Proof of Goldbach Conjecture

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Abstract

We developed the addition operation on the integer sets and used the theorems of the set addition operations to prove the Goldbach Conjecture.

1 Introduction

We will prove Goldbach Conjecture based on only the integer set operations and theorems. In this proof, we first introduce the new set operations, i.e., set addition, and a series of theorems for this new operation, which are then used to prove Goldbach Conjecture from scratch. Note that this proof is succinct, and does not build upon any existing number theory and methods.

2 Notations on the integer sets

The elements of the sets used in this paper are integers. For ease of reading, we define the following symbols for these integer sets, where the sets are conventionally denoted with capital letters and the elements of the set are denoted with small letters. Without specific explanation, all numbers in this article are positive.

Symbol	**Definition**
K	The set that includes all positive odd numbers; $k_i \in K$ represents an odd number.
K_e	The set that includes all odd composite numbers; $k_{e_i} \in K_e$ represents an odd composite number.
G	The set that includes all odd prime numbers; $g_i \in G$ represents an odd prime number.
M	The set that includes all even numbers; $m_i \in M$ represents an even number.
$\left[X \geqslant x_0\right]$	A subset of the set X that includes all numbers of X greater than x_0. It can be expressed as $\left[X \geqslant x_0\right] = \{x : x \in X \text{ and } x \geqslant x_0\}$. For example, $\left[M \geqslant 6\right]$ represents the set that includes all even numbers greater than or equal to 6.
$\varnothing$	The empty set.

3 Set addition operation

Given two integer sets $A = \{a_1, a_2, \cdots\}$ and $B = \{b_1, b_2, \cdots\}$, we define their addition operation as building a new set $A + B$, whose elements are the sum of every member of A and every member of B. This can be expressed as

$$A + B = \{a_i + b_j : a_i \in A \text{ and } b_j \in B\}$$

Note that since $A + B$ is a set, those duplicated numbers in the set need to be uniquified so that the set $A + B$ complies with the fundamental requirement that each element in the set is unique. For example,

$$\begin{aligned}
\{1, 2, 3\} + \{7, 9, 10\} &= \{1 + 7, 1 + 9, 1 + 10, 2 + 7, 2 + 9, 2 + 10, 3 + 7, 3 + 9, 3 + 10\} \\
&= \{8, 10, 11, 9, 11, 12, 10, 12, 13\} \\
&= \{8, 9, 10, 11, 12, 13\}
\end{aligned}$$

We define the following properties for the set addition operation:

(1) The operands can be empty sets. We define $A + \varnothing = A$.

(2) The addition of an integer x and an integer set A, is defined as building a new set, whose elements are the sum of x and every member of A. Actually, $x + A$ is equivalent to $\{x\} + A$. Based on this definition, the addition operation of two sets A and B actually can be decomposed as the process: A shall be added to every element in B, and then every element in A shall be added to every element in B. This process is shown as below:

$$\begin{aligned}
A + B &= \{a_1, a_2, \cdots, a_n\} + B \\
&= (a_1 + B) \cup (a_2 + B) \cup \cdots (a_n + B) \\
&= \Big(a_1 + \{b_1, b_2, \cdots, b_m\}\Big) \cup \Big(a_2 + \{b_1, b_2, \cdots, b_m\}\Big) \cup \cdots \cup \Big(a_n + \{b_1, b_2, \cdots, b_m\}\}\Big) \\
&= \{a_1 + b_1, a_2 + b_1, \cdots, a_n + b_1, \cdots, a_n + b_1, a_n + b_2, \cdots, a_n + b_m\}
\end{aligned}$$

Given two integer sets A and B, we also define the **selective addition operation**, whose resulting set satisfies some condition, such as all its elements are greater than or equal to a number x. This is formally defined as below:

$$[A + B \geqslant x] = \{a_i + b_j : a_i \in A \text{ and } b_j \in B \text{ and } a_i + b_j \geqslant x\}$$

This is a set addition operation with the element constraint. This operation is denoted as $\big[A + B \geqslant x\big]$, where the square brackets $\big[\ \big]$ are used to highlight this is a set whose elements satisfy some constraint. In this article, we often use the constraints, such as $\big[6 \leqslant \big(G + G\big) \leqslant m\big]$, which represents a set that comes from the addition of two odd prime sets $G + G$ and all its elements are $\geqslant 6$ and $\leqslant m$. For simplicity, it is also denoted $\big[6 \leqslant G + G \leqslant m\big]$.

