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The Universe in a Double-Well Potential
Conclusion

\(\psi_{ij}\) and \(\psi'_{ij}\) are on one side the ground state and its derivative of a quantum mechanical system, which describes quantum fluctuations about the trajectory of a three-dimensional metric in a double well potential. On the other side they represent velocity and acceleration associated with the trajectory of space. Just by knowing the trajectory \(h_{ij}(t)\) and its quantum fluctuations we could have recovered general relativity, because we connected the Ricci-Tensor and its trace to \(h_{ij}\) in a very specific way. Motivated by conclusions on the following pages the metric seems to cover the observed and theorised states of cosmic expansion very well. It contains the inflation period for small \(t\) and a Friedmann-like behaviour for large \(t\). The assumptions generating the conclusions are reasonable, so one could argue, that the field equations of general relativity, speaking the equations describing the development of spacetime, are strongly connected to the quantum effects concerning the trajectory. The fluctuations have a maximum at \(t = 0\) and decrease exponentially. The change of the fluctuation-value with respect to time and the acceleration and deceleration of expansion seem to be two sides of the same coin. Maybe general relativity is a theory of cosmological quantum mechanics. In the constructed system GR is equal to a nonlinear evolution equation of a quantum state. Unclear remains in which way the solutions should be interpreted for \(t < 0\) as well as the physical relevance of that domain. In fact, spacetime can be viewed as a tunnelling process between two Euclidean metrics, causing itself to expand and contract probably infinitely in the process.
The Modell

We’d like to describe the evolution of the universe as a trajectory of a three-dimensional Riemannian-Manifold \((\Sigma, h)\) with positive, constant scalar curvature equipped with metric tensor \(h_{ij}(x^k)\) in a special kind of double-well potential. We assume a four-dimensional Lorentzian-Manifold \((M, g)\) is foliated by the family \(\Sigma_t\) in such a way, that the basis vector \(e_t\) is perpendicular to \(\{e_i\}_{i=1}^3\), where \(e_t\) forms a basis on the hypersurface \(\Sigma_t\).

\[
N_i = e_t \cdot e_t = 0
\]

\[
g_{\mu\nu} = \begin{pmatrix} g_{tt} & N_i \\ N_i & h_{ij} \end{pmatrix},
\]

where \(N_i\) is called the Shift. \(\Sigma_t\) is the image of the embedding

\[
\mathcal{E}_t : \Sigma \to M
\]

so that \((\Sigma, h_t)\) is isomorphic to \(\Sigma_t\). One should be reminded of the absence of any implication, relating the parameter \(t\) to clock readings so far. Such relation can only be made, if we view spacetime as a one-parameter group of hypersurfaces \((\Sigma, h_t)\). The dynamics of the system are governed by the action in Planck-Units

\[
S[h_{ij}] = \int_M d^4x \sqrt{-g} \left( R - 2\Lambda \right) + \int_{\partial M} d^3x \sqrt{h} K + \int dt \frac{1}{2} \left( \partial_t h_{ij} \right)^2 + R_{ij}(h_{ij})
\]

where \(R_{ij}\) is the Ricci-Tensor, \(\partial_t h_{ij} = 2K_{ij}\) the second fundamental form of \(\Sigma_t\) and \(K\) its trace. The equations of motion

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = T_{\mu\nu}
\]

\[
\frac{\partial h_{ij}}{\partial t} = \sqrt{2R_{ij}(h_{ij})}
\]

are coupled via (1) and
\[ R_{ij} = R_0 (h_{ij}^2 - \delta_{ij})^2 \]  

(5)

with \( R_0 \) being a real constant, which can be interpreted as the maximal scalar curvature of \( h_{ij} \) and with \( \delta_{ij} \), the euclidean metric. Clearly the minima of the potential are located at \( \pm \delta_{ij} \) and they are asymptotically approached by the nontrivial solutions \( h_{ij}(x^k, t - t_0) \). More specifically

\[ h_{ij}(x^k, t - t_0) = \mp \delta_{ij} \tanh(\sqrt{R_0}(t - t_0)) \]  

(6)

which can be calculated using the Bogomolny-Equation (4). Ref. [1], [2] and [3]. In the following calculations we restrict to the positive sign and choose \( t_0 = 0 \). The solutions (6) generate a spacial line element

\[ ds^2 = \tanh^2(\sqrt{R_0}t) d\Sigma^2 \]  

(7)

with scaling factor \( a(t) = \tanh(\sqrt{R_0}t) \). Following from the assumptions made for the Ricci-Tensor, it is now possible to evaluate the dependence of \( R_{ij} \) on the parameter \( t \).

\[ R_{ij}(t) = R_0 (\delta_{ij}^2 \tanh(\sqrt{R_0}t) - \delta_{ij})^2 = R_0 \delta_{ij} \text{sech}^4(\sqrt{R_0}t) \]  

(8)

Taking the trace of \( R_{ij}(t) \) gives the \( t \)-dependent scalar curvature \( R(t) \)

\[ R(t) = trR_{ij}(t) = 3R_0 \text{sech}^4(\sqrt{R_0}t). \]  

