Abstract. This note is a proof of a strengthened form of the strong Goldbach conjecture.

Notations. Let \( \mathbb{N} \) denote the natural numbers starting from 1 and let \( \mathbb{P}_3 \) denote the prime numbers starting from 3.

Theorem (Strengthened strong Goldbach conjecture (SSGB)). Every even integer greater than 6 can be expressed as the sum of two different primes.

Proof. We define the set \( S_g := \{ (p_k, m_k, q_k) \mid k, m, p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \} \).

SSGB is equivalent to saying that every integer \( x \geq 4 \) is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers \( x \geq 4 \) appear as \( m \) in a middle component \( m_k \) of \( S_g \).

The negation \( \neg \)SSGB means that there is at least one \( n \geq 4 \) such that \( n_k \) is different from all the \( m_k \) for each \( k \geq 1 \), where all pairs \( (p, q) \) of odd primes that determine the numbers \( m \) are used in \( S_g \). For each \( k \geq 1 \), \( n_k \) can be written as some \( p_k \) when \( n \) is prime, as some \( p_k' \) when \( n \) is composite and not a power of 2, or as \( 4k' \) when \( n \) is a power of 2; \( p \in \mathbb{P}_3; k, k' \in \mathbb{N} \).

The expression \( p_k' \) for \( n_k \) with \( k' = k \) or \( k' \neq k \) is a first component of \( S_g \) triples and the expression \( 4k' \) for \( n_k \) is component of the triple \( (3k', 4k', 5k') \). Hence, for any \( n \geq 4 \) given by \( \neg \)SSGB we have

\[
(C): \forall k \in \mathbb{N} \quad \exists (p_k', m_k', q_k') \in S_g \quad n_k = p_k' \lor n_k = m_k'.
\]

Now, let \( S \) be a set such that \( SSGB \Rightarrow S_g = S \) and let \( S' \) be a set such that \( \neg SSGB \Rightarrow S_g = S' \). Then, since \( \neg \)SSGB means that there is an \( n \geq 4 \) such that \( n_k \neq m_k \) for all \( (p_k, m_k, q_k) \in S_g \) and since SSGB means that there is no such \( n \), we can infer:

By the non-existence of \( n \), \( S \) equals \( S_g \) as it is defined, i.e. without any condition, and \( S' \) also equals \( S_g \) as it is defined because, according to \( (C) \), every \( n_k \) given by \( \neg SSGB \) is a component of some triple that exists by definition of \( S_g \). So, by \( S = S' = S_g \), we have that the set \( S_g \) remains the same in the case \( n_k \) exists and in the case \( n_k \) does not exist.

Therefore, we obtain: \( \neg SSGB \Rightarrow SSGB \). This proves the theorem. \( \Box \)

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