The special functions and the proof of the Riemann's hypothesis

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Abstract : By studying the \( \zeta \) function whose integer zeros are the prime numbers, and being inspired by the article [2], I give a new proof of the Riemann hypothesis.

Résumé : En étudiant la fonction \( \zeta \) dont les zéros entiers sont les nombres premiers, et en m’inspirant de l’article [2], je donne une nouvelle preuve de l’hypothèse de Riemann.

I- INTRODUCTION

The Riemann’s hypothesis [2] conjectured that all non-trivial zeros of \( \zeta \) are in the line \( x = \frac{1}{2} \).

In this article, the study of the sghiar’s function \( \zeta \) which I introduced and whose integer zeros are the prime numbers inspired me to use the function Gamma \( \Gamma \). And miraculously a proof similar to that used in [2] allowed me to give a short and elegant proof of the Riemann hypothesis.

In order not to recall everything, I suppose known - among others - the functional identities \( \zeta \), Gamma \( \Gamma \) : 
\[
z \mapsto \int_0^{\infty} t^{z-1} e^{-t} dt \text{ and their properties (See [3] and [4]).}
\]

II- THE PROOF OF THE RIEMANN HYPOTHESIS :

Theorem 1 (The Riemann hypothesis) All non-trivial zeros of \( \zeta \) are in the line \( x = \frac{1}{2} \).

Lemma 1

\[
0 < Re(z) < 1 \implies \left| \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt \right| \neq 0
\]

Proof :

It suffices to prove that \( Re(\int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt) \neq 0 \) or \( Im(\int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt) \neq 0 \)

Let \( z = x + iy \), by change of variable, and by setting \( t^{z-1} = e^u \), we deduce :

\[
-Re\left(\int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt\right) = \int_{-\infty}^{\infty} \frac{e^u}{e^{xu} - 1} \cos(y \frac{u}{x-1}) \frac{1}{x-1} e^{\frac{uy}{x-1}} du
\]

Note :

As \( \frac{e^u}{e^{xu} - 1} \cos(y \frac{u}{x-1}) \frac{1}{x-1} e^{\frac{uy}{x-1}} \) is zero for \( u = (2k + 1) \frac{\pi}{2} \), \( k \in \mathbb{Z} \) and oscillates increasing in amplitude because \( g(u) = \frac{e^u}{e^{xu} - 1} e^{\frac{uy}{x-1}} \) is decreasing with \( u \), we deduce that :

\[
\int_{u=(2(k+1)+1) \frac{\pi}{2}}^{u=(2(k+1)+1) \frac{\pi}{2}} \cos(y \frac{u}{x-1}) \frac{1}{x-1} e^{\frac{uy}{x-1}} du
\]

is different from 0 and its sign does not depend on \( k \in \mathbb{Z} \) (we have the same result if \( k \in 2\mathbb{Z} + 1 \) :

Because : \( \int_{u=(2(k+1)+1) \frac{\pi}{2}}^{u=(2(k+1)+1) \frac{\pi}{2}} \cos(y \frac{u}{x-1}) \frac{1}{x-1} e^{\frac{uy}{x-1}} du = 0 \)

In this case :

\[
Re\left(\int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt\right) = \lim_{u_k \to \infty} \int_{-\infty}^{u_k} f(u) du
\]

If \( \int_{u}^{u+1} f(u) du \geq 0 \) :

So :

- Either \( f(u) \geq 0 \) ( \( f \) increasing in the vicinity of \( u \) )

In this case :

\[
Re\left(\int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt\right) = \int_{u}^{u+1} f(u) du + \int_{u}^{u+1} f(u) du + \sum_{k \in \mathbb{N}^*} \int_{u_k}^{u_k+1} f(u) du \geq 0
\]

- Or either \( f(u) \leq 0 \) ( \( f \) decreasing in the vicinity of \( u \) )

In this case :

\[
Re\left(\int_0^{\infty} \frac{t^{z-1}}{e^t - 1} dt\right) = \int_{u}^{u+1} f(u) du + \sum_{k \in \mathbb{N}^*} \int_{u_k}^{u_k+1} f(u) du \leq 0
\]

Similarly :

If \( \int_{u}^{u+1} f(u) du \leq 0 \) :

So :

- Either \( f(u) \leq 0 \),
In this case: 
- \(-\Re \left( \int_{0}^{\infty} t^{s-1}e^{-t}dt \right) = \int_{-\infty}^{\infty} f(u)du + \sum_{k=2N} \int_{u+1}^{u+1} f(u)du \leq 0\)

- Or either \(f'(u) \leq 0\). 

In this case: 
- \(-\Re \left( \int_{0}^{\infty} t^{s-1}e^{-t}dt \right) = \int_{-\infty}^{\infty} f(u)du + \int_{u+1} f(u)du + \sum_{k=2N} \int_{u+1}^{u+1} f(u)du \leq 0\)

**Proof of the theorem**

We know ([3,4]) that:

\[ \zeta(z) \Gamma(z) = \int_{0}^{\infty} \frac{t^{z-1}}{e^t - 1} dt \]

So:

\[ \zeta(z) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} \frac{t^{z-1}}{e^t - 1} dt \]

If \(\zeta(s) = 0\) with \(s\) a non trivial zero of \(\zeta\), then, by symmetry of the zeros about the critical line \(\Re(z) = \frac{1}{2}\), we assume that \(s = \frac{1}{2} - \alpha + i\beta\), with \(0 \leq \alpha \leq \frac{1}{2}\) (because it is known that any non-trivial zero belongs to the critical strip: \(\{ s \in \mathbb{C} : 0 < \Re(s) < 1 \}\)).

By tending \(z\) towards \(s\) and by using the **lemma** 1, we will have: \(|\Gamma\left(\frac{1}{2} - \alpha + i\beta\right)| = +\infty\)

As \(\Gamma(z+1) = z\Gamma(z)\), then \(|\Gamma\left(-\frac{1}{2} - \alpha + i\beta\right)| = +\infty\)

And consequently: \(|\Gamma\left(-\frac{1}{2} - \alpha\right)| = +\infty\)

The gamma function also checks the Legendre duplication formula [3]: \(\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \sqrt{\pi} \Gamma(2z)\).

By setting \(z = -\frac{1}{2} - \alpha\), we deduce that: \(|\Gamma\left(-1 - 2\alpha\right)| = +\infty\).

The study of **Gamma** - See Figure 1 - Shows that the only possible case is \(-1 - 2\alpha = -1\), so \(\alpha = 0\).

**Theorem 2**

The sghiar’s function and the prime numbers:

Let \(\mathcal{S}(z) = \zeta\left(-\frac{\Gamma(z)+1}{2\pi}\right)\).

If \(z \in \mathbb{N}^*\) then \(\mathcal{S}(z) = 0 \iff z\) is a prime number.

**Proof**

It follows from Wilson’s theorem [1] - which assures that \(p\) is a prime number if and only if \((p-1)! \equiv -1 \mod p\) - and the fact that the trivial zeros of \(\zeta\) are \(-2\mathbb{N}\).

**III- Conclusion:**

The Gamma function \(\Gamma\) and the Mertens function \(M\) are closely linked to the Riemann zeta function \(\zeta\).

What is curious is that by the same techniques the Mertens function allowed the proof of the Riemann hypothesis in [2], and the gamma function allowed also in this article a simple, short and elegant proof of the Riemann hypothesis.

**IV- References**