On the zeros of the Riemann zeta function,

twelve years later

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Abstract. The paper proves the Riemann Hypothesis.

Key words: Riemann zeta function, Riemann Hypothesis, complex analysis.

1 Introduction

Mathematicians like to say that extraordinary claims require extraordinary proofs, and journal editors add that they receive purported proofs of the Riemann Hypothesis every day. There is another saying claiming that the correct solution is often simple. Both sayings may have some truth in it, but only on occasion. The only extraordinary thing in this present proof of the Riemann Hypothesis is its extreme shortness and simplicity. It is also correct. If it indeed is so that journal editors receive every day short, simple and correct proofs of the Riemann Hypothesis and reject them without a review, then there must be something wrong with the review process. A very short and simple proof can and should be checked. In a complicated proof that uses new, innovative and difficult methods, it is necessary to have extremely hard precision. In the present proof the methods are very traditional and fully known. Lemmas 1 and 2 are obvious from elementary calculation. Lemma 3 is just partial differentiation of a complex function of two variables. Simplicity of a solution, however, does not imply that it was easy to find.

I will give the outline of this proof before mathematicians have a chance to point out to what they think are inaccuracies in notations and such formal issues.
The idea is to construct a function of two complex variables, $\xi(s, z)$, such that $\xi(s, z) = \varphi(x, s)^{-1} \zeta(s)$. Then this function has all zeros $s_0$ of the Riemann zeta function $\zeta(s)$. The function $\varphi(s, z)$ is constructed to be finite in a point $(s, z)$ where $s + z = s_0$ for some zero $s_0$ of $\zeta'(s)$ where $z$ is small but nonzero. Then $\xi(s, z)$ does not have a zero at $(s, z)$. Some derivative of $\zeta(s)$ must be nonzero at $s_0$ as $\zeta$ is not the zero function. We can express

$$\frac{d\zeta(s)}{ds} = h(s)\zeta(s) + g(s)\zeta(s)$$

where $g(s)$ is convergent at $s_0$. Then $h(s)$ must have a pole at $s_0$. The function $\varphi(s, z)$ is so constructed that $\xi(s, z)$ has the expression

$$\frac{\partial \xi(s, z)}{\partial z} = h(s + z)\xi(s, z) + u(s, z)\xi(s, z)$$

where $u(s, z)$ is convergent at $(s, z)$ where $s + z = s_0$. The pole of $h(s)$ at $s_0$ means that $h(s + z) = h(s_0)$ has a pole. As $\xi(s, z)$ is finite and not zero and $h(s, z)$ is finite, the partial derivative

$$\frac{\partial \xi(s, z)}{\partial z}$$

must be infinite. This is not possible as $\xi(s, z)$ is finite. Thus, the goal in the prove is to move the pole of $h(s)$ at $s_0$ to another place, to $(s, z), s \neq s_0$, where the pole cannot be cancelled by a zero of $\xi(s, z)$, which has a zero only when $\zeta(s)$ has a zero. We have to construct $\varphi(s, z)$ in Lemma 1, show that $\xi(s, z)$ is finite and nonzero at $(s, z), s + z = s_0$, in Lemma 2, and conclude that $h(s + z) = h(s_0)$ cannot have a pole at $(s, z)$ in Lemma 3. Theorem 1 is simply a consequence of what Riemann proved.
2 Definitions

The Riemann zeta function is defined as

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \]  

(1)

where \( s = x + iy \), \( x, y \in \mathbb{R} \), is a complex number. The zeta function can be continued analytically to the whole complex plane except for \( s = 1 \) where the function has a pole. The function has trivial zeros at even negative integers. It does not have zeros for \( x \geq 1 \) and the only zeros for \( x \leq 0 \) are the trivial ones. The nontrivial zeros lie in the strip \( 0 < x < 1 \), see e.g. [1].

Let

\[ P = \{ p_1, p_2, \ldots | p_j \text{ is a prime, } p_{j+1} > p_j > 1, j \geq 1 \} \]

be the set of all primes (larger than one). Let \( s = x + iy, x, y \in \mathbb{R} \) and \( x > \frac{1}{2} \).

