Non-existence of odd almost perfect numbers

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April 28th, 2020
Abstract
Let $b$ be an odd almost perfect number. Let the prime factors of $b$ which are different from each other be odd primes $p_1,p_2,...,p_r$ and let the exponent of $p_k$ be a positive integer $q_k$. If the product of the series of the prime factors is an odd integer $a$,

$$a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + ... + 1)$$

$$b = \prod_{k=1}^{r} p_k^{q_k}$$

If $b$ is an almost perfect number,

$$a = 2b - 1$$

holds. By a research of this paper, let $a_k$ and $b_k$ be odd integers and $c_k$ be a positive integer and the following equations are assumed to hold.

$$a_k = a/(p_k^{q_k} + ... + 1)$$

$$b_k = b/p_k^{q_k}$$

$$a_k = c_k(p_k + 1) + 2b_k - 1$$

When $r \geq 2$, By a proof which uses the product of $a_k/b_k$, in order for $a$ to be an odd almost perfect number the following equality must be satisfied when $r \geq 2$.

$$b = -4/7$$

We have obtained a conclusion that there are no odd almost perfect numbers other than 1 since this equality does not hold.

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1. Introduction
In mathematics, an almost perfect number (sometimes also called slightly defective or least deficient number) is a natural number n such that the sum of all divisors of n (the sum-of-divisors function \( \sigma(n) \)) is equal to \( 2n - 1 \). The only known almost perfect numbers are powers of 2 with non-negative exponents (sequence A000079 in the OEIS). Therefore the only known odd almost perfect number is \( 2^0 = 1 \). (Quoted from Wikipedia)
In this paper, we prove that there are no odd almost perfect numbers other than 1.

2. Proof

Let \( b \) be an odd almost perfect number. Let the prime factors of \( b \) which are different from each other be odd primes \( p_1, p_2, \ldots, p_r \) and let the exponent of \( p_k \) be a positive integer \( q_k \). If the product of the series of the prime factors is an odd integer \( a \),

\[
a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1) \quad \text{①}
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{②}
\]

If \( b \) is an almost perfect number,

\[
a = 2b - 1 \quad \text{③}
\]

holds.

Let \( a_k \) and \( b_k \) be odd integers,

\[
a_k = a / (p_k^{q_k} + \cdots + 1)
\]

\[
b_k = b / p_k^{q_k}
\]

\( p_k^{q_k} + \cdots + 1 \) is odd since \( a \) and \( a_k \) are odd integers. Thereby, \( q_k \) is an even integer for all \( k \).

From the equation ③,

\[
a_k(p_k^{q_k} + \cdots + 1) = 2b_k p_k^{q_k} - 1 \quad \text{④}
\]

1. When \( r = 1 \)

\[
p_1^{q_1} + \cdots + 1 = 2p_1^{q_1} - 1
\]

\( 1 \equiv -1 \pmod{p_1} \)

It becomes inconsistent since \( p_1 \geq 3 \). Therefore, odd almost perfect numbers do not exist when \( r = 1 \).
II. When $r \geq 2$

$$p_k^{q_k} + \cdots + 1 = (p_k^{q_k+1} - 1)/(p_k - 1) < p_k^{q_k+1}/(p_k - 1)$$

When $p_k \geq 3$,

$$p_k^{q_k} + \cdots + 1 < p_k^{q_k+1}/2$$

$$a_k(p_k^{q_k} + \cdots + 1) < a_kp_k^{q_k+1}/2$$

From the equation ④,

$$2b_k p_k^{q_k} - 1 < a_kp_k^{q_k+1}/2$$

Since $p_k \geq 3$ and $b_k p_k^{q_k} \geq 9$,

$$15b_k p_k^{q_k}/8 < a_k p_k^{q_k+1}/2$$

$$a_k/b_k > 15/(4p_k)$$

\[
\prod_{k=1}^{r} a_k/b_k > \prod_{k=1}^{r} \left(\frac{15}{4p_k}\right)
\]

\[
\prod_{k=1}^{r} p_k > (15/4)^{r}/(a/b)^{r-1} \quad \ldots ⑤
\]

Let $c_k$ be a positive integer. From the equation ④,

$$a_k \equiv 2b_k - 1 \pmod{p_k + 1}$$

$$a_k = c_k(p_k + 1) + 2b_k - 1 > c_k p_k$$

From the inequality ⑤,

\[
\prod_{k=1}^{r} a_k > \prod_{k=1}^{r} c_k p_k > (15/4)^{r} \prod_{k=1}^{r} c_k/(a/b)^{r-1}
\]

\[
a^{r-1} > (15/4)^{r} \prod_{k=1}^{r} c_k/(a/b)^{r-1} \quad \ldots ⑥
\]

\[
(4a^2/15b)^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k
\]

\[
a^{r-1} > (8a^2/15(a + 1))^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k
\]

\[
a^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k
\]
From the inequality \( \circ \), a set \( A \) and a set \( B \) each having \( a \) as an element are defined under the following conditions.

A: \( a^{r-1} > (15/4)^r \prod_{k=1}^{r} c_k / (a/b)^{r-1} \)

B: \( a^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k \)

Since \( A \Rightarrow B \), \( A \subseteq B \) holds. On the other hand \( A \supseteq B \) holds because \( B \land \neg A = \varnothing \) must be hold. Therefore, \( A = B \) must be satisfied.

\[
(15/4)^r / (a/b)^{r-1} = 15/4 \\
(15/4)/(a/b) = 1 \\
a/b = 15/4 \\
4(2b - 1) = 15b \\
b = -4/7
\]

This expression does not hold obviously when \( r \geq 2 \). Therefore, odd almost perfect numbers do not exist when \( r \geq 2 \). From the above \( I \) and \( II \), there are no odd almost perfect numbers other than 1.

3. Acknowledgement

We would like to thank the family members who sustained the research environment and the mathematicians who reviewed these studies in conducting this study.

4. References

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