Studies in Figured Tours of Knight in Two and Higher Dimensions

Awani Kumar, Forest Department, Uttar Pradesh, Lucknow 226001, India

E-mail: awanieva@gmail.com

Abstract

Tour of knight is over a millennium year old puzzle but 'Figured tour' of knight is a recent field of research. T. R. Dawson, an English chess problemist and the father of Fairy Chess, coined the term in 1940s. The name figured tour is appropriate for any numbered tour in which certain arithmetically related numbers are arranged in a geometrical pattern. Figured tours have been only looked into two-dimensional boards, mostly on 8x8 board. The author has constructed knight tour with square numbers in fiveleaper {3, 4}+ {0, 5}path and various other figured tours on 6x6 board and extended it in three and four dimensional space. Construction of figured tours is a mathematical recreation and can also be used in pedagogy of higher mathematics.

Key words: Knight, Tour, Higher dimensions.

Introduction: Knight's tour on chess board is an interesting ancient problem - over a millennium old. According to Dickins, [1] "The earliest known Knight's Tour dates from about 900 A.D. ..." Books related to recreational mathematics, namely, by Rouse Ball [2], Kraitchik [3], Pickover [4], Petkovic [5], Wells [6], Gardner [7] and by many others frequently cover knight's tour problem. It is basically a problem in graph theory — finding what is called a Hamiltonian path. Some great mathematical minds, such as, Raimond de Montmort, Abraham de Moivre, Jean-Jacques de Mairan, Vandermonde and Leonhard Euler have delved into the knight's tour problem. According to Jelliss, [8] "The name figured tour is appropriate for any numbered tour in which certain arithmetically related numbers are arranged in a geometrical pattern, since it combines in one concept both senses of the ambiguous term 'figure', which can mean either a numerical symbol or a geometrical shape. Tours in which all the entries participate in the arithmetical properties, for example arithmo-geometric tours in which the arithmetical properties derive from the geometrical structure such as symmetry, or magic tours which require calculation of rank and file totals, are not figured tours in this sense." Although tour of knight problem is over a millennium old, work on figured tour of knight began in 18th century. The term 'figured tour' was first used by the English chess problemist and 'father of Fairy Chess', T. R. Dawson in connection with his knight's tour showing the square numbers in closed, symmetrical circuits of knight moves in 1944. Perusal of literature reveals that 'figured tours' have been mostly looked into 8x8 board. What about 'figured tours' on 6x6 board? Can they be extended in 3 dimensions and even in 4 dimensions? The author plans to look into these questions.

1. Figured tours on 6x6 board: Like its conventional meaning, knight tour on a 6x6 board is consecutive knight moves where knight visits all the 36 cells without visiting any cell twice. First, knight's tours showing the square numbers in various formations are described and then knight's tours showing other numbers in formation are covered.

1A. Square numbers in formations: Figured tours with square numbers are comparatively easy to construct. It is because the number of intermediate cells between square numbers increases rapidly ; so there are fewer constraints as the tour progresses. Figure 1a by Jelliss [8] is the only tour with the square numbers 1, 4, 9, 16, 25, 36 arranged in the order of magnitude in a row. There are only 11 tours (one closed and the rest open) with the square numbers in the order of magnitude as wazir moves. Wazir {0,1}is a fairy chess piece which moves only one step, along the rank or file, and the tour by Jelliss is one of them. Three more such tours are shown in Figure 1. If the 'order of magnitude' criteria is relaxed then there are 23 tours (3 closed and the rest open) with square numbers as wazir moves. Two such tours are shown in Figure 2. The square numbers in a knight sequence forming a closed chain of knight's moves is another interesting field. They are also called 'Dawsonian Tours' in the honour of T. R. Dawson who introduced them in 1932 and compiled 100 such symmetrical tours on 8x8 board over a span of 16 years. There are 12 closed tours on 6x6 board, none of them symmetric, and Figure 3 shows four of them.

1	**4**	**9**	**16**	**25**	**36**
10	17	26	3	8	15
5	2	7	14	35	24
18	11	20	27	32	29
21	6	13	30	23	34
12	19	22	33	28	31

a. (Jelliss)

8	**1**	**4**	**25**	**36**	15
3	24	**9**	**16**	5	26
10	7	2	35	14	17
23	34	21	6	27	30
20	11	32	29	18	13
33	22	19	12	31	28

b.

17	**36**	15	24	19	34
8	**25**	18	35	14	23
5	**16**	7	22	33	20
26	**9**	**4**	**1**	30	13
3	6	11	28	21	32
10	27	2	31	12	29

c.

36	**25**	28	17	14	23
27	**16**	35	24	29	18
4	**9**	26	15	22	13
1	34	3	8	19	30
10	5	32	21	12	7
33	2	11	6	31	20

d.

Fig.1. Knight tour with square numbers in the order of magnitude as wazir {0,1} moves.

5	**36**	**25**	**16**	27	2
18	**9**	**4**	**1**	24	15
35	6	17	26	3	28
10	19	8	31	14	23
7	34	21	12	29	32
20	11	30	33	22	13

a.

5	**36**	**25**	**16**	27	2
24	**9**	**4**	**1**	18	15
35	6	17	26	3	28
10	23	8	31	14	19
7	34	21	12	29	32
22	11	30	33	20	13

b.

Fig.2. Knight tour with square numbers, in a rectangle, as wazir {0,1} moves.

There are 72 open tours with the six square numbers in knight's paths. In few tours, polygons, none of them regular, can be constructed by joining the starting and ending cells. Figure 4 shows two interesting tours in which the polygons have smallest and largest areas, being 3 units and 7 units respectively. 'Non attacking queens' is another famous problem. It is to place 'n' queens on an n x n board so that none of them attack each other. There are 12 fundamental solutions for 8x8 board. If the square numbers are taken to represent 'queens' then Dawson and Jelliss have given solutions for 8x8 board. Now, let us come to 6x6 board. Can there be a knight tour on 6x6 board in which the square numbers representing queens are non-attacking? To answer this question, let us have a look at the 6x6 board with six non-attacking queens as shown in Figure 5.

This is the only fundamental solution for non-attacking queens on 6x6 board. In order to have the six square numbers in this formation, three of them have to be on black squares and the other three on white squares, being odd and even. Since the non-attacking queens are four on black and two on white squares, so there can't be a knight tour on 6x6 board in which the square numbers representing queens are non-attacking.

7	10	23	**36**	21	12
24	**1**	8	11	26	35
9	6	**25**	22	13	20
2	31	**4**	17	34	27
5	**16**	29	32	19	14
30	3	18	15	28	33

a.

17	6	19	28	15	8
26	29	**16**	7	20	35
5	18	27	**36**	**9**	14
30	**25**	**4**	13	34	21
3	12	23	32	**1**	10
24	31	2	11	22	33

b.

17	6	23	32	15	8
24	3	**16**	7	30	33
5	18	31	22	**9**	14
2	**25**	**4**	13	34	29
19	12	27	**36**	21	10
26	**1**	20	11	28	35

c.

17	8	19	28	15	6
26	29	**16**	7	20	35
9	18	27	**36**	5	14
30	**25**	**4**	13	34	21
3	10	23	32	**1**	12
24	31	2	11	22	33

d.

Fig.3. Knight tours with square numbers in a closed chain of knight moves.

17	6	19	28	15	8
26	29	**16**	7	20	35
5	18	27	**36**	**9**	14
30	**25**	**4**	13	34	21
1	12	23	32	3	10
24	31	2	11	22	33

a. Area of polygon = 3 units

21	10	27	12	15	8
28	3	22	**9**	26	13
23	20	11	14	7	**16**
2	29	**4**	**25**	32	35
19	24	31	34	17	6
30	**1**	18	5	**36**	33

b. Area of polygon = 7 units

Fig.4. Square numbers as knight's path forming polygons with smallest and largest area.

Monogram tours, that is, knight tours delineating letter shapes, have an aesthetic appeal. Four such tours depicting letters J, K, L and X are shown in Figure 6. Figure 7 shows closed tours with the square numbers at knight's move around the starting cell 1. Magic sum properties are not usually considered in figured tours but as a bonus, three rows are also summing up to magic constant 111 in the tour shown in Figure 7a.

	Q				
			Q		
					Q
Q					
		Q			
				Q	

6 Non-attacking queens

Fig.5. Non-attacking queens on 6x6 board

	1				
			4		
					9
16					
		36			
				25	

No such tour possible because of 4 black and 2 white cells

13	34	11	24	15	**36**
30	23	14	35	10	**25**
33	12	31	26	**1**	**16**
22	29	20	7	**4**	**9**
19	32	27	2	17	6
28	21	18	5	8	3

a. J-shape

6	33	30	23	8	3
31	22	7	**4**	29	24
34	5	32	11	2	**9**
21	12	19	**16**	**25**	28
18	35	14	27	10	**1**
13	20	17	**36**	15	26

b. K-shape

27	22	35	14	11	24
34	13	26	23	**36**	15
21	28	33	12	**25**	10
32	7	30	19	**16**	3
29	20	5	2	**9**	18
6	31	8	17	**4**	**1**

c. L-shape

15	20	27	30	**1**	18
28	5	**16**	19	26	31
21	14	29	**4**	17	2
8	11	6	**25**	32	35
13	22	**9**	34	3	24
10	7	12	23	**36**	33

d. X-shape

Fig.6. Monogram tours (knight tours delineating letter shapes).

9	24	17	**36**	15	22	123
26	1	8	23	18	35	**111**
7	10	25	**16**	21	14	93
2	27	**4**	19	34	31	117
11	6	29	32	13	20	**111**
28	3	12	5	30	33	**111**

a.

7	18	27	2	5	14
28	1	6	13	26	3
17	8	19	**4**	15	12
36	29	**16**	23	32	25
9	20	31	34	11	22
30	35	10	21	24	33

b.

Fig.7. Closed knight tour with square number at knight's move from starting cell 1.

Figure 8 shows knight tours with square numbers along the sides of parallelogram. Figure 8b has two rows summing up to magic constant 111. Figure 9 shows closed knight tours with square numbers in the corners. Figure 10 shows knight tours with square numbers in the 4x4 corners.

1	34	**25**	12	**9**	6
26	13	2	7	32	11
35	24	33	10	5	8
14	27	20	3	18	31
23	**36**	29	**16**	21	**4**
28	15	22	19	30	17

a.

21	24	19	28	7	26	125
18	**1**	22	**25**	34	**9**	109
23	20	29	8	27	6	113
14	17	2	33	10	35	**111**
3	30	15	12	5	32	97
16	13	**4**	31	**36**	11	**111**

b.

Fig.8. Knight tours with square numbers along the sides of parallelogram.

1	6	13	28	31	**4**
14	27	2	5	12	29
23	36	7	30	3	32
26	15	24	19	8	11
35	22	17	10	33	20
16	25	34	21	18	**9**

a.

