Compressible Flow around a Sphere.

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Exact solutions of the Navier-Stokes equation are given which represents steady compressible flow of a viscous fluid past a sphere. Numerical discussions of the relevant functions as well as the structure of the flow field are made.

Key words. Navier-Stokes equations, exact solutions, compressible flow.
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For reasons related to the nonlinearity of the Navier-Stokes equation, very few flows are currently known that are its exact solution [1]. In most cases, to construct exact solutions, the appropriate ideal incompressible fluid flow $\vec{u}^0$ is used as the basis, which is at the same time a solution of both the Navier-Stokes equation, which is nonlinear, and the simpler, linear vortex-free flow equation

(1) $$\vec{\Omega}^0 = \vec{\nabla} \wedge \vec{u}^0 = \vec{0},$$

in which the velocity field is represented as a vector product of the gradients of its integral surfaces $\psi_i^0, i = 1,2$

(2) $$\vec{u}^0 = \vec{\nabla}\psi_1^0 \wedge \vec{\nabla}\psi_2^0 .$$

In a special form of surface ψ_i^0 representation typical of the method of separation of variables in linear partial differential equations

(3) $$\psi_1^0 = f_0^0(x_1) + f_1^0(x_1) f_2^0(x_2), \psi_2^0 = x_3 - f_3^0(x_1)$$

the variables in equation (1) are separated, which makes it possible to reduce it to a system of ordinary differential equations.

Further, in order to extend the ideal solution (3) to the case of a viscous flow, the form of the function $f_2^0(x_2)$ is preserved, and the remaining functions are searched again, assuming their asymptotic desire for their “ideal” inverse images $\{f_i^0\}$:

(4) $$\psi_1 = f_0(x_1) + f_1(x_1) f_2^0(x_2), \psi_2 = x_3 - f_3(x_1) .$$

When substituting expressions (4) into the stationary Navier-Stokes equation

(5) $$\vec{\nabla} \wedge \left(\mathrm{Re}\cdot \vec{\Omega} \wedge \vec{u} + \vec{\nabla} \wedge \vec{\Omega}\right) = \vec{0},$$

the number of independent equations for determining functions $\{f_i^0\}$ usually exceeds their number, and such a system turns out to be joint only in the case of an ideal fluid. In those rare cases when the number of equations coincides with the number of unknowns, or when redundant equations coincide with others from the system, it is possible to find the exact solution to the Navier-Stokes equation [2,3].

An obvious reason for the limited applicability of the described method is that during viscous fluid flow, at least all functions in representation (3) must change [4].

This article provides a method to expand the applicability of the standard approach to finding exact Navier-Stokes solutions by taking into account the compressibility of a viscous fluid. This method is illustrated by the example of laminar flow around a sphere.

We choose the Cartesian coordinates (x, y) in the equatorial plane of the sphere and the axis z in the direction of the incident flow.
When a free-stream flow around a sphere $\vec{u}$ with a density ρ

$$(6) \qquad \rho\vec{u} = \vec{\nabla}\psi_1 \wedge \vec{\nabla}\psi_2$$

in spherical coordinates

$$(7) \qquad [x, y, z] \to [r \cdot \sin(o)\cos(o_1), r \cdot \sin(o)\sin(o_1), r \cdot \cos(o)]$$

equations (4) take the following form

$$(8) \qquad \psi_1 = v_0 r f_1(r) \cdot \sin(o)^2, \psi_2 = o_1, \rho = \rho(r).$$

The stationary Navier-Stokes equation for a compressible fluid whose dynamic viscosity depends only on its density ρ has the form [5]

$$(9) \qquad \frac{\rho}{2}\vec{\nabla}(\vec{u}\cdot\vec{u}) + [\vec{\nabla}\wedge\vec{u}]\wedge\vec{u} = -\vec{\nabla}p + v_1\vec{\nabla}\rho(\vec{\nabla}\cdot\vec{u}) - v_0\vec{\nabla}\wedge\rho[\vec{\nabla}\wedge\vec{u}]$$

