Non-existence of odd almost perfect numbers

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Abstract

Let \(b\) be an odd almost perfect number. Let the prime factors of \(b\) which are different from each other be odd primes \(p_1, p_2, \ldots, p_r\) and let the exponent of \(p_k\) be a positive integer \(q_k\). If the product of the series of the prime factors is an odd integer \(a\),

\[
a = \prod_{k=1}^{r} (p_k^{q_k} + p_k^{q_k-1} + \cdots + 1)
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k}
\]

If \(b\) is an almost perfect number,

\[
a = 2b - 1
\]

holds. By a research of this paper, let \(a_k\) and \(b_k\) be odd integers and \(c_k\) be a positive integer and the following equations are assumed to hold.

\[
a_k = a / (p_k^{q_k} + \cdots + 1)
\]

\[
b_k = b / p_k^{q_k}
\]

\[
a_k = c_k(p_k + 1) + 2b_k - 1
\]

When \(r \geq 2\), By a proof which uses the product of \(a_k/b_k\), we found that it becomes a contradiction when odd almost perfect numbers exist other than 1. We have obtained a conclusion that there are no odd almost perfect numbers other than 1.

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1. Introduction

In mathematics, an almost perfect number (sometimes also called slightly defective or least deficient number) is a natural number n such that the sum of all divisors of n (the sum-of-divisors function \( \sigma(n) \)) is equal to \( 2n - 1 \). The only known almost perfect numbers are powers of 2 with non-negative exponents (sequence A000079 in the OEIS). Therefore the only known odd almost perfect number is \( 2^0 = 1 \).

(Quoted from Wikipedia)

In this paper, we prove that there are no odd almost perfect numbers other than 1.

2. Proof

Let \( b \) be an odd almost perfect number. Let the prime factors of \( b \) which are different from each other be odd primes \( p_1, p_2, \ldots, p_r \) and let the exponent of \( p_k \) be a positive integer \( q_k \). If the product of the series of the prime factors is an odd integer \( a \),

\[
a = \prod_{k=1}^{r} \left( p_k^{q_k} + p_k^{q_k-1} + \cdots + 1 \right) \quad \text{①}
\]

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{②}
\]

If \( b \) is an almost perfect number,

\[
a = 2b - 1 \quad \text{③}
\]

holds.

Let \( a_k \) and \( b_k \) be odd integers,

\[
a_k = a / (p_k^{q_k} + \cdots + 1)
\]

\[
b_k = b / p_k^{q_k}
\]

\( p_k^{q_k} + \cdots + 1 \) is odd since \( a \) and \( a_k \) are odd integers. Thereby, \( q_k \) is an even integer for all \( k \).

From the equation ③,

\[
a_k(p_k^{q_k} + \cdots + 1) = 2b_k p_k^{q_k} - 1 \quad \text{④}
\]

I. When \( r = 1 \)

\[
p_1^{q_1} + \cdots + 1 = 2p_1^{q_1} - 1
\]

\[
1 \equiv -1 \quad \text{(mod } p_1\text{)}
\]

It becomes inconsistent since \( p_1 \geq 3 \). Therefore, odd almost perfect numbers do not exist when \( r = 1 \).
II. When $r \geq 2$

\[ p_k^{q_k} + \cdots + 1 = \frac{(p_k^{q_k+1} - 1)}{(p_k - 1)} < \frac{p_k^{q_k+1}}{(p_k - 1)} \]

When $p_k \geq 3$,

\[ p_k^{q_k} + \cdots + 1 < \frac{p_k^{q_k+1}}{2} \]

\[ a_k(p_k^{q_k} + \cdots + 1) < \frac{a_k p_k^{q_k+1}}{2} \]

From the equation (4),

\[ 2b_k p_k^{q_k} - 1 < \frac{a_k p_k^{q_k+1}}{2} \]

Since $p_k \geq 3$ and $b_k p_k^{q_k} \geq 9$,

\[ 15b_k p_k^{q_k}/8 < \frac{a_k p_k^{q_k+1}}{2} \]

\[ a_k/b_k > \frac{15}{(4p_k)} \]

\[ \prod_{k=1}^{r} \frac{a_k}{b_k} > \prod_{k=1}^{r} \left( \frac{15}{(4p_k)} \right) \]

\[ \prod_{k=1}^{r} p_k > \left( \frac{15}{4} \right)^r/(a/b)^{r-1} \ldots (5) \]

Let $c_k$ be a positive integer. From the equation (4),

\[ a_k \equiv 2b_k - 1 \pmod{p_k + 1} \]

\[ a_k = c_k(p_k + 1) + 2b_k - 1 > c_k p_k \]

From the inequality (5),

\[ \prod_{k=1}^{r} a_k > \prod_{k=1}^{r} c_k p_k > \left( \frac{15}{4} \right)^r \prod_{k=1}^{r} c_k/(a/b)^{r-1} \]

\[ a^{r-1} > \left( \frac{15}{4} \right)^r \prod_{k=1}^{r} c_k/(a/b)^{r-1} \ldots (6) \]

\[ (4a^2/15b)^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k \]

\[ (8a^2/15(a + 1))^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k \]

Suppose the following expression holds. Let $x$ be a real number and $x \neq 0$ holds.

\[ xa < 15a + 1 \ldots (7) \]

\[ (8a/x)^{r-1} > 15/4 \times \prod_{k=1}^{r} c_k \]
In order to keep the direction of the inequality sign, $x > 0$ must be satisfied when $r$ is even.

$$a^{r-1} > \frac{15}{4} \times \left(\frac{x}{8}\right)^{r-1} \prod_{k=1}^{r} c_k$$

From the inequality 6, a set $A$ and a set $B$ each having $a$ as an element are defined under the following conditions.

$A$: $a^{r-1} > \left(\frac{15}{4}\right)^r \prod_{k=1}^{r} c_k / (a/b)^{r-1}$

$B$: $a^{r-1} > \frac{15}{4} \times \left(\frac{x}{8}\right)^{r-1} \prod_{k=1}^{r} c_k$

Since $A \Rightarrow B$, $A \subseteq B$ holds. On the other hand, $A \supseteq B$ holds because $B \land \neg A = \emptyset$ must be hold. Therefore, $A = B$ must be satisfied. It is not appropriate when $r$ is even and $x < 0$ hold since $A = B$ does not hold.

$$\left(\frac{15}{4}\right)^r / (a/b)^{r-1} = \frac{15}{4} \times \left(\frac{x}{8}\right)^{r-1}$$

$$\left(\frac{15}{4}\right) / (a/b) = x / 8$$

$$a/b = 30 / x$$

From this equation, $a = 2b$ holds when $x = 15$. However, since it becomes contrary to the equation 3 it is not appropriate when $x = 15$.

When $x \neq 15$,

$$x(2b - 1) = 30b$$

$$b = -x / (30 - 2x)$$

From the inequality 7,

$$(x - 15)(2b - 1) < 1$$

$$(x - 15)(-2x / (30 - 2x) - 1) < 1$$

$$14 < 0$$

It becomes a contradiction. Therefore, odd almost perfect numbers do not exist when $r \geq 2$. From the above I and II, there are no odd almost perfect numbers other than 1.
3. Acknowledgement

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4. References

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