9	14	31	28	3	**16**
30	21	10	15	32	27
13	8	29	2	17	4
22	1	20	11	26	33
7	12	35	24	5	18
36	23	6	19	34	**25**

b.

Fig.9. Closed knight tours with square numbers in the corners.

9	6	15	2	27	**4**
14	**1**	8	5	**16**	29
7	10	35	28	3	26
34	13	22	19	30	17
23	**36**	11	32	**25**	20
12	33	24	21	18	31

a.

9	34	15	2	7	**4**
14	**1**	8	5	**16**	27
35	10	33	26	3	6
32	13	22	19	28	17
23	**36**	11	30	**25**	20
12	31	24	21	18	29

b.

Fig.10. Knight tours with square numbers in the 4x4 corners.

Figure 11 shows knight tours with square numbers in the central squares. Figure 11b has 5 magic lines (4 columns and 1 row) as bonus. Figure 12 shows closed knight tours with powers of 2 at knight move from square 1. We have already seen squares in wazir {0,1}and knight {1,2}paths. In general, an {r,s}-leaper is a piece that moves (or leaps) from cell (x,y) to any of the cells (x ± r, y ± s) or (x ± s, y ± r). In fairy chess, there are three other leapers, namely, giraffe {1,4}, zebra {2,3}and antelope {3,4} that move alternately between white and black squares.

1	8	11	34	21	6
10	33	22	7	12	35
23	2	**9**	**36**	5	20
32	17	**4**	**25**	28	13
3	24	15	30	19	26
16	31	18	27	14	29

a.

21	2	11	32	23	**4**	93
12	33	22	3	10	31	**111**
1	20	**9**	**16**	5	24	75
34	13	**36**	**25**	30	27	165
19	8	15	28	17	6	93
14	35	18	7	26	29	129
101	**111**	**111**	**111**	**111**	121	

b.

Fig.11. Knight tours with square numbers in the central squares.

5	**16**	19	34	7	36
18	27	6	1	20	33
15	**4**	17	26	35	**8**
28	25	**2**	11	**32**	21
3	14	23	30	9	12
24	29	10	13	22	31

a.

7	**16**	25	34	5	36
18	27	6	1	24	33
15	**8**	17	26	35	**4**
28	19	**2**	11	**32**	23
9	14	21	30	3	12
20	29	10	13	22	31

b.

Fig.12. Closed knight tours with powers of 2 at knight move from square 1.

1	20	3	30	15	18
36	31	**16**	19	**4**	29
21	2	23	32	17	14
24	35	10	7	28	5
11	22	33	26	13	8
34	**25**	12	**9**	6	27

a. Area of polygon = 10 units

11	20	**1**	22	**9**	14
36	23	10	13	2	7
19	12	21	8	15	30
24	35	26	31	6	3
27	18	33	**4**	29	**16**
34	**25**	28	17	32	5

b. Area of polygon = 14 units

17	**36**	**1**	24	19	22
2	31	18	21	8	**25**
35	**16**	3	32	23	20
30	13	34	7	26	**9**
15	**4**	11	28	33	6
12	29	14	5	10	27

c.

31	18	**1**	22	**9**	20
16	23	32	19	2	7
33	30	17	8	21	10
24	15	26	11	6	3
29	34	13	**4**	27	**36**
14	**25**	28	35	12	5

d.

Fig.13. Knight tours with square numbers in giraffe {1,4} path.

The author has constructed over twenty tours with square numbers in giraffe paths and four examples are shown in Figure 13. The smallest polygon formed by the giraffe path has area of 10 units and the largest one is of 14 units. Readers may like to improve upon it. Jelliss [8] has constructed a closed giraffe path on 8x8 board but the author couldn't get one on 6x6 board. Probably, it doesn't exist. Readers are requested to prove or disprove the conjecture. The author has constructed 17 tours, 14 open and 3 closed, with square numbers in zebra paths and 4 tours are shown in Figure 14. The author couldn't get a closed zebra path. Readers are requested to look for its existence. Jelliss [8] jots, “Other tasks to be considered here are, for example, {3,4} or {2,5} or fiveleaper paths. Some of these look difficult, if not impossible.” Here, Jelliss is talking about 8x8 board. The author has constructed a knight tour on 8x8 board with square numbers in antelope {3,4}path as shown in Figure 15a. The author couldn't get a closed knight tour or a closed antelope {3,4}path. Readers are requested to look for them. It is really difficult with {2,5}leaper. The nearest, the author could get is shown in Figure 15b where all square numbers except 49-64 are in {2,5}leaper path. Perhaps, knight tour with square numbers in {2,5}leaper path doesn't exist on 8x8 board. Readers are requested to prove or disprove this conjecture. The author couldn't get a knight tour with square numbers in fiveleaper {3,4}+ {0,5}closed path and readers are requested to look into it. Knight tours with square numbers as antelope {3,4}paths or {2,5}leaper paths are not possible on 6x6 board. Readers can check that the board is just too small for them. However, knight tour with square numbers in fiveleaper {3, 4}+ {0, 5}path is possible and the author could only one tour as shown in Figure 16. Here, the symmetry of line diagram joining consecutive square numbers looks elegant.

1	12	**25**	6	3	30
24	7	2	31	26	5
11	32	13	**4**	29	**36**
8	23	10	27	**16**	19
33	14	21	18	35	28
22	**9**	34	15	20	17

a.

3	12	**25**	14	**1**	6
26	15	2	5	24	21
11	**4**	13	22	7	**36**
16	27	34	31	20	23
33	10	29	18	35	8
28	17	32	**9**	30	19

b.

3	14	**25**	22	**1**	12
26	23	2	13	32	21
7	**4**	15	24	11	**36**
16	27	6	33	20	31
5	8	29	18	35	10
28	17	34	**9**	30	19

c.

17	12	19	**36**	5	14
20	3	**16**	13	26	35
11	18	31	**4**	15	6
2	21	10	27	34	**25**
9	30	23	32	7	28
22	**1**	8	29	24	33

d.

Fig.14. Knight tours with square numbers in zebra {2,3} path.

47	52	23	**4**	45	**36**	21	6
24	3	46	35	22	5	10	37
53	48	51	44	11	34	7	20
50	**25**	2	57	32	19	38	**9**
1	54	**49**	12	43	8	33	18
26	13	56	61	58	31	42	39
55	60	15	28	63	40	17	30
14	27	62	59	**16**	29	**64**	41

a. antelope {3,4} path

43	48	45	28	41	26	23	52
46	29	42	**49**	22	51	40	**25**
1	44	47	62	27	24	53	60
30	63	**16**	21	50	61	8	39
17	2	31	64	35	10	59	54
32	15	20	11	56	7	38	**9**
3	18	13	34	5	**36**	55	58
14	33	**4**	19	12	57	6	37

b. {2,5} leaper path (nearest approach)

Fig.15. Knight tour with square numbers in (a) {3,4} and (b) {2,5} leaper path.

11	32	**9**	28	13	30
36	27	12	31	8	**25**
33	10	35	26	29	14
20	17	2	5	24	7
1	34	19	22	15	**4**
18	21	**16**	3	6	23

Fig.16. Knight tour with square numbers in fiveleaper {3,4} + {0,5} path.

1B. Other numbers in formations: Tours having numbers in arithmetic progression are also of interest. In general, it is more difficult to get figured tours with arithmetic progressions than the tours with square numbers because unlike latter, the number of intermediate cells is fixed. So the choice for subsequent moves gets restricted as the tour progresses. There are no tours with all the numbers along even diagonal in a multiple of 6. However, two such nearest approach tours are shown in Figure 17. There are no tours in arithmetic progression with common difference 6 along the even or odd diagonals. Two such nearest approach tours are shown in Figure 18. However, if the order of magnitude criterion is relaxed, odd diagonal in Figure 18b has all the numbers in arithmetic progression. Now let us come to tours with rows in arithmetic progression having common difference (CD) as 3 or 5 or 7. That is, tours with 1-4-7-10-13-16 (CD 3) or 1-6-11-16-21-26 (CD 5) or 1-8-15-22-29-36 (CD 7) along any row with any sequence. Jelliss [9] has shown that there is no tour with numbers in arithmetic progression along rows or columns "even if we drop the requirement for the numbers to be in order of magnitude." However, two nearest approach tours are shown in Figure 19.

1	28	13	20	3	**6**
30	21	2	5	**12**	19
27	14	29	**18**	7	4
22	31	**24**	9	34	11
15	26	35	32	17	8
36	23	16	25	10	33

a.

15	28	13	20	17	**6**
30	21	16	5	**12**	19
27	14	29	**18**	7	4
22	31	**24**	9	34	11
1	26	35	32	3	8
36	23	2	25	10	33

b.

Fig.17. Knight tour with multiples of 6 along even diagonal (nearest approach).

1	12	15	6	3	22
14	**7**	2	23	16	5
11	28	**13**	4	21	24
8	31	10	**19**	34	17
27	36	29	32	**25**	20
30	9	26	35	18	33

a.

1	12	15	8	23	10
14	**7**	36	11	16	21
35	2	**13**	22	9	24
6	27	4	**31**	20	17
3	34	29	18	**25**	32
28	5	26	33	30	**19**

b.

Fig.18. Knight tour with common difference 6 along odd diagonal (nearest approach).

1	**4**	**7**	**16**	**13**	22
6	17	2	21	26	15
3	8	5	14	23	12
18	31	20	25	34	27
9	36	29	32	11	24
30	19	10	35	28	33

a.

14	21	12	9	16	23
1	**8**	**15**	**22**	**29**	10
20	13	2	11	24	17
7	34	19	28	3	30
36	27	32	5	18	25
33	6	35	26	31	4

b.

Fig.19. Knight tour with a row in arithmetic progression (nearest approach).

The numbers of the form n(n+1)/2, that is, 1,3,6,10,15,21,28,36,… are called triangular numbers. Figure 20 shows knight tour with triangular numbers forming trapezium and in a compact formation. Figure 21 shows triangular numbers in triangular and octagonal formation. All the triangular numbers are equidistant from the centre of the 6 x 6 board in Figure 21b. We subtract 1 from 1,3,7,13,21,31 to get 0,2,6,12,20,30 which are called 'metasquare numbers' or 'Double triangle numbers' of form n(n+1). Figure 22 shows two tours with 'metasquare numbers' along long diagonals. The numbers of the form n(3n-1)/2, that is, 1,5,12,22,35,… are called pentagonal numbers. Figure 23 shows tours with pentagonal numbers forming smallest and largest pentagons having 7 units and 20.5 units of area respectively.