The Riemann zeta function can be expressed as

\[ \zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1} \]  

(2)

This infinite product converges absolutely if \( x = \text{Re}\{s\} > 1 \).

3 Lemmas and the theorem

Lemma 1. Let \( z = \sigma + i\delta, \sigma, \delta \in \mathbb{R}, s = x + iy, x, y \in \mathbb{R}, x > 1, \text{ and } \text{Re}\{s + z\} > 1 \). We define the absolutely convergent products

\[ \varphi(s, z) = \prod_{j=1}^{\infty} \left( 1 - p_j^{-s} + p_j^{-s-z} \right)^{-1} \]  

(3)

\[ \xi(s, z) = \prod_{j=1}^{\infty} \left( \frac{1 - p_j^{-s}}{1 - p_j^{-s} + p_j^{-s-z}} \right)^{-1} \]  

(4)
Then \((\sigma = \text{Re}\{z\})\)

\[
\zeta(s) = \varphi(s, z)\xi(s, z)
\]

\[
\varphi(s, 0) = 1 , \ \xi(s, 0) = \zeta(s),
\]

\[
\lim_{\sigma \to \infty} \varphi(s, z) = \zeta(s) , \lim_{\sigma \to \infty} \xi(s, z) = 1.
\]

**Proof.** Directly multiplying the absolutely convergent products shows (5). At \(z = 0\) the term \(1 - p_j^{-z} = 0\) and if \(\sigma \to \infty\) then \(p_j^{-z} \to 0\). The limits follow. \(\blacksquare\)

**Lemma 2.** Let \(z = \sigma + i\delta, \ \sigma, \delta \in \mathbb{R}, s = x + iy, x, y \in \mathbb{R}, \text{ and } x > \frac{1}{2}.\) Let us assume there exists a zero \(s_0\) of \(\zeta(s)\) with \(1 > \text{Re}\{s_0\} > \frac{1}{2}.\) Then there exists \(\epsilon > 0\) such that if \(0 < |z| < \epsilon\) and \(s + z = s_0\) then \(\xi(s, z)\) is analytic at the point \((s, z)\) as a function of \(z\). The function \(\xi(s, z)\) is finite and nonzero at such a point \((s, z)\) and the partial derivative of \(\xi(s, z)\) with respect to \(z\) is finite at such a point.

**Proof.** Let \(\epsilon > 0\) be chosen smaller than \(x - \frac{1}{2}\) and \(|z| < \epsilon\). Then \(\text{Re}\{s + z\} \geq x - |\sigma| > \frac{1}{2}\). We are only interested in values of \((s, z)\) when \(s + z = s_0\), thus we insert \(s + z = s_0\) to \(\varphi(s, z)\) i (3):

\[
\varphi(s, z) = \prod_{j=1}^{\infty} (1 - p_j^{-s} + p_j^{-s_0})^{-1}.
\]

Clearly, this is a function of \(s\), if \(s = s_0\) the function is one. If \(\epsilon\) is sufficiently small, this function is close to one. The function \(\varphi(s, z)\) is finite and nonzero at \((s, z)\) where \(s + z = s_0\). The Riemann zeta function \(\zeta(s)\) is analytic as a function of \(s\) when \(1 > \text{Re}\{s\} > \frac{1}{2}\). As \(\varphi(s, z)\) is finite and nonzero, the function \(\xi(s, z) = \varphi(s, z)^{-1}\zeta(s)\) can be continued analytically as a function of \(s\) to \(1 > \text{Re}\{s\} > \frac{1}{2}\). If \(s + z = s_0\), \(0 < |z| < \epsilon\) then \(\xi(s, z)\) is finite and nonzero. Then the partial derivative of \(\xi(s, z)\) with respect to \(z\) is finite at \((s, z)\). \(\blacksquare\)
**Lemma 3.** Let $\zeta(s_0) = 0$ for $s_0$ such that $\text{Re}\{s_0\} > \frac{1}{2}$. Then for $n \geq 1$ and $D = \frac{d}{ds}$