(9)

The field equations of general relativity for \( g_{\mu\nu} \) with the Stress-Energy-Tensor \( T_{\mu\nu} \) of a perfect fluid

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = T_{\mu\nu} \]

are then reduced to

\[ R_0 \text{sech}^4(\sqrt{R_0}t) \left[ 1 - \frac{3}{2} \tanh(\sqrt{R_0}t) \right] + \Lambda \tanh(\sqrt{R_0}t) = p(t)_r + p(t)_m - p(t)_\Lambda \]  

(10)

and
\[
\rho = -p(t)A + ap(t)r + bp(t)m.
\] (11)

With \(a, b \in \mathbb{R}\) and \(\rho\) the energy density. For the cosmological constant \(\Lambda\) it follows with \(\tanh(t) \neq 0\), that

\[
\Lambda(t) = \frac{1}{2} \left( \cosh(\sqrt{R_0} t) \rho(t) - 2R_0 \cosh(\sqrt{R_0} t) \sech^3(\sqrt{R_0} t) + 3R_0 \sech^4(\sqrt{R_0} t) \right).
\] (12)

This is a rather remarkable function with interesting properties. For \(t \to 0\) the function approaches negative infinity and for small values of \(t\) it has a sharp positive maximum, whose location with respect to the \(t\)-axis and value is determined by \(R_0\). The width of the peak is proportional to \((4R_0)^{-1/2}\). For larger \(t\) the cosmological constant depends on the behaviour of the first term, which is directly determined by the values of \(\rho\). With regard to the fact, that \(R_0\) is the curvature of a positively curved Riemannian-Manifold with a volume of the magnitude of the Plank-Volume, the peak of \(\Lambda(t)\) has an approximated width of \(\Delta t = 10^{-35}\) and thus can be associated with the period of cosmic inflation. The height of the peak is in the vicinity of \(R_0\) and the maximum is followed by a rapid exponential decrease to very small values close to \(\rho(t)\).

\[\text{Figure 1: Plot of } \Lambda(t) \text{ with } R_0 = 10^{70} \text{ and } \rho(t) = 0.\]
Now we would like to evaluate the Hubble-Parameter. Writing equation (8) in terms of the scaling factor gives

\[
\frac{\dot{a}(t)^2}{a(t)} - \frac{3}{2}(\dot{a}(t))^2 + \Lambda = \frac{\rho(t)}{a(t)} \tag{13}
\]

\[
= a(t)H(t)^2 - \frac{3}{2}(\dot{a}(t))^2 + \Lambda = \frac{\rho(t)}{a(t)}
\]

\[
= H(t)^2 - \frac{3}{2} \frac{\dot{a}(t)^2}{a(t)} + \frac{\Lambda}{a(t)} = \frac{\rho(t)}{a(t)^2} \tag{14}
\]

It follows that

\[
H(t) = \sqrt{\frac{3(\ddot{a}(t))^2}{2a(t)} - \frac{\Lambda}{a(t)} + \frac{\rho(t)}{a(t)^2}}
\]

\[
= \sqrt{-\Lambda \coth(\sqrt{R_0} t) + \rho \coth^2(\sqrt{R_0} t) + \sqrt{R_0} \frac{3}{2} \frac{\text{sech}^4(\sqrt{R_0} t)}{\tanh(\sqrt{R_0} t)}} \tag{15}
\]

and for large \( t \) holds

\[
H(t) = \sqrt{\rho(t) - \Lambda} \approx \sqrt{\frac{\Lambda}{3}} \approx 10^{-61}, \tag{16}
\]

which is in the vicinity of the measured value. It should be emphasised, that although quantum corrections are missing in the previous calculations, quantum fluctuations about

\[
h_{ij}(x^k, t) = \delta_{ij} \tanh(\sqrt{R_0} t) \tag{17}
\]

are described by the fluctuation operator.
where $S_E$ is the euclidean action.

Something interesting follows: By virtue of the nature of the constructed system, general relativity emerges from quantum mechanics. More precisely: The system creates its own general relativity. That is because the freedom of choosing $t_0$ causes $h_{ij}(t - t_0)$ to form an equipotential curve in field configuration space. For every $t_0$ the action is minimized, so that along the curve the fluctuation operator vanishes, concluding that the ground state $\psi_{ij}$ of the quantum system is the derivative of $h_{ij}$ with respect to $t$ up to a normalization constant.

$$\psi_{ij} = \left. \frac{\partial h_{ij}}{\partial t} \right|_{t_0} = \sqrt{R_0} \delta_{ij} \text{sech}^2(\sqrt{R_0} t)$$

