Abstract. This note is a proof of a strengthened form of the strong Goldbach conjecture.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1 and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem (Strengthened strong Goldbach conjecture (SSGB)). Every even integer greater than 6 can be expressed as the sum of two different primes.

Proof. We define the set \( S_9 := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q)/2 \} \).

SSGB is equivalent to saying that every integer \( x \geq 4 \) is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a middle component \( mk \) of \( S_9 \).

Let us assume \( \neg \text{SSGB} \) now. Then, there is at least one \( n \geq 4 \) such that \( nk \) is different from all the \( mk \) for each \( k \geq 1 \), where all pairs \( (p, q) \) of odd primes, that determine the numbers \( m \), are used in \( S_9 \). For each \( k \geq 1 \), \( nk \) can be written as some \( pk \) when \( n \) is prime, as some \( pk' \) when \( n \) is composite and not a power of 2, or as \( 4k' \) when \( n \) is a power of 2; \( p \in \mathbb{P}_3; k, k' \in \mathbb{N} \).

The expression \( pk' \) for \( nk \) with \( k' = k \) or \( k' \neq k \) is a first component of \( S_9 \) triples and the expression \( 4k' \) for \( nk \) is component of the triple \( (3k', 4k', 5k') \). So, since \( nk \) equals some triple component \( pk' \) or \( 4k' \) that exists by definition of \( S_9 \) and since the \( S_9 \) triples are generated by the first and third components, the triples are always the same, regardless of whether \( nk \) as a component of them exists or not. That is, the set \( S_9 \) remains the same, regardless of whether the assumed \( nk \) actually exists or not. Therefore, we obtain:

\( \neg \text{SSGB} \Rightarrow \text{SSGB} \).

This is a contradiction to the assumed existence of \( nk \) and so SSGB is proved.

\( \square \)

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