$$D^n \zeta(s_0) = 0.$$ 

**Proof.** Let $z = \sigma + i \delta$, $\sigma, \delta \in \mathbb{R}$, $s = x + iy$, $x, y \in \mathbb{R}$, and $x > \frac{1}{2}$. Let $\epsilon > 0$ be chosen smaller than $1 > x - \frac{1}{2}$ and $|z| < \epsilon$. Then $\text{Re}\{s + z\} \geq x - |\sigma| > \frac{1}{2}$. We also assume that $\epsilon > 0$ is chosen so small that $\varphi(s, z)$ and $\xi(s, z)$ have the analytic continuations from Lemma 2 as a function of $z$.

Let $f_j(s, z)$, $j \geq 1$, be any continuously differentiable functions of $s$ and $z$ in an open neighborhood of $(s, z)$ and let the infinite product

$$f(s, z) = \prod_{j=1}^{\infty} (1 - f_j(s, z))^{-1}$$

be absolutely convergent, then

$$\frac{\partial f(s, z)}{\partial z} f(s, z)^{-1} = \partial \ln f(s, z) = -\frac{\partial}{\partial z} \sum_{j=1}^{\infty} \ln(1 - f_j(s, z))$$

$$= \sum_{j=1}^{\infty} \frac{\partial f_j(s, z)}{\partial z} + \sum_{j=1}^{\infty} \frac{f_j(s, z)}{1 - f_j(s, z)} \frac{\partial f_j(s, z)}{\partial z}. \quad (7)$$

We can express $\xi(s, z)$ as

$$\xi(s, z) = \prod_{j=1}^{\infty} \left(1 - p_j^{-s-z} \left(1 + p_j^{-s} \frac{1 - p_j^{-z}}{1 - p_j^{-s}(1 - p_j^{-z})}\right)\right)^{-1}. \quad (8)$$

For $\text{Re}\{s + z\} > 1$ we select in (7)

$$f_j(s, z) = p_j^{-s-z} \left(1 + p_j^{-s} \frac{1 - p_j^{-z}}{1 - p_j^{-s}(1 - p_j^{-z})}\right).$$

Thus $f(s, z) = \xi(s, z)$. Let us write

$$h(z) = -\sum_{j=1}^{\infty} \ln(p_j)p_j^{-z}$$
and
\[ u(s, z) = \frac{\partial}{\partial z} \left( \sum_{j=1}^{\infty} p_j^{s-2z} \frac{1-p_j^{-z}}{1-p_j^{-s}(1-p_j^{-z})} \right) + \sum_{j=1}^{\infty} \frac{f_j(s, z)}{1-f_j(s, z)} \frac{\partial f_j(s, z)}{\partial z}. \]

Here \( f(s, z) = \xi(s, z) \). The leading term for
\[ \frac{\partial f_j(s, z)}{\partial z} \]
is \( \ln(p_j)p_j^{-s-z} \) and for
\[ f_j(s, z)/(1 - f_j(s, z)) \]
the leading term is \( p_j^{-s-z} \). From the total leading term \( \ln(p_j)p_j^{2s-2z} \) it follows that \( u(s, z) \) is absolutely convergent if \( \text{Re}\{s + z\} > \frac{1}{2} \). From the definition of \( h(z) \) follows that \( h(z) \) is absolutely convergent if \( \text{Re}\{s + z\} > 1 \). For \( \text{Re}\{x + z\} > 1 \) holds
\[ \frac{\partial \xi(s, z)}{\partial z} = h(s + z)\xi(s, z) + u(s, z)\xi(s, z). \] (9)

The function \( h(z) \) converges absolutely if \( \text{Re}\{z\} > 1 \). In the equation (9) \( \partial \xi(s, z) / \partial z \) is analytic as a function of \( z \) in if \( s + z = s_0 \), \( s_0 \) is a zero of \( \zeta(s) \), \( 1 < |z| < \epsilon \) and \( 1 > \text{Re}\{s\} > \frac{1}{2} \) by Lemma 2. We can continue \( h(z) \) analytically to \( 1 > \text{Re}\{z\} > \frac{1}{2} \) as a function of \( z \) by (9), except to points where \( \zeta(s) = 0 \). At those points \( h(s) \) may have a pole.