9	20	27	24	11	22
26	**3**	**10**	**21**	**28**	**1**
19	8	25	2	23	12
4	33	**6**	**15**	**36**	29
7	18	31	34	13	16
32	5	14	17	30	35

a. Isoceles trapezium

7	2	5	12	9	20
4	13	8	19	26	11
1	**6**	**3**	**10**	**21**	**36**
14	31	34	25	18	27
33	24	29	16	35	22
30	**15**	32	23	**28**	17

b. Isoceles trapezium

9	32	35	2	11	16
34	**3**	**10**	**15**	**36**	**1**
31	8	33	12	17	14
4	**21**	**6**	19	**28**	25
7	30	23	26	13	18
22	5	20	29	24	27

c. Right angled trapezium

9	20	29	4	11	2
30	5	**10**	**1**	**36**	27
19	8	**21**	**28**	**3**	12
22	31	**6**	**15**	26	35
7	18	33	24	13	16
32	23	14	17	34	25

d. Compact formation

Fig.20. Knight tour with triangular numbers forming trapezium and in a compact formation.

9	24	27	2	11	18
26	**3**	**10**	17	**28**	**1**
23	8	25	12	19	16
4	33	**6**	**15**	**36**	29
7	22	31	34	13	20
32	5	14	**21**	30	35

a. Triangular formation

13	16	**1**	**36**	7	18
24	29	14	17	2	35
15	12	23	8	19	**6**
28	25	30	5	34	**3**
11	22	27	32	9	20
26	31	**10**	**21**	4	33

b. Octagonal formation

Fig.21. Triangular numbers in triangular and octagonal formation.

2	19	22	11	8	33
23	**12**	1	32	21	10
18	3	**20**	9	34	7
13	24	15	**0**	31	28
4	17	26	29	**6**	35
25	14	5	16	27	**30**

a.

6	19	22	11	8	33
21	**12**	7	32	23	10
18	5	**20**	9	34	1
13	28	15	**0**	31	24
4	17	26	29	**2**	35
27	14	3	16	25	**30**

b.

Fig.22. Knight tour with 'Metasquare numbers' along long diagonals.

9	20	**35**	28	11	14
36	29	10	13	**22**	27
19	8	21	34	15	**12**
30	**1**	32	**5**	26	23
7	18	3	24	33	16
2	31	6	17	4	25

a. Area of pentagon = 7 units

1	16	19	30	**5**	10
18	31	4	9	20	29
15	2	17	6	11	8
32	25	34	3	28	21
35	14	23	26	7	**12**
24	33	36	13	**22**	27

b. Area of pentagon = 20.5 units

Fig.23. Knight tour with pentagonal numbers forming smallest and largest pentagons.

Numbers in the sequence 1, 2, 3, 5, 8, 13, 21, 34 … are called Fibonacci numbers, named after the 12th century Italian mathematician Leonardo of Pisa, later known as Fibonacci. Figure 24 shows tours with Fibonacci numbers forming smallest and largest polygons having 5 units and 12 units of area respectively. It is obvious from the figures that a larger polygon is not possible; however, readers may look for a smaller polygon. Numbers in the sequence 2, 1, 3, 4, 7, 11, 18, 29 … are called Lucas numbers, named after the 19th century French mathematician Edouard Lucas. Figure 25 shows tours with Lucas numbers forming smallest and largest polygon having 5 units and 19 units of area respectively. Readers may look to improve upon them. Prime numbers have been fascinating humankind for over 2000 years. Figure 26 shows odd primes in rectangle and trapezium formation. It is interesting to note that Figure 26b has five rows adding to 113. 'Domination by queen' is another classic problem and Figure 27 shows all the primes being dominated by queen at cell 13 and 11 respectively. As a bonus, Figure 27b has two magic rows too.

13	16	33	22	7	4
34	**21**	14	**5**	32	23
15	12	17	**8**	**3**	6
28	35	20	**1**	24	31
11	18	29	26	9	**2**
36	27	10	19	30	25

a. Area of polygon = 5 units

23	14	35	12	25	20
36	11	24	**21**	**34**	29
15	22	**13**	28	19	26
10	**5**	**8**	**1**	30	33
7	16	**3**	32	27	18
4	9	6	17	**2**	31

b. Area of polygon = 5 units

3	28	31	18	**5**	**8**
32	19	4	7	30	17
27	**2**	29	16	9	6
20	33	22	25	12	15
1	26	35	14	23	10
34	**21**	24	11	36	**13**

c. Area of polygon = 23 units

3	26	29	10	**5**	**8**
28	17	4	7	30	11
25	**2**	27	18	9	6
16	35	20	23	12	31
1	24	33	14	19	22
34	15	36	**21**	32	**13**

d. Area of polygon = 23 units

Fig. 24. Knight tour with Fibonacci numbers forming smallest and largest polygons.

5	36	**7**	10	19	**2**
8	**11**	**4**	**1**	30	17
35	6	9	**18**	**3**	20
12	25	22	31	16	**29**
23	34	27	14	21	32
26	13	24	33	28	15

a. Area of polygon = 5 units

7	**4**	21	32	15	**2**
22	13	6	**3**	20	31
5	8	33	14	**1**	16
12	23	10	19	30	27
9	34	25	28	17	36
24	**11**	**18**	35	26	**29**

b. Area of polygon = 19 units

Fig. 25. Knight tour with Lucas numbers forming smallest and largest polygon.

27	6	**11**	2	21	4
10	**19**	28	**5**	12	1
7	26	9	20	**3**	22
32	**29**	18	15	36	**13**
25	8	**31**	34	**23**	16
30	33	24	**17**	14	35

a.

1	36	21	14	**11**	30	113
20	15	2	**29**	22	**13**	101
35	6	**19**	12	**31**	10	113
18	**3**	16	**23**	28	25	113
7	34	**5**	26	9	32	113
4	**17**	8	33	24	27	113

b.

3	10	**5**	22	1	12
36	**23**	2	**11**	6	21
9	4	**7**	20	**13**	30
24	35	26	**31**	16	**19**
27	8	33	18	**29**	14
34	25	28	15	32	**17**

c.

7	10	**5**	22	1	12
36	**23**	8	**11**	4	21
9	6	**17**	30	**13**	2
24	35	26	**3**	20	**31**
27	16	33	18	**29**	14
34	25	28	15	32	**19**

d.

Fig. 26. Odd primes in (a, b) rectangle and in (c, d) trapezium.

17	22	15	10	**7**	24
14	**11**	18	**23**	36	9
21	16	**13**	8	25	6
12	**31**	**2**	**19**	28	35
3	20	33	30	**5**	26
32	1	4	27	34	**29**

a.

7	10	21	14	**31**	28	**111**
20	**13**	8	**29**	22	15	107
9	6	**11**	32	27	30	115
12	**19**	**2**	**23**	16	33	105
5	24	35	18	**3**	26	**111**
36	1	4	25	34	**17**	117

b.

Fig. 27. Queen at cell 13 in (a) and at cell 11 in (b) dominating all the primes.

2. Figured tours in three dimensions: For almost a millennium, knight's tour was confined to two-dimensional board and was extended into three dimensions in 18th century. A.T. Vandermonde [10], a mathematician, musician and chemist, was the first to construct a three-dimensional knight's tour, in a 4×4×4 cube, published in 1771. Other 3D examples have been provided by Schubert [11], Gibbins [12], Stewart [13], Jelliss [14], Petkovic [15] and DeMaio [16]. More recently, Awani Kumar [17] [18] [19] [20], looked into the possibilities of knight's tours in cubes and cuboids, having 'magic' properties. In three dimensions, the knight is assumed to move in its usual fashion in each of the three mutually perpendicular planes through the cell it initially occupies. If the three coordinate directions are x, y, z then the three planes can be represented by the pairs of coordinates xy, xz, yz. Thus the mobility of the knight is multiplied three-fold; on a two-dimensional board the knight has a choice of up to 8 cells to which it can move, but in three dimensions it can have as many as 24 cells to move. However, on small boards, or near the edges of larger boards, the number of moves will of course be less than this maximum, since blocked by the board edges or faces. The knight cannot move at all in a 2×2×2 cube, or from the central cell of a 3×3×3 cube. So the smallest cubical board on which the knight is mobile on every cell is the 4×4×4 cube and the author plans to look for 'Figured tours' in it. Figure 28 shows the 12 possible moves of a knight from an inner cell of a 4×4×4 cube.

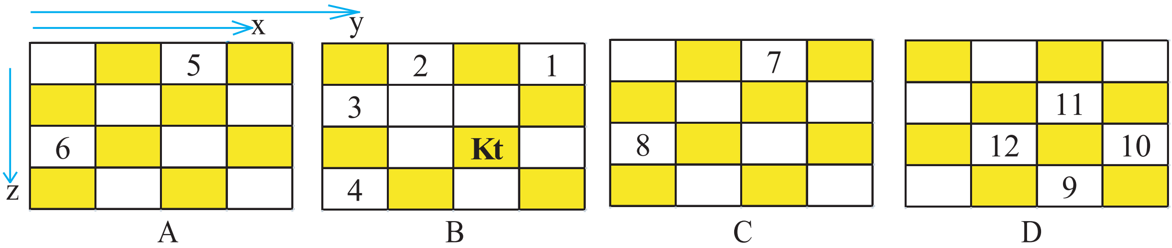


Fig. 28. Possible knight's move in 4x4x4 cube.

Readers can visualize this 3D board as a stack of four 2D boards, one above the other, lettered in alphabetical order A to D. As in the 2D case the cubical cells (or their floors) can be coloured alternately light and dark so that a knight at a light cell can only jump to a dark cell and vice versa. This remains true in higher dimensions too. Figure 29a and Figure 29b show square numbers in the corner cells and in the central cells of the cube, respectively.

1	42	29	**4**
30	17	38	41
39	28	3	32
16	31	40	**9**

62	27	2	13
15	12	53	50
26	51	14	11
59	10	37	52

21	34	5	46
6	43	20	33
63	18	45	8
44	7	58	19

64	47	22	**49**
61	24	57	54
56	35	48	23
25	60	55	**36**

a.

33	2	11	54
10	53	48	27
39	34	63	12
52	13	38	57

40	59	62	3
47	**4**	**9**	58
32	**1**	**16**	43
15	46	51	28

31	18	55	26
24	**49**	**36**	19
17	**64**	**25**	56
50	35	14	37

22	41	8	61
5	60	23	42
30	21	44	7
45	6	29	20

b.

Fig. 29. Knight tour with square numbers in (a) corner cells and (b) central cells.

Figure 30a and Figure 30b show cubic numbers and square numbers in the corner cells and in the central cells of the cube, respectively. Figure 31a and Figure 31b show square numbers in a circular formation on the surface and inside the cube, respectively. Figure 32a and Figure 32b show square numbers in a closed chain of knight moves in square and diamond quarte and in Beverley quarte, respectively. It is just to remind the readers that William Beverley was the first person to construct a magic knight tour on 8x8 board using what is now known as 'beverley quarte', in 1848.

1	12	39	**8**
22	45	28	13
19	14	9	40
64	31	20	**27**

18	37	6	33
63	32	23	48
54	41	34	7
21	46	61	30

55	50	11	38
2	59	62	29
15	44	51	10
60	3	26	47

36	17	52	**49**
53	24	5	58
56	35	16	43
25	42	57	**4**

a.