Here v_0, v_1 are the kinematic viscosities.
Substitution of (8) into (9) gives ordinary differential equations for $f_1(r)$ and $\rho(r)$. Thus

$$(10) \qquad f_1'' = \frac{-f'^3 + 7ff'^2 - 6f^3 + C_2 f'(-f' + 6f + 5f^2 + 2C_2 f)}{f'(f' + 2f) - f(3f + C_2)}$$

$$(11) \qquad \rho' = e^l \frac{C_1 e^3 + f'' + f' - 2f + C_2}{f + f'}\rho$$

Here $(\,)' = \frac{d}{dl}, l = \ln(r), \{C_i\}$ are arbitrary constants.
The components of the material flow field $\vec{J} = \rho\vec{u} = [\rho u_x, \rho u_y, \rho u_{z]}]$ in the new coordinates (7) have the form

$$(12) \qquad \rho u_x = \frac{v_0 e^{-l}\sin(2o)\cos(o_1)(f - f')}{2}$$

$$(13) \qquad \rho u_y = \frac{v_0 e^{-l}\sin(2o)\sin(o_1)(f - f')}{2}$$

$$(14) \qquad \rho u_z = 2 \cdot v_0 e^{-l}\left(f\sin\left(\frac{o}{2}\right)^2 + f'\cos\left(\frac{o}{2}\right)^2\right)$$

Differential equation (10) under boundary conditions

(15) $$f_1(0)=0, f_1'(0)=0$$

has two independent solutions

(16) $$f_1^{(1)}=C_2 l^2\left(\frac{1}{2}+\frac{l}{8}+\frac{23l^2}{384}+\ldots\right)$$

(17) $$f_1^{(2)}=C_2 l^2\left(\frac{1}{2}+\frac{5l}{12}-\frac{23l^2}{96}+\ldots\right)$$

The streamlines of these flows in the plane $[y,z]$ are shown in Fig. 1 and Fig. 2

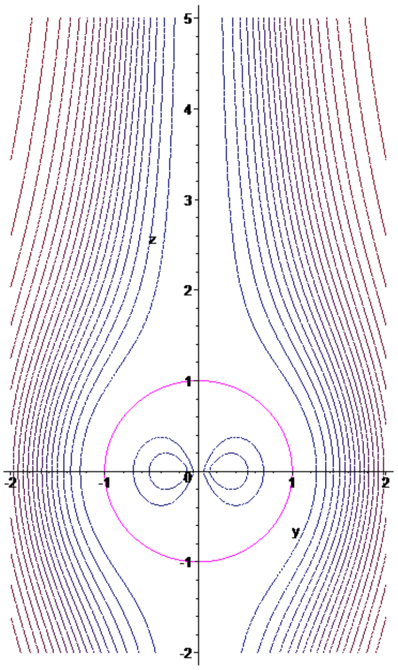

Fig. 1

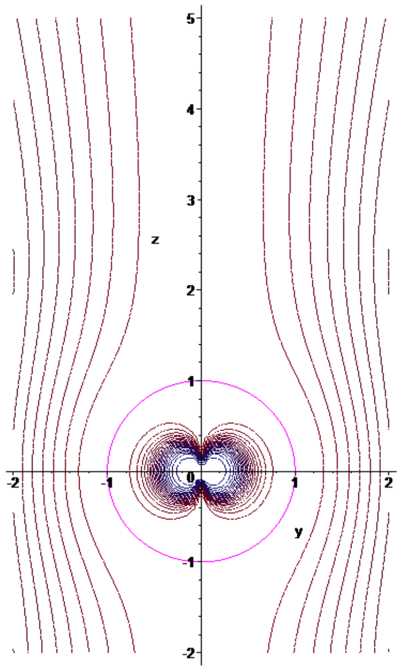

Fig. 2

The distribution of the components of the material flow depending on the radius (with a fixed value of the coordinate o) is shown in Fig. 3 and Fig. 4, which allow us to conclude that the first solution corresponds to the constant value of the material flow at infinity, and the second to its zero value.