Thus, if \( 1 > x > \frac{1}{2} \) and \( |z| < \epsilon \), then holds
\[ \frac{\partial \xi(s, z)}{\partial z} = h(s + z)\xi(s, z) + u(s, z)\xi(s, z). \] (10)

The term \( u(s, z)\xi(s, z) \) is finite because \( \xi(s, z) \) is finite and \( u(s, z) \) converges and is therefore finite.

In a similar way, let \( \text{Re}\{s + z\} > 1 \). We take \( f(s, z) = \zeta(z) \), calculate the expansion from (7) and then write \( z \) instead of \( s \) in the result, as \( \zeta(z) \) does not
have two parameters $s, z$. The result is that for $\text{Re}\{x + z\} > 1$ holds

$$D\zeta(s) = h(s)\zeta(s) + g(s)\zeta(s),$$

(11)

where

$$g(s) = -\sum_{j=1}^{\infty} \frac{\ln(p_j)p_j^{-2s}}{1 - p_j^{-s}}.$$  

The function $g(s)$ converges absolutely if $x > \frac{1}{2}$. The term $g(s)\zeta(s)$ is finite because $\zeta(s)$ is finite and $g(s)$ converges and is therefore finite.

If the derivative $D\zeta(s_0)$ is nonzero, then $h(s)$ must have a pole at $s_0$. If so, then $h(s + z)$ has a pole at any point where $s_0 = s + z$. It follows from (10) that the partial derivative $\frac{\partial k(s, z)}{\partial z}$ must be infinite at such a point because the term $u(s, z)\zeta(s, z)$ is finite and $\xi(s, z)$ is finite and nonzero.

This is not possible because $\xi(s, z)$ is finite and therefore its partial derivative with respect to $z$ is finite. It follows that $h(s)$ does not have a pole at $s = s_0$ and therefore $D\zeta(s_0) = 0$.

Let us assume $D^{(j)}\zeta(s_0) = 0$ is shown by induction for $j \leq n$. The initial step $n = 2$ is proven above. The convergent part $g(s)$ and all its derivatives are finite at $(s_0)$ and vanish when multiplied by a zero. As $h(s_0)$ is finite, all its derivatives must also be finite at $s_0$. At $(s_0)$ the function $h(s)$ and all its derivatives vanish when multiplied by a zero. Because of the induction assumption $D^{(j)}\zeta(s_0) = 0$ for $j \leq n$. All terms of the $n$th derivative of $\zeta(s)$ vanish when the terms are expanded:

$$D^{(n)}(g(s)\zeta(s))_{s=s_0}$$

and

$$D^{(n)}(h(s)\zeta(s))_{s=s_0}$$

Therefore

$$D^{(n+1)}\zeta(s)|_{s=s_0}.$$
The claim of the lemma follows by induction on \( n \). \( \square \)

**Theorem 1.** If \( s = x + iy, \, x, y \in \mathbb{R}, \, 0 < x < 1, \) and \( \zeta(s) = 0 \) then \( x = \frac{1}{2} \).