34	37	26	47
31	46	35	24
6	33	48	17
45	22	5	42

9	14	55	18
54	**1**	**8**	15
61	**64**	**27**	56
28	57	60	23

30	51	38	3
7	**36**	**25**	50
32	**49**	**4**	43
21	44	41	16

39	2	13	52
10	53	40	19
29	58	63	12
62	11	20	59

b.

Fig. 30. Knight tour with square numbers and cubic numbers in the (a) corner cells and (b) central cells.

39	**36**	**1**	56
16	63	48	**25**
49	58	15	**64**
52	**9**	**4**	47

22	57	40	37
53	8	5	62
44	23	32	59
17	60	43	24

35	38	31	2
50	11	34	55
33	14	3	26
10	51	46	61

30	21	6	41
7	54	29	20
18	27	42	13
45	12	19	28

a.

7	44	13	2
54	3	6	33
45	8	55	12
26	53	28	5

60	**1**	**36**	21
49	52	61	**64**
16	35	48	**9**
47	**4**	**25**	32

39	20	57	14
46	43	40	11
59	56	15	34
42	27	62	29

50	37	22	19
41	18	63	30
38	51	58	23
17	24	31	10

b.

Fig. 31. Knight tour with square numbers in a circle (a) on the surface and (b) inside the cube.

5	24	29	62
32	61	8	19
59	6	33	30
34	31	48	7

56	**1**	22	27
35	40	55	**16**
4	15	58	63
47	54	**9**	18

23	28	57	**64**
50	**25**	52	41
45	60	**49**	20
36	51	46	53

12	39	2	21
3	42	13	26
38	11	44	17
43	14	37	10

a.

1	12	31	18
22	19	**4**	27
11	44	21	**16**
20	**9**	26	5

30	37	2	59
3	42	53	6
54	15	60	35
23	34	41	8

13	62	**49**	28
64	45	32	17
61	**36**	63	50
10	43	46	**25**

38	29	56	51
55	52	39	58
14	57	48	7
47	24	33	40

b.

Fig. 32. Squre numbers in a closed chain of knight moves in (a) square and diamond, (b) Beverley quarte.

Figure 33 shows knight tour with square numbers at the corners of a truncated pyramid. Figure 34 shows knight tour with square numbers at the corners of a three-dimensional trapezoid (or trapezoidal prism). Figure 35 shows closed knight tour with square numbers in zebra {2,3} path. Figure 36 shows knight tour with square numbers in closed zebra {2,3}path. Figure 37 shows closed tour of knight with powers of 2 at knight move from cell 1.

1	46	53	**4**
28	59	8	55
41	54	3	60
16	43	58	**9**

34	19	2	51
13	50	35	38
18	45	26	5
27	12	17	44

29	52	47	20
62	39	56	7
15	42	61	48
40	63	10	57

14	21	30	37
33	**64**	**25**	22
24	**49**	**36**	31
11	32	23	6

Fig. 33. Knight tour with square numbers at the corners of a truncated pyramid.

1	30	41	**64**
10	37	56	59
35	58	21	8
18	**9**	**36**	57

44	5	20	31
19	60	51	6
46	27	40	63
11	48	17	14

29	26	39	54
2	55	42	13
43	38	33	22
34	23	12	7

4	53	32	**25**
45	24	3	50
28	61	52	15
47	**16**	**49**	62

Fig. 34. Knight tour with square numbers as three-dimensional trapezoid (or trapezoidal prism).

25	52	19	2
44	59	56	51
57	26	47	**36**
32	35	58	55

12	**1**	24	17
31	34	43	62
10	23	20	27
41	60	39	**4**

45	18	53	**64**
42	63	50	3
49	46	37	54
38	33	48	61

30	13	**16**	7
11	6	29	14
22	15	8	5
9	40	21	28

Fig. 35. Closed Knight tour with square numbers in zebra {2,3} path in 4x4x4 cube.

1	44	17	10
14	11	32	47
45	2	13	**36**
12	31	46	33

16	41	6	61
51	60	15	40
58	37	50	**9**
29	34	59	22

7	18	43	**64**
28	63	48	19
49	8	3	62
52	21	30	35

42	5	56	**25**
57	24	39	54
4	55	26	23
27	38	53	20

Fig. 36. Knight tour with square numbers in closed zebra {2,3} path in 4x4x4 cube.

1	30	15	46
24	45	**4**	29
13	**2**	31	26
54	25	12	3

14	47	**64**	37
63	52	7	44
32	39	60	51
11	42	55	28

23	**16**	49	20
8	41	62	27
61	48	5	38
34	53	10	43

58	21	6	17
33	18	57	36
22	59	50	19
9	40	35	56

Fig. 37. Closed tour of knight with powers of 2 at knight move from starting cell 1.

Figure 38 shows tours of knight with cell 1 equidistant from multiples of (a) 6, (b) 8, (c) 10, and (d) 12, respectively. Here Figure 38a is an open tour. Readers are requested to improve upon it to get a closed knight tour. There are 4 space diagonals in a cube. Figure 39 shows two tours with square numbers along the space diagonals.

41	32	3	44
2	39	62	33
25	56	43	38
22	37	**12**	53

24	57	50	29
63	28	1	52
42	35	58	55
11	**54**	27	**36**

31	40	47	16
48	61	4	45
23	46	49	34
26	21	**60**	13

64	17	**30**	51
5	8	59	**18**
10	19	**6**	15
7	14	9	20

a.

39	18	41	2
62	31	38	19
37	**40**	61	30
46	63	58	13

42	1	44	15
45	50	59	**32**
24	29	**56**	3
57	12	53	20

17	28	35	**8**
36	7	54	49
55	**64**	23	14
52	47	60	11

34	43	**16**	27
51	**48**	33	6
22	5	26	9
25	10	21	4

b.

14	1	6	33
3	48	15	**10**
30	17	**50**	5
49	4	47	16

7	32	13	**20**
62	37	56	59
27	64	29	18
38	61	36	55

2	21	**40**	9
39	**60**	51	34
42	31	22	11
23	52	43	46

41	8	57	12
24	53	44	19
63	28	25	58
26	45	54	35

c.

34	17	2	41
3	52	49	18
24	35	40	5
51	4	7	62

1	42	39	14
50	57	64	61
33	**60**	25	22
8	63	54	19

16	37	**48**	21
45	30	53	58
36	23	28	47
29	46	31	6

43	**12**	15	38
10	59	44	13
27	56	11	20
32	9	26	55

d.

Fig.38. Knight tour with multiples of (a) 6, (b) 8, (c) 10, and (d) 12 at knight move from starting cell 1.

1	6	37	60
40	59	52	5
43	62	23	38
64	39	14	53

48	61	22	3
27	**4**	41	34
18	**49**	44	57
13	58	17	50

7	2	47	56
42	55	**36**	51
63	46	**9**	24
28	15	54	35

32	21	10	**25**
11	26	33	30
8	31	20	45
19	12	29	**16**

a.

1	42	31	54
52	29	12	33
13	32	53	30
18	11	14	**25**

38	35	44	5
19	**4**	61	28
2	43	**36**	45
51	24	17	58

41	8	55	34
62	**49**	20	59
37	56	**9**	26
10	15	48	23

64	39	6	27
3	60	63	22
40	7	46	57
47	50	21	**16**

b.

Fig.39. Knight tour with square numbers along space diagonals in 4x4x4 cube.

Figure 27 showed 'domination by queen' in two dimensions. Figure 40 shows possible queen's move in three dimensions and Figure 41 shows queen at cell 5 dominating all the primes in 4x4x4 cube.

•	•	•	
•	•	•	
•	•	•	

	•		•
•	•	•	
•	Q	•	•
•	•	•	

•	•	•	
•	•	•	
•	•	•	

	•		•
	•		•

Fig. 40. Possible queen's move in 4x4x4 cube (shown as dots).

1	14	33	20
18	**53**	58	15
59	48	**19**	6
52	**17**	62	27

32	**47**	40	**7**
51	44	63	4
2	**5**	60	**41**
61	54	**3**	16

39	30	57	12
64	**13**	34	21
31	56	49	28
50	**43**	26	55

46	**37**	8	**23**
9	22	45	36
38	**29**	24	**11**
25	10	35	42

Fig. 41. Queen (at cell 5) dominating all the primes in 4x4x4 cube.

Monogram tours, that is, knight tours delineating letter shapes, have an aesthetic appeal. Fourteen such tours depicting letters C, D, J, L, T and X are shown in Figure 42. Letter O has already been shown in Figure 31.