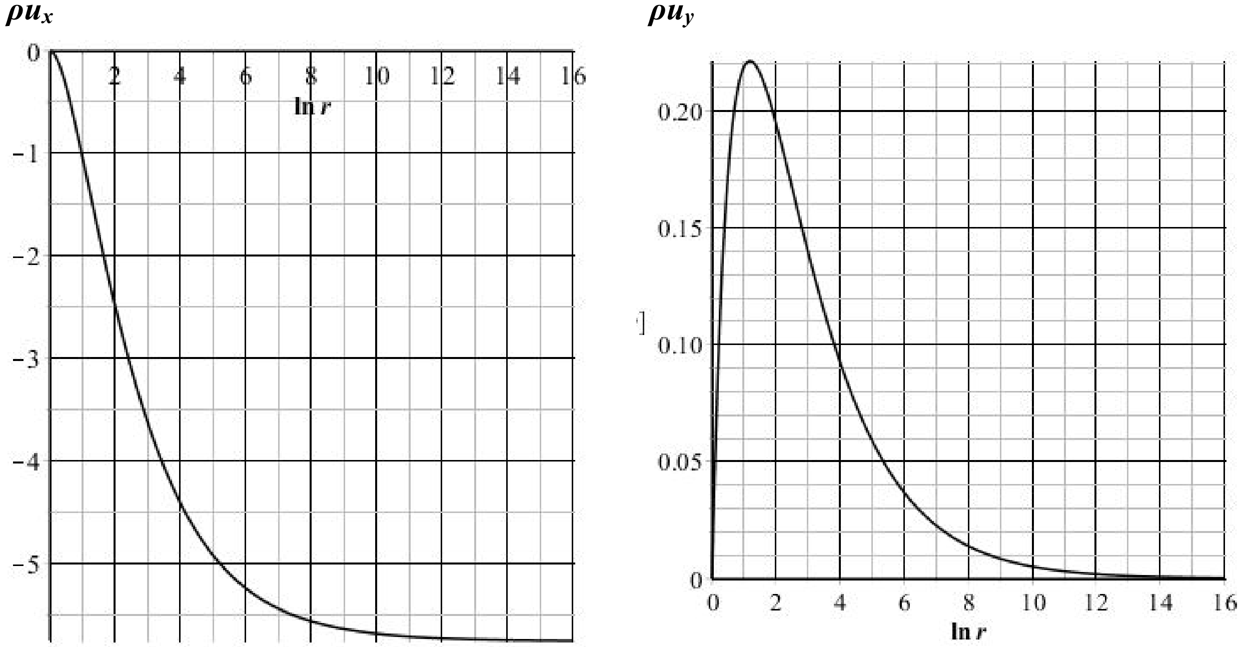
ρu_x
0
-1
-2
-3
-4
-5
2
4
6
8
10
12
14
16
ln r
ρu_y
0.20
0.15
0.10
0.05
0
0
2
4
6
8
10
12
14
16
ln r