**Proof.** Riemann showed that

\[
2^{1-s} \Gamma(s) \zeta(s) \cos \left( \frac{1}{2} s \pi \right) = \pi^s \zeta(1 - s).
\]

Thus, if there exists a zero \( s_0 = x_0 + iy_0 \) of \( \zeta(s) \) with \( 0 < x_0 < \frac{1}{2} \) then there exists a zero of \( \zeta(s) \) at a symmetric point in \( \frac{1}{2} < x < 1 \). Therefore we only need to look at the strip \( \frac{1}{2} < x < 1 \). Let us assume there exists \( s_0 = x_0 + iy_0 \) with \( \zeta(s_0) = 0 \) and \( \frac{1}{2} < x_0 < 1 \). By the assumption

\[
\left( \frac{\partial^n}{\partial x^n} + \frac{\partial^n}{\partial y^n} \right) \zeta(s_0) = 0
\]

for every \( n \geq 1 \). As \( \zeta(s) \) is analytic if \( s \neq 1 \), it has a converging Taylor series in some neighborhood of every point \( s \neq 1 \). We can analytically continue \( \zeta(s) \) along the \( x + iy_0 \) line starting from \( x = x_0 \). Since \( \zeta(s_0) = 0 \) and all derivatives with respect to \( s \) vanish, \( \zeta(s) = 0 \) for every point \( s \) to which the function is continued. Thus \( \zeta(x + iy_0) = 0 \) for some \( x > 1 \) which is a contradiction. Therefore no such \( s_0 \) exists. \( \square \)

Theorem 1 proves the Riemann Hypothesis.

### 4 The issues found by Bombieri in 2008

Twelve years ago I put to arXiv a preprint [2] with the short abstract: The paper proves the Riemann Hypothesis. I had worked on the problem for five months and found what looked like a proof. I could not find any mathematician who would have agreed to comment the proof. Finally I saw no other way and sent the paper to Enrico Bombieri, who had authored the CMI problem statement for the Riemann Hypothesis problem. Bombieri answered and came up very fast with two
problems in the paper, but as a condition for reading it he had demanded that I will not send him any improved versions, which I promised and felt obliged to comply to.

Addressing the issues Bombieri had raised would have required new text in the paper, thus an improved version. As this was excluded, and there seemed to be no option to get the paper read by a mathematician, I decided to check it myself, but in order to do it, I had to be able to read the paper with fresh eyes: there was no chance of finding anybody to help me by reading and commenting the text.

I had worked intensively on the paper a quite long time and needed to look at it as an outsider. I estimated that it will take at least ten years before I have forgotten the problem and what I wrote, and could read the paper with fresh eyes. I changed the claim of having solved the problem in the arXiv paper to a harmless claim of a new characterization to the Riemann Hypothesis and then did not think of this problem any more until now.

It is twelve years since that time. I managed to forget it all and to read the manuscript with fresh eyes. I did carefully consider the objections that Bombieri had raised in 2008. They were quite relevant to the original version of the proof, but the idea of the proof did not need any changes and the paper again looks to me like a proof.

The first problem Bombieri mentioned was that a function of two complex variables does not always have the same limit if a point is approached from different directions. As an example, let

$$f(s, z) = (1 - z)(s - s_0).$$

The limit at $(s_0, 0)$ depends on how it is taken:

$$\lim_{z \to 0} f(s_0, z) = 0 \quad \lim_{s \to s_0} f(s, 0) = 1.$$
The original proof of Lemma 2 in [2] has a place where this concern was valid. The argument for \( \psi(s, z) \) is made in [2] by using the expression

\[
\psi(x, s) = \phi(s, z) \frac{1}{\xi(s, z)} \zeta(s).
\]

From this expression it is not possible to know how \( \psi \) behaves in the zeros of \( \zeta(s) \), which also are zeros of \( \xi(s) \). Now \( \varphi(s, z) = \phi(s, z)\psi(s, z) \), which is simpler, and the proof of Lemma 2 is different. Yet, the proof of Lemma 3 does not need any knowledge of how \( \varphi(s, z) \) behaves at \( (s_0, 0) \) and this problem is not relevant to the proof.

The second issue Bombieri mentioned was more general. He was worried if the method proves that \( \zeta(s) \) has no zeros even in \( x = \frac{1}{2} \) (which would be false) and if the method proves that an arbitrary function has no zeros (which also would be false).