9	**4**	**1**	38
16	37	32	41
25	40	13	2
36	**49**	**64**	31

22	55	12	3
57	50	63	30
8	61	56	39
17	58	33	48

5	10	23	28
44	15	42	47
35	24	27	14
26	43	46	19

54	21	6	11
7	62	53	20
60	51	34	29
45	18	59	52

a. C-shape

2	41	10	19
11	50	45	58
40	3	18	53
17	52	47	44

9	**4**	**1**	60
16	59	30	43
25	54	37	20
36	**49**	**64**	57

12	55	38	33
39	42	63	56
8	51	46	61
15	62	7	48

5	32	23	28
24	29	6	31
13	22	27	34
26	35	14	21

b. C-shape

1	**36**	39	22
4	41	**64**	11
9	38	**49**	40
16	**25**	10	31

46	51	48	7
63	28	61	32
50	35	8	57
3	62	15	12

5	34	21	52
2	37	44	23
45	42	53	30
26	17	24	43

20	47	6	33
27	60	19	56
54	29	58	13
59	14	55	18

c. D-shape

10	55	50	23
63	44	3	54
2	27	46	59
29	62	17	26

1	**36**	41	48
4	53	**64**	39
9	60	**49**	24
16	**25**	8	61

42	51	22	37
11	56	43	52
30	45	58	47
57	28	31	18

5	20	35	40
34	13	32	19
15	6	21	38
12	33	14	7

d. D-shape

5	48	**1**	10
64	11	**4**	47
49	46	**9**	12
36	**25**	**16**	43

2	51	30	55
37	58	63	40
52	33	38	61
17	42	35	26

29	22	3	60
6	45	28	13
21	50	23	44
24	15	8	41

20	59	54	31
53	32	19	56
18	57	62	39
7	34	27	14

e. J-shape

2	55	8	45
35	30	57	38
10	47	50	53
17	34	31	58

13	46	**1**	54
64	37	**4**	61
49	62	**9**	44
36	**25**	**16**	33

42	29	56	39
3	60	7	52
18	43	48	59
11	32	51	24

19	14	41	28
12	27	22	5
63	20	15	40
26	23	6	21

f. J-shape

1	28	43	18
4	17	2	27
9	50	15	**64**
16	**25**	**36**	**49**

54	51	38	29
57	26	47	32
52	63	56	19
5	48	11	24

39	44	55	22
34	3	42	13
59	8	37	30
10	35	14	7

58	21	46	31
53	62	33	20
40	45	60	23
61	6	41	12

g. L-shape

20	33	58	7
17	60	31	48
32	63	6	43
5	44	61	26

1	8	19	54
4	51	22	35
9	34	15	**64**
16	**25**	**36**	**49**

18	59	30	57
21	56	47	42
28	41	62	55
45	50	27	24

29	2	53	12
46	13	38	23
3	52	11	40
10	39	14	37

h. L-shape

1	10	13	56
4	15	**64**	11
9	22	**49**	14
16	**25**	**36**	21

8	55	2	59
17	60	35	46
42	53	58	51
5	44	41	26

3	62	57	12
38	45	28	63
19	50	23	48
24	37	20	33

18	7	30	47
29	54	39	34
6	61	52	31
43	32	27	40

i. L-shape

40	31	18	47
17	58	51	44
32	41	54	57
5	52	15	42

1	48	61	28
4	45	**64**	35
9	62	**49**	46
16	**25**	**36**	63

30	59	50	19
39	34	23	56
6	53	60	43
33	14	55	24

7	2	29	12
22	13	38	27
3	8	11	20
10	21	26	37

j. L-shape

1	**16**	**9**	26
36	**25**	**4**	23
17	**64**	35	10
34	**49**	24	5

32	27	54	11
53	50	61	6
46	57	8	51
37	52	43	20

39	2	29	22
62	15	42	3
33	30	63	58
18	59	48	13

28	31	40	55
41	60	7	12
38	45	56	21
47	14	19	44

k. T-shape

10	39	46	17
43	50	53	24
40	47	60	45
31	44	23	52

1	**16**	**9**	62
36	**25**	**4**	51
11	**64**	57	18
42	**49**	30	33

38	63	56	19
41	48	37	54
58	55	8	61
7	32	59	22

27	2	5	14
6	15	28	3
35	26	13	20
12	21	34	29

l. T-shape

1	10	35	**4**
34	**25**	**64**	37
21	**36**	**9**	24
16	33	22	**49**

60	3	28	39
15	42	61	48
2	57	40	51
43	50	7	56

11	26	59	52
20	63	8	5
31	46	23	38
14	17	32	45

30	53	12	27
13	18	29	62
54	41	58	47
19	44	55	6

m. X-shape

62	29	14	33
15	56	41	50
18	45	60	13
59	20	17	40

1	32	61	**4**
58	**25**	**64**	39
63	**36**	**9**	44
16	57	42	**49**

30	55	34	27
47	28	51	24
52	31	46	35
19	48	21	12

53	26	3	38
2	23	8	5
7	54	37	10
22	11	6	43

n. X-shape

Fig. 42. Monogram tours (knight tours delineating letter shapes) in 4x4x4 cube.

3. Figured tours in four dimensions: Manning [21] mentions that "The notion of geometries of n dimensions began to suggest itself to mathematicians about the middle of the 19th century. Cayley, Grassmann, Riemann, Clifford and some others introduced it into their mathematical investigations."Awani Kumar [22] extended knight's tour in hyperspace and constructed magic tours in four and five dimensions. Before we go for figured tours in four dimensions it is important for the reader to visualize a 4D hypercube. If we move a unit square a unit distance orthogonal to its plane in 3D space and join the corresponding corners, we get a cube. Analogously, we can imagine moving this unit cube a unit distance in an 'orthogonal' direction in 4D space to produce the 4D equivalent of a 3D cube, which is known as a 'hypercube' However we can only show this by means of a perspective drawing as shown in Figure 43.

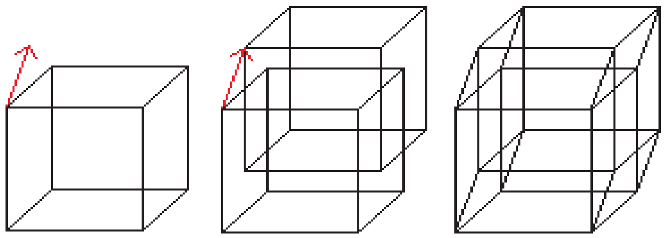


Fig. 43. Visualisation of a hypercube.

This way, we can visualize a hypercube and be comfortable (and confident) with its 'look'. The hypercube has 16 corners (derived from 2 cubes), 32 edges (2 cubes and joining lines) and 24 square faces. Figure 44 shows all the possible knight's moves from an inner cell in a 4×4×4×4 hypercube. Once the readers can visualize the jumps of the knight in hyperspace, they can count the possible number of knight moves from the nine distinct cell positions (lettered A to I) in the six planes (*xy*, *xz*, *xw*, *yz*, *yw*, *zw*) determined by pairs of the four coordinates *x*, *y*, *z*, *w*, as shown in Figure 45. On a two-dimensional board, knight has a choice of minimum 2 cells (when it is in the corner) and maximum up to 8 cells to which it can move. Now, readers can see that in 4D, knight has a choice of minimum 6x2 (= 12 cells) and maximum up to 6x8 (= 48 cells) to which it can move. In general, knight has a choice of minimum n(n-1) cells and maximum up to 4n(n-1) cells to which it can move in a *n*-dimensional hypercube.

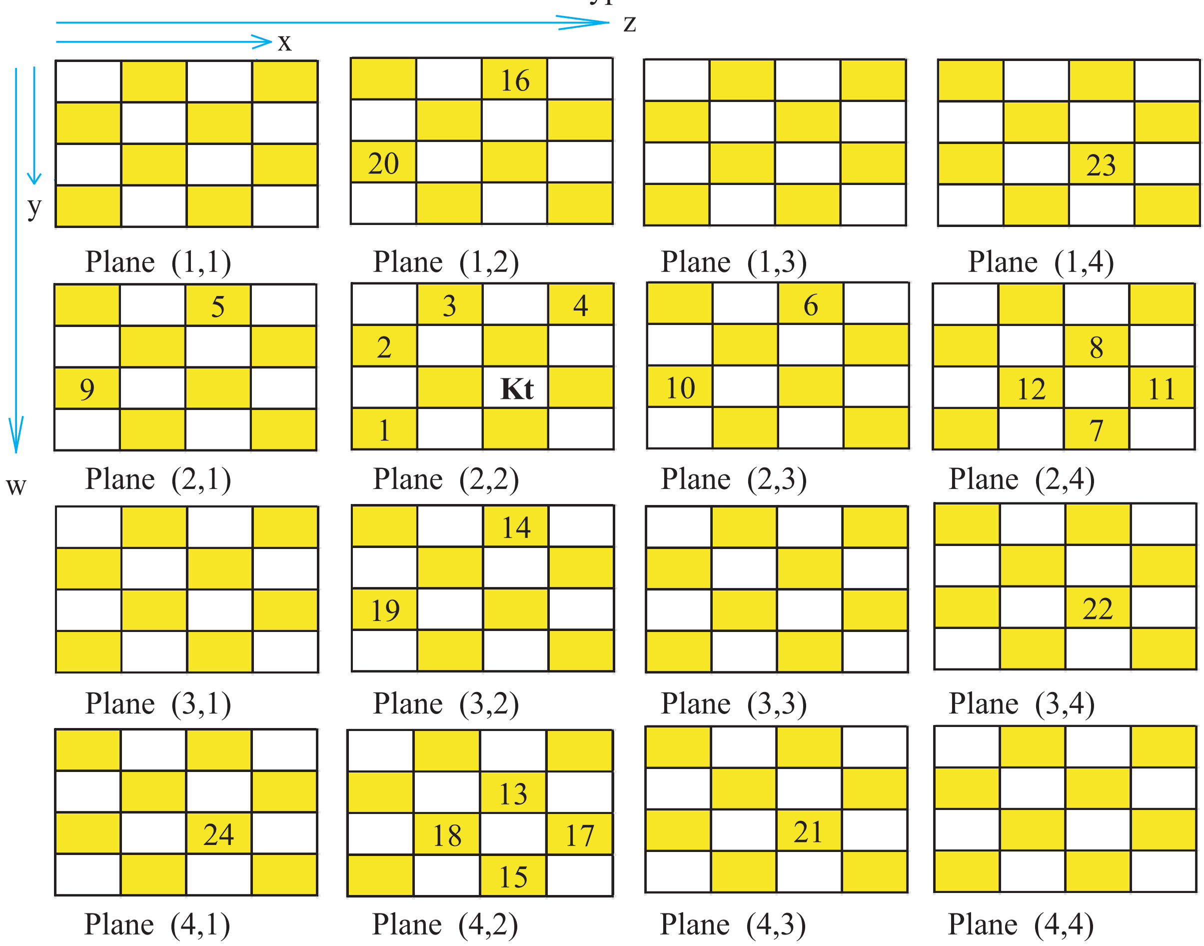
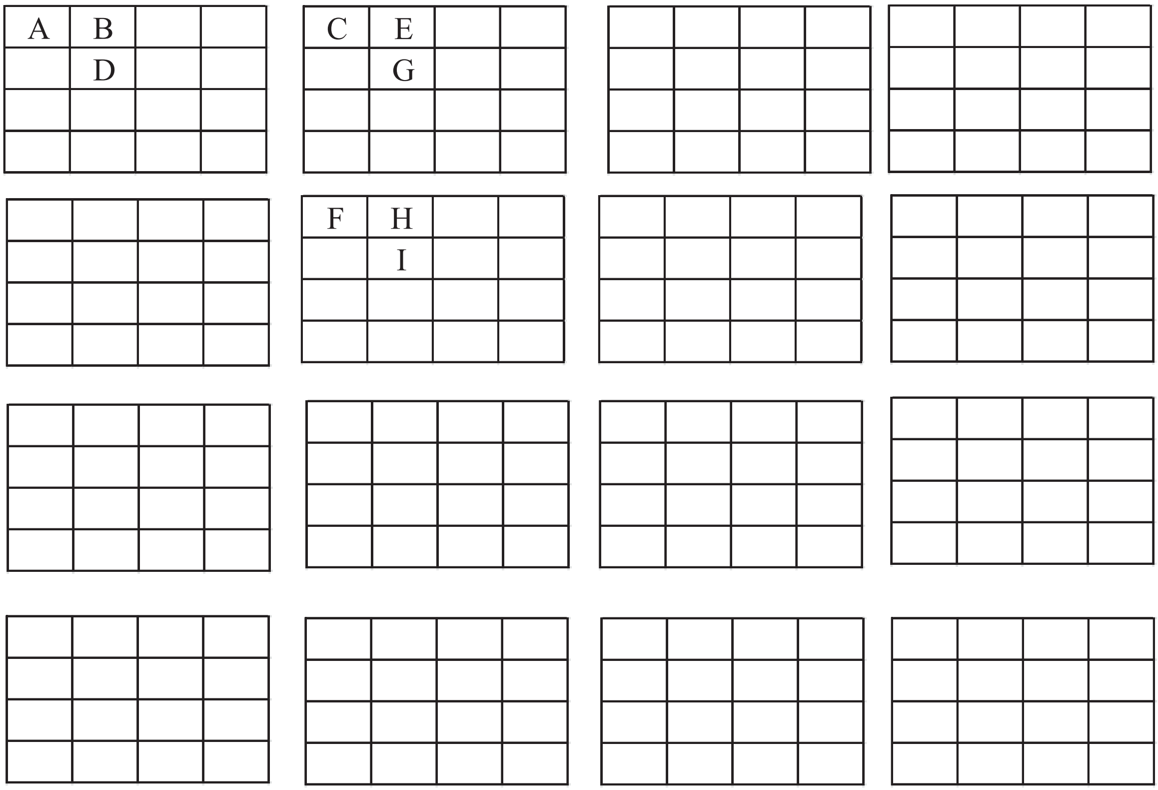


Fig.44. Possible knight's move in 4x4x4x4 hypercube.



