On the Riemann hypothesis

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A proposed proof of the Riemann hypothesis.

1. Introduction

The Riemann zeta function is

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1} \]

for \( \sigma = \text{Re}(s) > 1 \). For other values of \( s \) it is defined uniquely by analytic continuation, see [1]. The function \( \zeta(s) \) has trivial zeros at \( s = -2l \) for \( l \in \mathbb{N} = \{1, 2, 3, \ldots\} \). It is known that the nontrivial zeros \( s = \sigma + it \) of \( \zeta(s) \) satisfy the following properties.

I: If \( s = \sigma + it \) is a nontrivial zero of \( \zeta(s) \) then \( s = \sigma - it \) is a nontrivial zero of \( \zeta(s) \).

II: If \( s = \sigma + it \) is a nontrivial zero of \( \zeta(s) \) then \( \sigma \in (0, 1) \).

III: If \( s = \sigma + it \) is a nontrivial zero of \( \zeta(s) \) then \( s = 1 - \sigma + it \) is a nontrivial zero of \( \zeta(s) \).

2. Proof of the Riemann hypothesis

Theorem

All nontrivial zeros of \( \zeta(s) \) have real part equal to \( \frac{1}{2} \).

Proof

In light of [2] consider

\[ \psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log_e(2\pi) - \frac{1}{2} \log_e(1 - x^{-2}) \tag{2} \]

for \( x \in (n, n + 1) \) and \( n \in \mathbb{N} \). Here \( \psi(x) \) is a weighted prime counting function

\[ \psi(x) = \sum_{\rho < x} \log_e \rho \tag{3} \]

where \( \rho \) is prime and the sum is over all prime powers. The sum in the second term on the right of (2) is over all \( \rho \) such that \( s = \rho \) is a nontrivial zero of \( \zeta(s) \). The exact function \( \psi(x) \) is constant on the domain between any two consecutive integers. The approximation of \( \psi(x) \) with finitely many \( \rho \) values displays a Gibbs phenomenon. Differentiating (2) with respect to \( x \) yields

\[ 0 = 1 - \sum_{\rho} x^{\rho-1} - \frac{1}{x^3 - x} \tag{4} \]

Rearranging (4) gives

\[ \sum_{\rho} x^{\rho-1} \left( \frac{x^3 - x}{x^3 - x - 1} \right) = 1. \tag{5} \]

Differentiating (5) with respect to \( x \) yields

\[ \sum_{\rho} x^{\rho-1} [\rho - 1] \left( \frac{x^2 - 1}{x^3 - x - 1} \right) - \left( \frac{3x^2 - 1}{(x^3 - x - 1)^2} \right) = 0. \tag{6} \]
Now
\[ \sum_{\rho} (\rho - 1)x^{\rho - 1} = \sum_{\beta + iy} (\beta + iy - 1)x^{\beta + iy - 1}. \] (7)