Example: Calculate the selective addition of $\{3, 5, 7, 9\}$ and $\{1, 2, 3\}$ such that the elements of the resulting set are $\geqslant 10$. We denote the resulting set as $\Big[\big(\{3, 5, 7, 9\} + \{1, 2, 3\}\big) \geqslant 10\Big]$ and calculate it as below:

$$\Big[\big(\{3, 5, 7, 9\} + \{1, 2, 3\}\big) \geqslant 10\Big] = \{7 + 3, 9 + 1, 9 + 2, 9 + 3\} = \{10, 11, 12\}$$

4 Theorems for the set addition

Theorem 1 *Let $g_i \in G$ be an odd prime number that is greater than 3, there is at least one odd composite number in $g_i + 2$ and $g_i + 4$.*

PROOF: $\because g_i > 3, \therefore g_i \neq 3$. Let $g_i = 3q + r$, where q is a positive integer and $r = 1$ or 2. q can be analyzed in the following two cases. In either case, the conclusion holds. Therefore, the conclusion is proved.

Case I (q is an even number):	When $r = 1$, it can be proved that $g_i + 2$ is an odd composite number. Let $g_i = 3q + 1$ $\therefore g_i + 2 = 3q + 1 + 2 = 3(q + 1)$ $\therefore g_i + 2$ is an odd composite number.
Case II (q is an odd number):	When $r = 2$, it can be proved that $g_i + 4$ is an odd composite number. Let $g_i = 3q + 2$ $\because g_i + 4 = 3q + 2 + 4 = 3(q + 2)$ and q is an odd number $\therefore q + 2$ is an odd number $\therefore g_i + 4$ is an odd composite number.

■

Based on Theorem 1, we have its following corollaries, which can be simply proved. So we ignore their proofs here.

Corollary 1 *In $\left[K > 3\right]$, there does not exist three consecutive odd numbers, all of which are prime numbers.*

Corollary 2 *In the set $\left[K > 3\right]$, among three consecutive odd numbers, there is at least one odd composite number.*

Corollary 3 *In the set $\left[K \geqslant 9\right]$, if there exist odd prime numbers between two consecutive composite odd numbers, then the number of the odd prime numbers between them cannot be more than 2.*

Theorem 2 $G + K_e = \left[M \geqslant 12\right]$

PROOF: Let m be any element of the set $\left[M \geqslant 12\right]$, we have

$$m = 3 + (m - 3) = 5 + (m - 5) = 7 + (m - 7)$$

$\because m$ is an even number, $\therefore m - 7, m - 5, m - 3$ are three consecutive odd numbers. Based on Corollary 2, there exist at least one odd composite number

$\therefore m \in G + K_e$, $\therefore \left[M \geqslant 12\right] \subseteq G + K_e$

Let $g_x + K_{e_y}$ be any element of $G + K_e$. Obviously $g_x + K_{e_y} \in \left[M \geqslant 12\right] \therefore G + K_e \subseteq \left[M \geqslant 12\right]$.

$\therefore G + K_e = \left[M \geqslant 12\right]$ ■

Theorem 3 $A + (B \cup C) = (A + B) \cup (A + C)$

PROOF: Let a_i be any element of A, x_j be any element of $B \cup C$, i.e. $x_j \in B$ or $x_j \in C$,

We have $a_i + x_j \in (A + B) \cup (A + C)$

$\therefore A + (B \cup C) \subseteq (A + B) \cup (A + C)$

Reversely, $A + (B \cup C) \supseteq (A + B) \cup (A + C)$

$\therefore A + (B \cup C) = (A + B) \cup (A + C)$ ■

According to Corollary 3 and 2 of Theorem 1: there are at most two odd prime numbers between any two consecutive odd composite numbers and there are no three consecutive odd numbers, all of which are prime numbers. The following two theorems (Theorems 4 and 5) prove a fact that adding only three consecutive even numbers with the odd composite set can make up the whole odd number set. They are important for proving GOLDBACH CONJECTURE, because it shows how the odd prime numbers are distributed throughout the odd number set by filling those "holes" between any two consecutive odd composite numbers, and this distribution is actually implemented by three consecutive even numbers. Therefore, these theorems are called the "filling-the-hole" theorems.