	A	B	C	D	E	F	G	H	I
xy=	2	3	2	4	3	2	4	3	4
xz=	2	3	3	3	4	3	4	4	4
yz=	2	2	3	3	3	3	4	3	4
xw=	2	3	2	3	3	3	3	4	4
yw=	2	2	2	3	2	3	3	3	4
zw=	2	2	3	2	3	4	3	4	4
Total	**12**	**15**	**15**	**18**	**18**	**18**	**21**	**21**	**24**

Fig.45. Possible number of knight's moves from various cells in a 4x4x4x4 hypercube.

After having a clear picture of the possible knight moves in hyperspace, we come to the tour of knight in it. Knight cannot move at all in a 2x2x2x2 cube and can neither move-in nor move-out from the central cell of a 3x3x3x3 cube. So, 4x4x4x4 is the smallest cube in which both closed and open tours are possible in 4-dimensions and the author plans to look for 'Figured tours' in it. Figure 46a and Figure 46b show knight tour with square numbers in the corner cells and in the central cells of 4x4x4x4 hypercube, respectively. There are eight 4D space diagonals in a 4x4x4x4 hypercube. Figure 47a and Figure 47b show knight tour with square numbers along the 4D diagonals. Figure 48a, Figure 48b and Figure 48c show knight tour with square numbers in square, diamond and Beverley quartes, respectively. Knight tour with square numbers as three-dimensional trapezoid (or trapezoidal prism) has been shown in Figure 34. Its extension as 4-dimensional parallelotope is in Figure 49. Figure 50 shows knight tour with square numbers in closed zebra {2,3}path in 4x4x4x4 hypercube. Figure 51 shows open knight tour with square numbers in zebra {2,3}path in 4x4x4x4 hypercube. Here, only one square number is in each layer. Readers are requested to look for closed tour. Figure 52 shows closed tour of knight with square numbers and cubic numbers at knight move from cell 1.

1	60	11	**256**	18	21	14	59	13	58	17	20	**16**	19	46	**81**
12	57	50	85	83	86	151	148	152	147	84	87	79	82	15	38
51	22	111	106	154	149	208	205	145	234	153	48	118	47	80	45
144	107	96	**49**	117	200	143	150	220	183	146	199	**169**	78	125	**36**

52	89	110	179	141	228	189	242	2	251	244	255	137	140	3	44
55	92	97	88	116	201	142	215	239	214	211	198	170	193	126	123
26	109	102	241	209	188	245	190	246	229	240	243	187	136	129	42
95	56	113	108	114	191	210	195	213	194	221	182	192	35	122	39

27	10	101	254	226	253	248	231	247	230	237	252	186	135	128	41
94	61	112	217	167	216	207	204	212	233	218	181	119	172	185	76
23	90	105	180	164	173	236	249	235	250	223	232	168	139	134	37
54	93	98	203	155	202	165	206	222	219	184	197	171	120	77	124

100	91	104	**9**	163	174	227	160	238	159	224	177	**225**	138	133	**4**
53	8	99	178	156	161	166	175	7	176	157	74	162	75	130	43
28	69	24	103	115	72	131	68	158	67	70	5	71	132	127	40
25	62	29	**196**	30	65	32	73	33	6	63	66	**64**	31	34	**121**

Fig.46a. Knight tour with square numbers in the corners of 4x4x4x4 hypercube.

51	6	53	42	44	75	2	7	3	8	43	46	76	45	74	11
54	41	84	95	77	80	145	82	48	83	96	79	97	78	123	68
29	52	147	152	146	151	176	181	177	182	47	150	166	133	130	73
40	85	94	149	227	180	213	230	208	231	178	183	179	184	69	18

88	155	158	153	187	192	251	218	244	219	188	191	165	138	129	72
39	114	93	120	242	217	214	253	211	232	255	216	136	99	126	19
26	89	154	159	175	250	**1**	**64**	**256**	**9**	210	239	139	174	137	10
55	92	119	238	212	237	**144**	**121**	**25**	**100**	215	254	98	135	122	65

5	50	203	190	202	247	**196**	**169**	**49**	**4**	201	220	186	173	168	67
30	91	198	221	197	236	**81**	**16**	**36**	**225**	222	235	167	134	131	12
37	86	157	194	228	193	204	249	209	246	195	200	172	185	124	17
28	115	148	199	205	224	229	234	226	233	206	223	207	132	127	70

38	87	156	117	141	170	163	248	164	245	142	189	171	102	125	20
27	116	113	160	162	241	252	63	243	34	161	240	140	105	128	71
56	23	118	111	107	110	143	22	24	101	112	109	103	108	21	66
31	90	57	60	58	61	32	15	35	14	59	62	106	33	104	13

Fig.46b. Closed knight tour with square numbers in the centre of 4x4x4x4 hypercube.

1	38	21	102
106	103	2	117
59	116	147	124
104	149	154	**25**

134	5	40	125
3	118	57	24
156	133	98	151
207	152	157	192

39	22	37	42
58	127	120	101
135	150	155	126
158	193	206	153

4	41	6	23
119	56	67	54
132	99	74	43
163	128	55	**100**

20	137	142	177
17	62	183	240
136	123	174	143
61	26	145	182

213	224	173	218
184	**81**	216	203
199	246	**225**	222
190	197	204	247

250	219	210	239
211	**196**	249	220
214	221	**36**	245
205	248	215	194

131	96	75	70
162	165	72	45
189	198	97	8
166	161	66	11

107	122	175	238
60	115	146	181
105	108	237	176
18	63	148	241

200	255	226	235
227	**64**	201	254
208	223	**256**	243
231	242	191	202

209	234	251	244
230	**121**	228	195
233	252	**169**	236
168	229	232	253

188	171	130	7
167	160	65	10
170	129	68	53
159	164	73	44

16	109	138	141
19	112	91	114
92	15	86	83
27	90	93	**144**

187	172	217	178
80	89	94	111
87	82	47	140
30	33	28	13

212	179	186	139
185	110	113	180
50	85	78	35
79	14	31	84

49	76	69	52
88	95	48	71
77	34	51	46
32	29	12	**9**

Fig.47a. Knight tour with square numbers along the 4D diagonals in 4x4x4x4 hypercube.

1	14	27	**256**
28	95	88	91
13	2	155	140
196	139	160	**169**

70	45	42	47
87	90	71	94
154	141	172	163
171	148	197	228

153	48	15	44
150	93	26	89
173	162	151	92
198	195	170	161

86	43	46	41
69	72	83	6
152	85	68	73
149	142	105	84

54	57	136	209
31	138	157	214
10	55	166	203
29	158	115	252

179	236	233	204
192	**225**	**36**	251
237	**144**	**9**	234
220	253	238	217

186	211	208	247
221	248	213	224
212	167	246	215
245	216	223	168

177	188	107	76
184	143	66	37
187	176	23	8
222	147	82	39

135	132	165	202
12	117	96	255
33	58	133	164
116	3	156	159

240	205	230	235
231	254	239	218
206	241	232	229
181	226	219	250

207	200	243	210
244	**49**	**16**	201
199	**4**	**25**	242
194	249	182	227

190	175	24	7
193	146	103	40
174	189	106	5
183	104	67	74

32	137	134	131
53	56	113	118
30	79	34	59
11	52	97	114

191	130	35	124
180	145	98	127
101	126	123	80
18	61	20	51

178	125	122	129
185	128	119	112
120	111	78	63
21	50	17	60

121	110	77	**64**
102	99	108	75
109	62	65	38
100	19	22	**81**

Fig.47b. Knight tour with square numbers along the 4D diagonals in 4x4x4x4 hypercube.

31	60	3	86
50	85	80	59
33	94	91	104
48	51	108	101

62	**1**	66	29
89	82	107	**256**
64	105	200	175
201	184	**81**	192

65	30	61	2
84	93	90	87
199	176	187	92
188	193	198	177

24	67	28	15
63	88	83	68
142	139	106	27
165	178	141	70

34	55	78	159
53	58	99	156
154	103	160	97
43	96	155	190

153	208	219	170
230	233	244	191
205	238	247	218
246	227	232	241

204	223	162	157
245	228	197	234
248	215	222	237
229	236	189	214

167	**16**	23	18
164	179	182	**169**
25	168	163	20
180	183	**144**	13

111	**4**	161	172
32	95	110	**225**
49	56	79	102
52	47	**100**	109

206	239	254	217
243	226	231	240
250	209	220	255
185	242	251	174

253	216	211	224
212	235	252	173
221	194	249	210
202	213	186	195

152	207	148	137
181	138	145	14
166	147	26	69
143	140	71	146

54	57	112	125
35	46	77	98
44	5	116	113
37	42	45	118

151	124	135	158
132	127	130	123
129	10	73	126
74	119	38	41

134	**9**	150	171
203	122	133	**196**
36	115	76	117
39	6	**121**	114

149	136	17	22
128	131	72	19
75	8	21	12
120	11	40	7

Fig.48a. Closed knight tour with square numbers in square quartes in 4x4x4x4 hypercube.

25	126	91	6
92	79	**144**	125
27	**64**	83	128
86	67	124	**81**

90	5	24	119
87	120	123	140
122	139	146	173
183	200	213	180

23	118	7	130
184	141	224	143
89	206	129	178
214	181	142	199

88	131	22	117
157	116	53	132
152	147	138	21
185	158	133	54

28	75	14	193
15	62	227	170
12	127	2	175
93	80	85	198

253	192	233	174
226	203	212	255
189	254	249	194
246	195	202	171

16	231	228	221
239	218	**169**	204
248	**1**	240	167
217	168	247	**256**

187	234	17	56
150	191	156	19
245	188	151	50
190	201	52	155

13	78	229	176
26	65	84	197
3	68	145	164
66	63	82	161

4	235	250	163
243	208	**225**	172
252	**121**	236	179
215	182	211	**196**

251	222	237	230
216	205	242	177
241	166	223	162
210	207	160	165

244	115	8	135
209	136	149	154
186	153	134	55
159	148	137	20

74	61	108	77
29	76	73	104
42	71	38	69
11	30	41	72

109	114	99	220
98	105	102	113
37	60	95	34
94	33	44	103

238	219	232	111
101	112	107	48
10	47	70	39
43	40	31	46

9	110	57	18
106	59	**100**	35
97	**36**	51	58
32	45	96	**49**

Fig.48b. Closed knight tour with square numbers in diamond quartes in 4x4x4x4 hypercube.