This is not the case. The proof of Lemma 3 depends on the particular form of the Riemann zeta function and it cannot be extended to \( x = \frac{1}{2} \). Let us look at a simple example. For the time being, let \( \zeta(s) \) not denote the Riemann zeta function. Let \( \zeta(s), \xi(s, z) \) and \( \varphi(s, z) \) be defined as

\[
\zeta(s) = s - s_0 \quad \xi(s, z) = (1 - z)^{-1}(s - s_0) \quad \varphi(s, z) = 1 - z. \]

We have the equation as in Lemma 1:

\[
\xi(s, z) = \varphi(s, z)^{-1} \zeta(s).
\]

Claims similar to the ones in Lemma 2 also hold. If we can follow the steps of Lemma 3 then the method proves that \( \zeta = s - s_0 \) does not have a zero at \( s = s_0 \). Obviously that cannot be true.
But when trying to follow the steps of Lemma 3 with these functions, we notice that these functions do not work in Lemma 3. We get

$$\ln \xi(s,z) = -\ln \varphi(s,z) + \ln \zeta(s)$$

$$\frac{\partial}{\partial z} \ln \xi(s,z) = -\varphi(s,z)^{-1} \frac{\partial}{\partial z} \varphi(s,z).$$

Thus

$$\frac{\partial}{\partial z} \xi(s,z) = -\varphi(s,z)^{-1} \frac{\partial}{\partial z} \varphi(s,z) \xi(s,z)$$

$$= h(s,z)\xi(s,z) + u(s,z)\xi(s,z).$$

Notice that here we cannot write $h(s,z) = h(s+z)$ as in Lemma 3. The convergent function $u(s,z)$ can be taken as zero. Then

$$h(s,z) = -\varphi(s,z)^{-1} \frac{\partial}{\partial z} \varphi(s,z) = (1 - z)^{-1}.$$ 

For $\zeta(s)$ we can calculate in a similar way

$$\frac{d}{ds} \ln \zeta(s) = (s - s_0)^{-1}$$

$$\frac{d}{ds} \zeta(s) = (s - s_0)^{-1} \zeta(s) = h(s)\zeta(s) + g(s)\zeta(s).$$

The convergent function $g(s)$ can be taken as zero. Thus

$$h(s) = (s - s_0)^{-1}.$$ 

Clearly, $1 = h(s,0) \neq h(s + 0) = (s - s_0)^{-1}$ as in Lemma 3. The case here is that the function $\zeta = (s - s_0)$ has an explicit zero at $s = s_0$. In fact, we do not initially know if it has a zero. We select $\varphi(s,z)$ which is finite and nonzero for small $z$, as the method requires, and set $\varphi(s,z) = 1 - z$, but this $\varphi$ does not give $h(s,z) = h(s+z)$. The Riemann function does not have an explicit zero at $s = s_0$. 


and we find in Lemma 3 a suitable $\varphi(s, z)$ that gives $h(s, z) = h(s + z)$ and we can show that $h(s)$ must be finite at $s = s_0$, from which the proof follows.

Notice that if we take $\zeta(s) = (s - s_0)^k$ to have a multiple zero at $s_0$, it does not help; some derivative of $h(s)$ must be infinite in this case and it implies that $h(s)$ must have a pole at $s = s_0$.

Thus, the paper does not prove that an arbitrary function does not have zeros. It is also not possible to extend the proof to show that the Riemann zeta function does not have zeros at $x = \frac{1}{2}$. The functions $g(s)$ and $u(s, z)$ in Lemma 3 converge only for $x > \frac{1}{2}$. These functions are also not closely related, like $h(s + z)$ and $h(s)$ in Lemma 3. With $u(s, 0)$ and $g(s)$ we cannot establish a similar equality. Therefore the proof extends only to the area where $g(s)$ and $u(s, z)$ converge and thus stay finite and are reduced to zero by multiplying with zero ($\zeta(s_0) = 0$ and $\xi(s_0, z) = 0$).

The proof presented in this paper is essentially the same as in [2], only rewritten to be more readable and to address Bombieri's comments.

References