On using Euler’s identity
\[ e^{i\theta} = \cos(\theta) + i \sin(\theta) \] (8)
equation (7) becomes
\[ \sum_{\rho} (\rho - 1)x^{\rho - 1} = \sum_{\beta + iy} (\beta + iy - 1)[\cos(\gamma \log_e x) + i \sin(\gamma \log_e x)] \] (9)
which expands to
\[ \sum_{\rho} (\rho - 1)x^{\rho - 1} = \sum_{\beta + iy} \beta^{\rho - 1}[\cos(\gamma \log_e x)(\beta - 1) - \sin(\gamma \log_e x)\gamma] + \sum_{\beta + iy} x^{\rho - 1}[\sin(\gamma \log_e x)(\beta - 1) + \cos(\gamma \log_e x)\gamma]. \] (10)
The second term on the right of (10) disappears due to I. Then (10) becomes
\[ \sum_{\rho} (\rho - 1)x^{\rho - 1} = \sum_{\beta + iy} \beta^{\rho - 1}[\cos(\gamma \log_e x)(\beta - 1) - \sin(\gamma \log_e x)\gamma]. \] (11)
Also
\[ \sum_{\rho} x^{\rho - 1} = \sum_{\beta + iy} x^{\beta + iy - 1}. \] (12)
On using Euler’s identity equation (12) becomes
\[ \sum_{\rho} x^{\rho - 1} = \sum_{\beta + iy} x^{\beta - 1} \cos(\gamma \log_e x) + i \sum_{\beta + iy} x^{\beta - 1} \sin(\gamma \log_e x). \] (13)
The second term on the right of (13) disappears due to I. Then (13) becomes
\[ \sum_{\rho} x^{\rho - 1} = \sum_{\beta + iy} x^{\beta - 1} \cos(\gamma \log_e x). \] (14)
Equation (6) is then
\[ \sum_{\beta + iy} x^{\beta - 1}[\cos(\gamma \log_e x)(\beta - 1) - \sin(\gamma \log_e x)\gamma] \left( \frac{x^2 - 1}{x^3 - x - 1} \right) - \sum_{\beta + iy} x^{\beta - 1} \cos(\gamma \log_e x) \left( \frac{3x^2 - 1}{(x^3 - x - 1)^2} \right) = 0. \] (15)
Let \( x = y + c \) where \( 0 \ll y \ll 1 \) and \( c \) is a constant such that \( x \in (n, n + 1) \). Then (15) implies
\[ \sum_{\beta + iy} (y + c)^{\beta - 1}[\cos(\gamma \log_e (y + c))(\beta - 1) - \sin(\gamma \log_e (y + c))\gamma] \left( \frac{(y + c)^2 - 1}{(y + c)^3 - (y + c) - 1} \right) \]
\[ - \sum_{\beta + iy} (y + c)^{\beta - 1} \cos(\gamma \log_e (y + c)) \left( \frac{3(y + c)^2 - 1}{(y + c)^3 - (y + c) - 1} \right) = 0. \] (16)
On using a Taylor expansion (16) becomes
\[ \sum_{\beta + iy} (y + c)^{\beta - 1}[\cos(\gamma \log_e c)(\beta - 1) - \sin(\gamma \log_e c)\gamma] \left( \frac{c^2 - 1}{c^3 - c - 1} \right) \]
\[ + [\sin(\gamma \log_e c)(\beta - 1)\gamma] \left( \frac{c^2 - 1}{c^3 - c - 1} \right) \]
\[ + [\cos(\gamma \log_e c)(\beta - 1) - \sin(\gamma \log_e c)\gamma] \left( \frac{-ec^4 + 2c^2 - 2c - 1}{(c^3 - c - 1)^2} \right) \]
\[ - \sum_{\beta + iy} (y + c)^{\beta - 1} \cos(\gamma \log_e (y + c)) \left( \frac{3c^2 - 1}{(c^3 - c - 1)^2} \right) \]
\[ + \cos(\gamma \log_e c) \left( \frac{-12c^7 + 18c^5 + 6c^4 - 8c^3 + 8c + 2}{(c^3 - c - 1)^4} \right) \]
\[ + \cos(\gamma \log_e c) \left( \frac{-12c^7 + 18c^5 + 6c^4 - 8c^3 + 8c + 2}{(c^3 - c - 1)^4} \right) y + O(y^2) = 0. \] (17)
Now (17) must be true independent of \( y \). We then must take coefficients of \((y + c)\) in (17), for \( \beta \in (0, 1) \) in accordance with II, and set them to zero. Now (17) has the form

\[
\sum_{\beta \in \mathbb{R}} \sum_{y \in \mathbb{R}(\beta)} (y + c)^{\beta - 1} \sum_{l=0}^{\infty} \left( f_l(y, c)(\beta - 1) + g_l(y, c)\right)(y + c)^l = 0. \tag{18}
\]

So for example, taking the the \( O((y + c)^{\beta - 1}) \) coefficient in (18) gives

\[
\sum_{y \in \mathbb{R}(\beta)} [f_0(y, c)(\beta - 1) + g_0(y, c)] = 0 \tag{19}
\]

which implies

\[
\beta = -\frac{\sum_{y \in \mathbb{R}(\beta)} g_0(y, c)}{\sum_{y \in \mathbb{R}(\beta)} f_0(y, c)} + 1 = -\frac{\sum_{y \in \mathbb{R}(1 - \beta)} g_0(y, c)}{\sum_{y \in \mathbb{R}(1 - \beta)} f_0(y, c)} + 1 = 1 - \beta \tag{20}
\]
on using III. Therefore without loss of generality \( \beta = \frac{1}{2} \). □

References