1	46	21	56
22	55	**4**	95
47	2	85	**16**
54	**9**	134	3

60	57	44	13
87	10	77	92
172	79	98	167
189	166	171	78

45	14	59	94
48	93	86	15
193	140	173	96
174	191	194	139

58	11	76	43
61	88	91	12
164	97	62	75
195	188	165	90

20	65	142	215
5	104	135	178
84	17	24	143
23	126	179	138

201	80	99	204
246	161	242	217
221	250	203	214
232	241	222	239

64	141	154	157
231	228	**49**	236
200	213	230	**25**
229	**36**	235	218

163	160	63	26
220	245	72	89
249	162	243	42
244	219	240	73

83	132	129	206
106	125	180	137
53	128	133	130
8	105	136	127

208	255	202	211
181	120	223	238
248	209	170	205
119	190	233	168

199	210	207	254
224	237	234	177
175	198	253	212
192	227	176	197

256	183	252	41
247	186	**225**	74
182	251	184	**169**
185	**196**	187	226

52	103	82	131
19	66	107	124
6	51	102	69
33	18	7	108

81	150	145	216
118	123	**100**	149
101	70	115	**144**
114	**121**	32	29

152	155	158	147
113	148	153	156
34	67	50	109
37	30	35	68

159	146	151	110
38	111	116	27
117	122	71	40
112	39	28	31

Fig.48c. Knight tour with square numbers in Beverley quartes in 4x4x4x4 hypercube.

1	32	29	**144**
30	85	24	41
27	40	89	94
86	**25**	**256**	103

22	39	2	33
153	42	165	172
210	171	252	175
215	152	209	202

31	84	23	38
164	37	154	83
253	174	163	200
208	201	212	173

4	21	34	**81**
43	82	3	20
170	135	80	35
211	**36**	**169**	52

28	97	88	95
87	66	149	102
12	93	96	145
67	104	187	148

191	194	221	176
150	183	248	203
193	190	239	220
186	217	214	235

198	181	162	195
207	204	227	182
240	199	254	205
213	206	249	188

179	184	131	18
192	139	44	51
185	132	57	76
216	189	138	45

11	92	143	106
68	105	166	147
91	146	255	142
26	69	90	73

230	223	242	219
247	218	237	234
238	233	222	243
151	236	251	158

245	232	229	224
228	167	246	155
231	244	241	160
250	159	168	141

130	133	178	61
5	136	79	46
134	77	60	19
137	140	53	78

100	107	98	**225**
65	126	101	74
10	99	108	63
13	**64**	**9**	70

129	120	177	62
124	111	118	157
119	116	123	110
8	71	54	15

180	197	128	161
127	156	125	226
122	109	56	75
55	14	7	72

121	114	59	**196**
112	117	48	17
115	58	113	50
6	**49**	**16**	47

Fig.49. Knight tour with square numbers as 4-dimensional parallelotope in 4x4x4x4 hypercube

1	32	57	38
56	39	54	15
41	58	149	**256**
18	141	138	177

34	93	148	11
53	140	35	178
186	179	154	147
139	146	187	**36**

31	10	33	14
182	143	52	37
155	218	181	214
144	213	142	217

92	13	30	**49**
51	94	91	12
180	103	50	95
207	188	145	102

42	59	80	175
19	78	203	162
2	163	174	79
77	122	135	**16**

81	194	241	220
252	221	230	199
247	242	193	240
192	239	250	201

156	219	204	**9**
229	254	251	202
232	215	246	255
253	228	231	198

205	104	157	98
158	97	106	27
105	108	99	8
100	191	90	107

151	164	173	**196**
40	55	150	165
43	60	137	176
136	17	76	123

248	243	234	237
185	238	249	226
152	235	244	223
75	222	153	166

233	236	245	172
210	227	168	197
183	224	209	216
208	167	212	**169**

184	109	48	29
111	190	89	96
206	87	110	**225**
189	112	101	88

44	195	130	119
3	118	133	124
20	45	120	61
121	62	21	134

129	160	115	**64**
82	63	128	161
73	68	83	200
70	23	74	67

4	171	84	131
117	132	125	170
46	85	116	65
211	66	71	22

159	86	127	114
126	113	26	7
5	24	47	28
72	69	6	**25**

Fig.50. Knight tour with square numbers in closed zebra {2,3} path in 4x4x4x4 hypercube.

59	40	71	44
14	43	74	145
41	60	95	138
10	99	132	**169**

46	93	128	57
73	76	131	126
94	127	166	191
165	168	247	**256**

15	58	45	**144**
130	125	72	75
235	140	129	124
246	231	164	139

92	47	78	83
77	80	91	56
216	123	82	79
81	244	167	90

70	17	102	137
11	98	171	174
38	103	136	177
13	170	**9**	172

1	214	183	190
238	203	254	223
227	208	241	202
248	255	222	207

236	209	**16**	175
253	230	233	178
240	201	176	173
229	204	251	232

215	2	189	88
228	243	158	53
237	188	3	84
4	157	242	55

39	116	135	146
42	61	96	117
69	134	101	162
100	97	68	133

218	193	198	**225**
249	224	219	192
194	199	226	197
221	206	195	118

121	200	143	186
234	205	252	119
217	120	163	142
250	141	220	161

152	187	48	85
159	156	151	154
122	153	86	89
245	160	155	**196**

18	147	**36**	115
37	34	67	62
12	19	104	35
21	8	33	26

211	182	213	184
66	63	106	181
105	108	65	114
64	27	6	107

148	185	210	179
239	180	149	112
22	113	32	109
7	20	**25**	28

49	212	87	110
150	111	52	29
31	50	23	54
24	5	30	51

Fig.51. Open knight tour with square numbers in zebra {2,3} path in 4x4x4x4 hypercube.

1	6	55	52	136	133	128	7	187	**8**	17	132	18	89	92	9
54	51	**16**	5	139	126	137	134	**216**	213	186	177	91	86	19	88
15	**4**	53	56	**196**	135	14	127	211	246	193	214	188	13	90	25
50	57	80	73	125	138	195	176	194	215	212	185	173	182	87	20
70	67	**100**	131	101	204	159	248	**256**	247	206	233	205	198	151	24
49	62	69	72	124	199	236	203	235	238	229	200	172	181	160	21
36	71	66	61	255	252	221	232	220	231	234	249	197	164	93	12
65	60	83	202	84	237	228	251	227	250	219	230	190	85	170	161
35	**144**	129	78	2	253	222	241	145	242	143	208	142	165	94	27
64	59	82	75	223	240	225	244	226	245	218	239	189	168	171	10
3	76	79	130	140	243	254	209	217	210	207	178	166	141	26	163
58	63	74	81	191	224	175	184	174	183	192	201	169	162	167	22
48	77	68	99	123	98	153	158	154	119	150	179	149	152	157	96
39	42	47	44	102	115	122	113	121	180	155	118	156	97	148	23
34	45	38	41	107	112	147	116	146	117	120	109	103	108	95	28
37	40	43	46	104	31	114	111	33	110	105	30	106	29	32	11

Fig.52. Closed tour of knight with square numbers and cubic numbers at knight move from starting cell 1.

Figure 53 shows closed tour of knight with powers of 2 at knight move from starting cell 1. Figure 54 shows open knight tour with multiples of 12 at knight move from starting cell 1. Readers are requested to look for a closed tour. Figure 55 shows a closed tour of knight with multiples of 16 at knight move from starting cell 1.

1	70	37	74	190	217	**16**	65	193	**64**	189	214	188	17	26	15
40	73	**4**	69	129	68	221	212	10	233	192	63	191	202	11	18
71	**2**	75	38	216	211	66	239	61	252	215	210	194	187	60	25
76	39	72	3	67	238	231	220	234	201	62	253	203	232	19	176
80	95	6	43	205	228	197	240	**256**	251	244	209	195	186	59	24
5	42	87	78	196	241	236	219	235	208	223	254	204	163	174	177
8	79	44	89	227	218	229	198	206	255	226	243	179	172	185	14
41	88	77	86	230	199	222	237	9	242	207	200	184	175	178	173
83	36	81	48	**128**	181	246	213	245	124	225	250	180	171	152	27
32	47	84	131	167	130	127	134	224	249	138	123	183	160	21	12
45	82	7	140	182	125	166	169	121	170	247	132	158	151	58	153
136	85	46	133	165	168	135	126	248	137	122	139	161	164	159	20
96	49	112	141	113	146	101	56	120	155	106	111	157	150	57	154
91	94	33	52	100	53	98	147	107	110	119	116	162	145	148	23
50	35	90	93	97	102	55	142	114	117	108	105	149	156	143	28
31	92	51	34	54	99	30	103	109	104	115	118	144	29	22	13

Fig.53. Closed tour of knight with powers of 2 at knight move from starting cell 1.

57	40	63	16
64	15	138	103
39	4	233	164
14	65	136	183

98	3	56	41
13	102	59	**96**
94	97	100	127
101	**60**	95	124

93	42	17	62
58	61	92	125
99	126	165	246
236	195	232	199

18	55	52	43
91	2	19	54
122	119	82	51
149	90	123	20

186	167	142	163
11	34	185	**168**
32	141	166	193
9	**36**	33	140

143	162	227	**192**
202	1	242	177
213	254	207	**228**
12	243	**204**	239

212	225	218	245
237	244	221	**216**
220	217	234	255
203	238	215	194

159	**24**	83	88
144	89	**48**	21
151	**84**	145	176
214	47	150	45

67	106	187	130
8	129	104	139
105	66	137	128
38	5	184	135

210	189	208	229
173	248	205	**240**
206	209	190	253
201	**252**	241	182

219	230	223	188
222	197	250	247
235	256	231	198
196	249	200	251

174	161	146	23
147	**120**	53	44
160	175	118	87
121	148	81	50

154	131	68	107
31	6	169	134
10	35	76	69
7	30	37	74

157	**108**	191	70
72	171	**156**	181
155	**132**	71	178
172	73	170	29

224	111	226	179
153	**180**	113	110
112	77	116	133
115	28	75	78

211	158	25	86
114	109	80	49
117	26	85	22
152	79	46	27

Fig.54. Tour of knight with multiples of 12 at knight move from starting cell 1.

