+\frac{F_{CA}\cdot F_{BC}}{r_{CA}\cdot r_{BC}}\cdot\vec{r}_C+\frac{F_{CA}\cdot F_{AB}}{r_{CA}\cdot r_{AB}}\vec{r}_A)\times\vec{r}_B=0$$

$$(\frac{F_{AB}\cdot F_{BC}}{r_{AB}\cdot r_{BC}}\cdot\vec{r}_B+\frac{F_{CA}\cdot F_{BC}}{r_{CA}\cdot r_{BC}}\cdot\vec{r}_C+\frac{F_{CA}\cdot F_{AB}}{r_{CA}\cdot r_{AB}}\vec{r}_A)\times\vec{r}_C=0$$

Vector

$$\frac{F_{AB}\cdot F_{BC}}{r_{AB}\cdot r_{BC}}\cdot\vec{r}_B+\frac{F_{CA}\cdot F_{BC}}{r_{CA}\cdot r_{BC}}\cdot\vec{r}_C+\frac{F_{CA}\cdot F_{AB}}{r_{CA}\cdot r_{AB}}\vec{r}_A$$

is coplanar with $\vec{r}_A$, $\vec{r}_B$ and $\vec{r}_C$, and $\vec{r}_A$, $\vec{r}_B$ and $\vec{r}_C$ intersect at P. So it's not hard to get

$$\frac{F_{BA}\cdot F_{BC}}{r_{BA}\cdot r_{BC}}\cdot\vec{r}_B+\frac{F_{CA}\cdot F_{CB}}{r_{CA}\cdot r_{CB}}\cdot\vec{r}_C+\frac{F_{AC}\cdot F_{AB}}{r_{AC}\cdot r_{AB}}\vec{r}_A=0$$

(7)

In other coordinate system ξ

$$\vec{r}_{\xi A}=\vec{r}_{\xi P}+\vec{r}_{PA}$$

$$\vec{r}_{\xi B}=\vec{r}_{\xi P}+\vec{r}_{PB}$$

$$\vec{r}_{\xi A}=\vec{r}_{\xi P}+\vec{r}_{PB}$$

substituting into equation **(7)**

$$\vec{r}_{\xi P}=\frac{\frac{F_{BA}\cdot F_{BC}}{r_{BA}\cdot r_{BC}}\cdot\vec{r}_{\xi B}+\frac{F_{CA}\cdot F_{CB}}{r_{CA}\cdot r_{CB}}\cdot\vec{r}_{\xi C}+\frac{F_{AC}\cdot F_{AB}}{r_{AC}\cdot r_{AB}}\vec{r}_{\xi A}}{\frac{F_{BA}\cdot F_{BC}}{r_{BA}\cdot r_{BC}}+\frac{F_{CA}\cdot F_{CB}}{r_{CA}\cdot r_{CB}}+\frac{F_{AC}\cdot F_{AB}}{r_{AC}\cdot r_{AB}}}$$

(8)

Conclusion.-

For the three body interaction force, the force center P is inevitable existence, and the solution to the restricted three body problem can be transformed into the condition to solve $\vec{a}_P\equiv 0$.

Reference.-

[1] Jacobi,Jacobi principle,1843.
[2] Isaac Newton,Mathematical Principles of Nature Philosophy,1687.