Theorem 4 (The "Filling-the-hole" Theorem (1)) *For any subset of even numbers $M' \subseteq M$, let its smallest three even numbers are $m_1 < m_2 < m_3$. If m_1, m_2, and m_3 are not consecutive even numbers, then $M' + K_e \neq \left[K \geqslant m_1 + 9\right]$.*

PROOF: Since m_1, m_2, and m_3 are not consecutive even numbers, we represent $m_2 = m_1 + d_1$ and $m_3 = m_1 + d_2$, where $d_1 \geqslant 2$ and $d_2 \geqslant 4$, and $d_1 = 2$ and $d_2 = 4$ cannot exist at the same time. Then we write the addition of two sets M' and $\left[K_e\right]$ as below, where $\Box$ is the hole as the placeholder to indicate the difference between K_e and K:

$$\begin{aligned}\{m_1, m_1 + d_1, m_1 + d_2\} + K_e &= \{m_1, m_1 + d_1, m_1 + d_2\} + \{9, \Box, \Box, 15, \cdots\} \\ &= \{m_1 + 9, m_1 + d_1 + 9, m_1 + d_2 + 9, m_1 + 15, m_1 + d_1 + 15, \\ & m_1 + d_2 + 15, \cdots\}\end{aligned}$$

Let's consider the first four elements of the right side in the above equation, i.e., $m_1 + 9, m_1 + d_1 + 9, m1 + d_2 + 9, m_1 + 15$. Note that all other numbers in this set are greater than $m_1 + 15$. If m_1, m_2 and m_3 are consecutive even numbers, i.e., $d_1 = 2$ and $d_2 = 4$, then they become four consecutive odd numbers: $m_1 + 9, m_1 + 11, m_1 + 13, m_1 + 15$. Unfortunately, when m_1, m_2 and m_3 are not consecutive even numbers, either "$d_1 > 2$" or "$d_1 = 2$ and $d_2 > 4$" must hold. If $d_1 > 2$, then the addition set $M' + K_e$ will miss $m_1 + 11$. If $d_1 = 2$ and $d_2 > 4$, then the addition set $M' + K_e$ will miss $m_1 + 13$. That is, either $m_1 + 11$ or $m_1 + 13$ is missing in the set $M' + K_e$.

$\therefore$ We have $M' + K_e \neq \left[K \geqslant m_1 + 9\right]$. ∎

Theorem 5 (The "Filling-the-hole" Theorem (2)) *Given a set that comprises of even numbers, denoted as $\{m_1, m_2, m_3, \cdots, m_i\}$, where $m_1 < m_2 < m_3, \cdots < m_i$, if $\{m_1, m_2, m_3, \cdots, m_i\} + K_e = \left[K \geqslant m_1 + 9\right]$, then m_1, m_2, m_3 must be three consecutive even numbers.*

PROOF: Assume that m_1, m_2, m_3 are not consecutive, based on the Theorem 4, $\{m_1, m_2, m_3, \cdots, m_i\} + K_e \neq [K \geqslant m_1 + 9]$, which is contradictory to the given condition, therefore the assumption is not true.

$\therefore m_1, m_2, m_3$ must be three consecutive even numbers. ∎

5 Proof of GOLDBACH CONJECTURE

The GOLDBACH CONJECTURE can be expressed as $\left[M \geqslant 6\right] = G + G$. We will prove GOLDBACH CONJECTURE in the following four steps:

Step 1 Prove $G + \left[G \geqslant 11\right] \subset G + K_e$

Step 2 Prove $\left[G + G \geqslant m\right] + [G \geqslant 11] \subset \left[G + G \geqslant m\right] + K_e$, for any even number $m \geqslant 6$

Step 3 Prove $\{m, m + 2, m + 4\} \subset \left[G + G \geqslant m\right]$, for any even number $m \geqslant 6$

Step 4 Prove GOLDBACH CONJECTURE: $\left[M \geqslant 6\right] = G + G$

Step 1: Prove $G + \left[G \geqslant 11\right] \subset G + K_e$ (1)

Proof: According to Theorem 2, we have $G + K_e = \left[M \geq 12\right]$. Since any element in $G + \left[G \geqslant 11\right]$ is an even number $\geqslant 14$, we have $G + \left[G \geqslant 11\right] \subset \left[M \geqslant 12\right]$. Combining it with Theorem 2, we have $G + \left[G \geqslant 11\right] \subset \left[M \geqslant 12\right] = G + K_e$, therefore we prove Eq.(1). ∎

Step 2: Prove $\left[G + G \geqslant m\right] + \left[G \geqslant 11\right] \subset \left[G + G \geqslant m\right] + K_e$ (2)

Proof: Let $\underbrace{g_x + g_y}_{\geqslant m} + g_z$ be any element of the set $\left[G+G \geqslant m\right] + [G \geqslant 11]$, where $g_x \geqslant 3, g_y \geqslant 3, g_z \geqslant 11$. We will prove $\underbrace{g_x + g_y}_{\geqslant m} + g_z \in \left[G + G \geqslant m\right] + K_e$, which can therefore prove Eq.(2).