45	30	47	**64**
62	1	82	33
31	46	63	**48**
16	83	**32**	81

60	65	58	113
17	114	61	**96**
146	179	162	195
163	**80**	201	178

145	**112**	129	34
128	199	**160**	111
161	**144**	127	130
200	177	164	197

18	59	66	57
115	110	19	24
180	99	70	35
147	116	73	4

86	139	132	211
133	84	155	**192**
44	49	138	245
137	**176**	203	106

171	252	229	140
154	97	248	251
253	246	219	206
202	205	184	247

232	143	126	131
235	2	165	158
220	233	244	107
185	198	159	204

149	98	69	36
166	117	74	3
153	108	67	118
182	119	20	25

29	**208**	87	226
224	135	**256**	209
15	**240**	225	194
134	105	136	213

228	239	218	207
255	8	215	238
216	173	254	95
183	94	79	196

189	242	227	210
186	223	236	191
241	190	187	174
222	175	214	237

152	109	56	23
181	120	71	10
148	9	122	5
121	100	75	72

88	85	156	193
43	50	89	212
28	53	14	39
13	40	51	104

217	172	141	230
170	249	78	91
77	92	169	54
42	7	90	93

142	231	188	157
221	234	125	250
168	243	52	103
27	12	41	38

167	150	123	102
124	101	68	37
151	6	55	22
76	21	26	11

Fig.55. Closed tour of knight with multiples of 16 at knight move from starting cell 1.

'Domination by queen', as shown Figure 41, can be further extended in higher dimensions. Figure 56 shows possible queen's move in four dimensions and Figure 57 shows queen at cell 7 dominating all the primes in 4x4x4x4 hypercube.

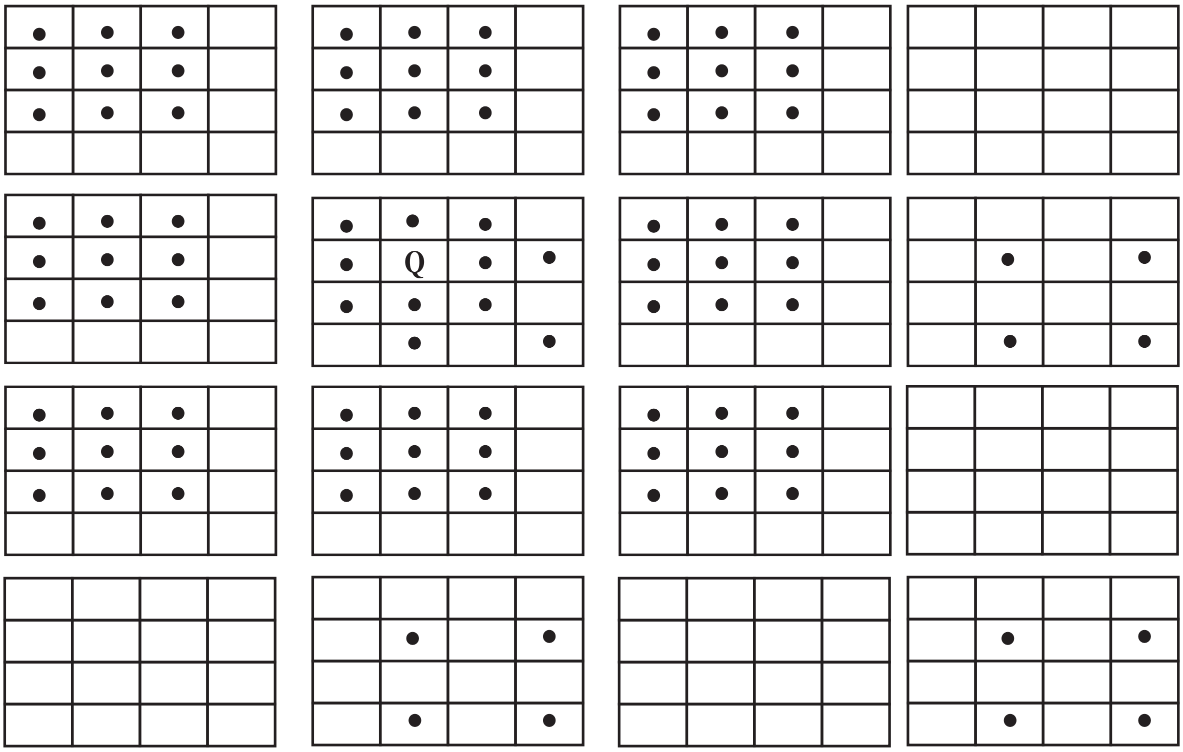


Fig.56. Possible queen’s move in 4x4x4x4 hypercube (shown as dots).

89	152	**199**	14
114	**13**	150	153
151	90	**59**	106
12	115	74	25

198	**131**	48	105
31	214	**113**	24
216	**197**	200	91
123	30	213	116

217	104	**227**	66
44	**193**	118	15
231	182	**73**	154
194	117	26	183

32	85	120	49
119	52	45	192
196	215	50	121
51	122	195	16

68	187	166	65
139	206	**233**	144
168	**107**	138	175
57	176	169	204

167	64	**3**	92
234	**7**	252	**191**
137	**2**	**211**	4
212	**5**	236	**17**

128	**67**	218	155
251	156	**173**	54
56	**103**	232	219
235	108	55	174

111	130	93	190
244	**239**	112	**83**
129	110	1	94
240	**29**	242	**109**

101	88	**41**	186
164	**149**	170	207
11	102	**163**	60
140	205	58	143

132	**47**	228	87
229	224	**23**	180
124	**181**	254	201
237	28	75	208

43	86	**127**	40
256	**53**	226	145
71	162	**61**	184
34	225	22	203

126	189	46	95
33	84	255	202
230	125	72	209
243	238	27	18

42	69	100	39
171	8	165	142
70	177	10	185
9	148	141	80

63	188	253	98
158	**179**	172	**223**
133	6	99	210
76	**79**	134	**19**

250	161	62	221
35	222	249	38
136	147	246	81
21	178	77	146

159	96	247	82
248	**157**	36	**97**
245	160	135	220
78	**241**	20	**37**

Fig.57. Queen (at cell 7) dominating all the primes in 4x4x4x4 hypercube.

Conclusion: Construction of figured tours is a mathematical recreation. It combines the logic of mathematics with the beauty of art. While compiling exhilarating figured tours, Jelliss [8] exhorts "There are many other possibilities yet to be investigated." The author has shown few of them and readers are requested to look for more and improve upon the configurations as mentioned earlier. Jelliss [23] laments, "Despite … efforts to raise further interest in this [Figured Tour] subject I seem to have been the only composer active in this field recently". Well, you are no more alone Jelliss; we are with you. Kaku [24] asserts that "There is a growing acknowledgment among physicists worldwide, including several Nobel laureates, that the universe may actually exist in higher-dimensional space." Pickover [25] declares "Various modern theories of "hyperspace" suggest that dimensions exist beyond the commonly accepted dimensions of space and time. The entire universe may actually exist in a higher-dimensional space. This idea is not science fiction …" Musser [26] muses that "As fantastic as extra dimensions of space sound, they might really exist… various mysteries of the world around us give the impression that the known universe is but the shadow of a higher-dimensional reality." Watkins [27] exclaims "… there is no reason to stop with chessboards in only three dimensions!" Accordingly, the author has shown possibility of figured tours of knight in higher dimensions and hopes that its study will help in unraveling secrets of hyperspace. It can also be used in pedagogy of higher mathematics, namely, differential geometry and topology.

Acknowledgement: The author is grateful to Atulit Kumar for his help in preparation of this article.

References

1. A. Dickins; *A Guide to Fairy Chess*, Dover Publications, 1971, p. 27.
2. W. W. Rouse Ball; *Mathematical Recreations and Essays*, The Macmillan Company, 1947, pp. 174-185.
3. M. Kraitchik; *Mathematical Recreations*, Dover Publications, 1953, pp. 257-266.
4. C. A. Pickover; *The Zen of Magic Squares, Circles and Stars*, Princeton University Press, 2002, pp. 210-220, 232-235.
5. M. S. Petkovic; *Famous Puzzles of Great Mathematicians*, American Mathematical Society, 2000, pp. 257-281
6. D. Wells; *Games and Mathematics: Subtle Connections*, Cambridge University Press, 2012, pp. 76-96.
7. M. Gardner; *Mathematical Magic Show*, Mathematical Association of America, 1989, pp. 188-202.
8. G. Jelliss; *Figured Tours, A Mathematical Recreation*, 1997, available at www.mayhematics.com
9. G. Jelliss; Figured Tours, *Mathematical Spectrum*, Vol. 25, No. 1, 1992/93, pp. 16-20.
10. A. T. Vandermonde; Remarques sur les Problemes de Situation, *Memoires de l'Academie des Sciences* 1771.
11. H. Schubert, *Mathematische Mussestunden* Eine Sammlung von Geduldspielen, Kunststücken und Unterhaltungs-aufgaben mathematischer Natur. (Leipzig), 1904, p.235-7 4×4×4 closed tours, p.238 3×4×6 closed tour.
12. N. M. Gibbins, Chess in 3 and 4 dimensions, *Mathematical Gazette*, May 1944, pp 46-50.
13. J. Stewart, Solid Knight's Tours, *Journal of Recreational Mathematics* Vol.4 (1),

January 1971, p.1.
14. G. P. Jelliss and T. W. Marlow, 3d tours *Chessics* 2(29/30), 1987, p.162.
15. M. Petkovic, *Mathematics and Chess*, Dover Publications, New York, 1997, p.65.
16. J. DeMaio, Which Chessboards have a Closed Knight's Tour within the Cube? *The Electronic Journal of Combinatorics*, 14, (2007) R32.
17. Awani Kumar, Studies in Tours of Knight in Three Dimensions, *The Games and Puzzles Journal* (online) #43, January-April 2006 available at http://www.mayhematics.com/j/j.htm
18. Awani Kumar, Magic knight's tours for chess in three dimensions, *Mathematical Gazette*, Vol.92 (March 2008), pp 111-115.
19. Awani Kumar, Construction of Magic Knight's Towers, *Mathematical Spectrum* Vol.42 (1), 2009, pp 20-25.
20. Awani Kumar, Magic Tours of Knight in 4x4x4 Cube, available at https://arxiv.org/abs/1708.06237.
21. H. P. Manning, *The Fourth Dimension Simply Explained*, Dover Publications, New York, 2005, p.14
22. Awani Kumar, Magic Knight's Tours in Higher Dimensions, available at https://arxiv.org/abs/1201.0458.
23. G.P. Jelliss, *Knight's Tours Notes*, Volume 11, Alternative Worlds, 2019, p.44 available at http://www.mayhematics.com.
24. Michio Kaku, *Hyperspace: A Scientific Odyssey Through Parallel Universes, Time Warps, and the Tenth Dimension*, Anchor Books, 1995, New York, p.vii.
25. C. A. Pickover, *Wonders of Numbers: Adventures in Mathematics, Mind, and Meaning*, Oxford University Press, New York, 2001, p.100.
26. G. Musser, Extra Dimensions, *Scientific American India*, Vol.5, No.6 (June 2010), p.19.
27. J. J. Watkins, *Across the Board, The Mathematics of Chessboard Problems*, Universities Press (India) Private Limited, 2005, pp.86-92.