Fig.4

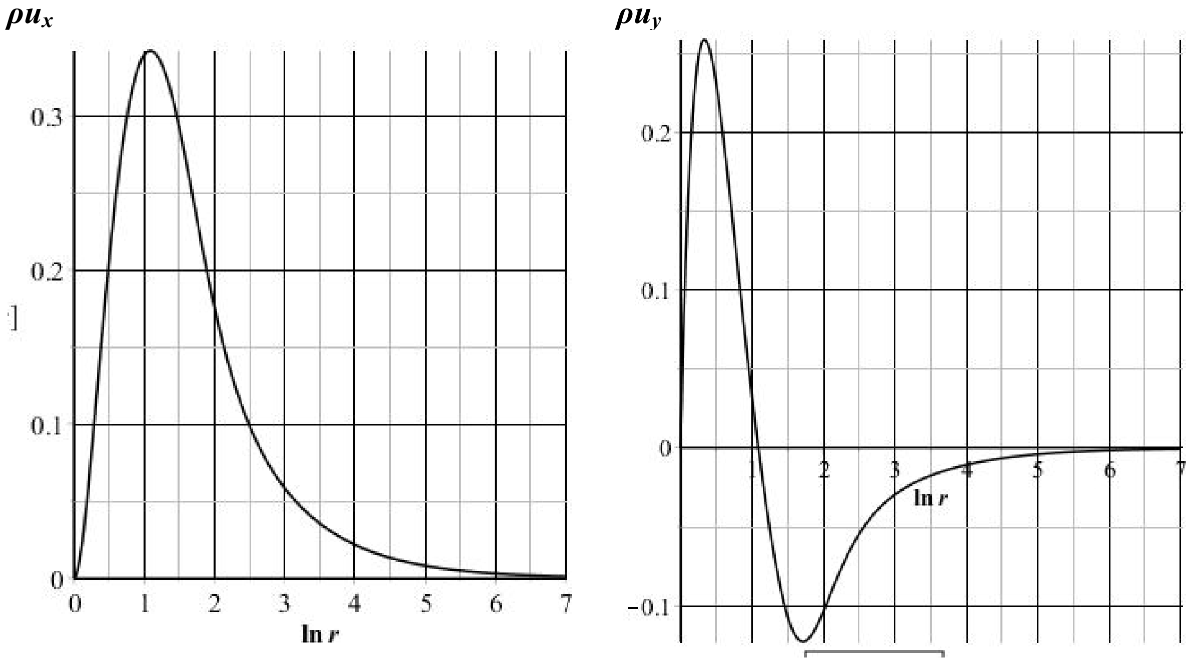
ρu_x
0.3
0.2
0.1
0
0
1
2
3
4
5
6
7
ln r
ρu_y
0.2
0.1
0
-0.1
1
2
3
4
5
6
7
ln r

Fig.3

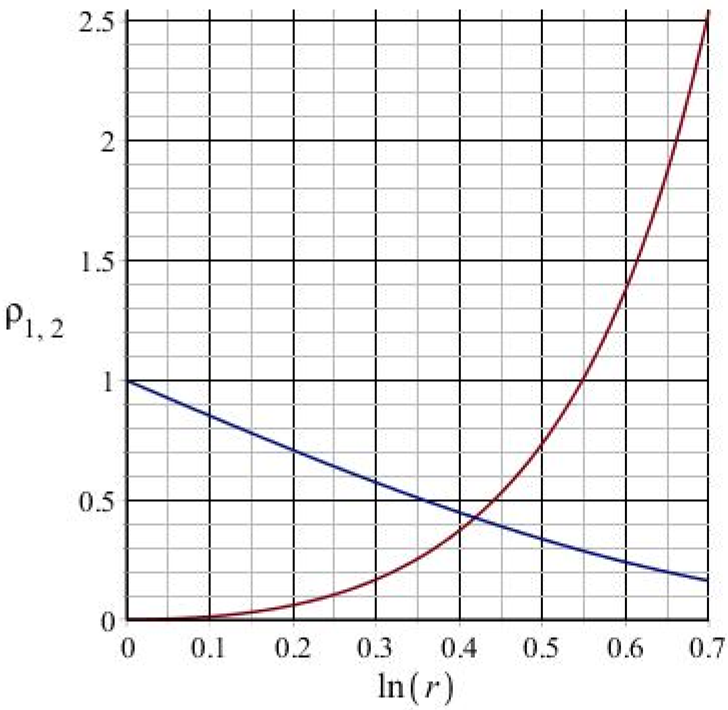

Fig.5

The density distribution of material flows, the first of which vanishes on the surface of the sphere, and the second decreases unlimitedly with distance from it (see Fig.5), partially disavow the physical value of the obtained solutions.
At the same time, the solutions considered can be of particular importance for the purposes of mathematical hydrodynamics.

References

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