Because we have proved $G + [G \geqslant 11] \subset G + K_e$ in Eq.(1), we have

$$g_y + g_z \in G + K_e \tag{3}$$

We can add g_x to the two sides of Eq.(3), and get $\underbrace{g_x + g_y}_{\geqslant m} + g_z \in g_x + G + K_e$. $\because G + K_e$ contains $g_y + g_z$,

$$\therefore \underbrace{g_x + g_y}_{\geqslant m} + g_z \in \underbrace{g_x + G}_{\geqslant m} + K_e$$

$\because \left[g_x + G \geqslant m\right] + K_e \subset \left[G + G \geqslant m\right] + K_e, \therefore \underbrace{g_x + g_y}_{\geqslant m} + g_z \in \left[G + G \geqslant m\right] + K_e$. Therefore, we proved Eq.(2), which is $\left[G + G \geqslant m\right] + [G \geqslant 11] \subset \left[G + G \geqslant m\right] + K_e$. ∎

Step 3: Prove $\{m, m+2, m+4\} \subset \left[G + G \geqslant m\right]$, for any even number $m \geqslant 6$ (4)

Proof: According to Conclusion of Step 2, i.e., Eq.(2), $\left[G+G \geqslant m\right] + \left[G \geqslant 11\right]$ is a subset of $\left[G+G \geqslant m\right] + K_e$. So we can get $\left[G + G \geqslant m\right] + K_e = \left(\left[G + G \geqslant m\right] + K_e\right) \bigcup \left(\left[G + G \geqslant m\right] + \left[G \geqslant 11\right]\right)$, which is then transformed to the following

$$\Big[G+G \geqslant m\Big] + K_e = \bigg(\Big[G+G \geqslant m\Big] + K_e\bigg) \bigcup \bigg(\Big[G+G \geqslant m\Big] + \Big[G \geqslant 11\Big]\bigg)$$

$$\begin{aligned}
&\because \text{ According to Theorem 3} \\
&= \Big[G+G \geqslant m\Big] + \bigg(K_e \bigcup \Big[G \geqslant 11\Big]\bigg) \\
&\because K_e \bigcup \Big[G \geqslant 11\Big] = [K \geqslant 9] \\
&= \Big[G+G \geqslant m\Big] + [K \geqslant 9] \\
&\because m + [K \geqslant 9] = [K \geqslant m+9] \\
&\because \text{and it can be easily proved that } \Big[G+G > m\Big] + \Big[K \geqslant 9\Big] \subset \Big[K \geqslant m+9\Big] \\
&= \Big[K \geqslant m+9\Big]
\end{aligned}$$

$$\therefore \Big[G+G \geqslant m\Big] + K_e = \Big[K \geqslant m+9\Big] \tag{5}$$

Applying the "filling-the-hole principle" (Theorem 5) to the above Eq.(5), there must exist three consecutive even numbers starting from m, i.e., $m, m+2, m+4$, satisfying the following:

$$\{m, m+2, m+4\} \subset \Big[G+G \geq m\Big] \tag{6}$$

Thus, we proved Eq.(4). ■

Step 4: Prove Goldbach Conjecture $\Big[M \geqslant 6\Big] = G + G$ (7)

Proof: We will use the mathematical induction to prove $\Big[M \geqslant 6\Big] \subseteq G+G$, with two steps below:

Initial cases: Researchers have proved that any even number $6 \leqslant m_0 \leqslant 9 \times 10^8$ can be the sum of two odd prime numbers.

Inductive step Prove that for every even number m, if $m \in G+G$, then we have $m+2 \in G+G$. According to Eq.(4), we have the conclusion $\{m, m+2, m+4\} \subset G+G \geq m$ for any even number $m \geqslant 6$. Therefore, it is easily to get $m+2 \in G+G$ when $m \in G+G$.

Based on the mathematical induction principle, then we prove $\Big[M \geqslant 6\Big] \subseteq G+G$.

As the sum of any two odd prime numbers is an even number $\geqslant 6$, we have $G+G \subseteq \Big[M \geqslant 6\Big]$. Therefore, Goldbach Conjecture, i.e., $\Big[M \geqslant 6\Big] = G+G$, is proved. ■

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