Additionally, for the family $h_{ij}(t - t_0)$ there exists a period of inflation for every $t_0$. If one would assume the initial scalar curvature $R_0$ to be function of $t$, then there would be infinitely many different inflations for $h_{ij}(t - t_0)$. Now we are able, to make the following observations:

$$\psi_{ij} = R_{ij}$$

$$\frac{3}{8\sqrt{R_0}} (\psi_{ij}^2)' = -\frac{1}{2} h_{ij} R$$

$$\Lambda \int \psi_{ij} dt = \Lambda h_{ij}.$$ 

Einstein’s field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}$$

take the following form

$$\psi_{ij}^2 + \frac{3}{8\sqrt{R_0}} (\psi_{ij}^2)' + \Lambda \int \psi_{ij} dt = \rho(t) \delta_{ij}$$
and

$$\rho = -p(t)\Lambda + ap(t)_r + bp(t)_m$$  \hspace{1cm} (23)$$

For this system General Relativity can be represented in terms of the zero mode of the quantum system describing fluctuations about the trajectory of $h_{ij}$. This comes very naturally because $\psi_{ij}^2$ is the energy density of the kink and a strong correlation between energy density and curvature should be expected. Rewriting (22) gives

$$\rho(t) + \frac{3}{8\sqrt{R_0}}\rho(t)' + \Lambda \int \sqrt{\rho(t)} \, dt = \rho(t)$$

$$\frac{3}{8\sqrt{R_0}}\rho(t)' + \Lambda \int \sqrt{\rho(t)} \, dt = 0$$

Integration of $R_{ij}$ over $t$ yields the one-kink action.

$$S_{ij} = \int_{-\infty}^{\infty} R_{ij} \, dt = \int_{-\infty}^{\infty} \psi_{ij}^2 \, dt = \int_{-\infty}^{\infty} \left( \frac{\partial h_{ij}}{\partial t} \right)^2 \, dt = \int_{-\infty}^{\infty} R_0 \delta_{ij} \text{sech}^4(\sqrt{R_0}t) \, dt = \frac{4\sqrt{R_0}}{3} \delta_{ij}$$  \hspace{1cm} (24)$$

(22) can be expressed also in terms of extrinsic curvature $K_{ij}$.

$$4K_{ij}^2 + \frac{3}{2\sqrt{R_0}}(K_{ij}^2)' + 2\Lambda \int K_{ij} \, dt = \rho(t)$$  \hspace{1cm} (25)$$

$$\rho = -p(t)\Lambda + ap(t)_r + bp(t)_m$$

The connection between the fields $K_{ij}, \psi_{ij}$ and $h_{ij}$ is

$$2K_{ij} = \psi_{ij} = \partial_t h_{ij}.$$  \hspace{1cm} (26)$$
Tunneling-Amplitude

In the path-integral approach to quantum gravity it is proposed that one only needs to specify the metric on the boundary $\Sigma_t$.

$$Z(h_f, h_i, t) = \langle h_f | e^{-iHt} | h_i \rangle = N \int D[h_{ij}] e^{iS[h_{ij}]}$$

(27)

The Gibbons-Hawking boundary term ensures the composition rule

$$\langle h_{n+2}, \Sigma_{n+2} | h_n, \Sigma_n \rangle \sum_{h_{n+1}} \langle h_{n+2}, \Sigma_{n+2} | h_{n+1}, \Sigma_{n+1} \rangle \langle h_{n+1}, \Sigma_{n+1} | h_n, \Sigma_n \rangle$$

(28)

to hold, so that the amplitude is obtained by summing over all states on the intermediate surface. In the presented model $h_{ij}$ can be viewed as an instanton, connecting to Euclidean manifolds which are the vacua of the double-well potential. The analytically continued generating functional includes the determinant of the fluctuation operator (19). For a detailed derivation we refer to [4]. The tunnelling solution satisfies the boundary conditions

$$h_{ij} \left( -\frac{\tau}{2} \right) = h_i \quad h_{ij} \left( \frac{\tau}{2} \right) = h_f$$

(30)

for $\tau \to \infty$ and the propagator in Euclidean time can be expressed as

$$Z(\delta_{ij}, -\delta_{ij}) = N e^{-S_E[h_{ij}]} \sqrt{\frac{S_E[h_{ij}]}{2\pi}} \tau (det' F[h_{ij}])^{-\frac{1}{2}}$$

(31)

where the zero mode is of course dropped in the calculation of the determinant. Evaluating the components of the expression leads to the tunnelling propagator up to $O(\hbar)$ in semiclassical approximation for large $\tau$.

$$Z(\delta_{ij}, -\delta_{ij}) = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{\pi} \tau} \sqrt{\frac{6S_E[h_{ij}]}{\pi}} e^{-S_E[h_{ij}]}$$

(32)

The derivation includes only saddle points of a single instanton, but we need to cover all additional saddle points, because they contribute to the amplitude for large $\tau$. 
The fluctuations about an ordered superposition of instantons are of the form

\[ \psi(\tau) = \psi_0(\tau) + \sum_{n=1}^{N} \psi_k(\tau) \]  

(33)

where \( \psi_0(\tau) \) represents fluctuations about the trivial solutions \( h_{ij} = \pm \delta_{ij} \). For the multi-instanton propagator, we have to take all possible values of the position \( \tau_0 \) into account by integrating the contribution of a single instanton to the propagator (32) over all values for \( N \) well separated (anti-) instantons, where \( \tau_{0,n-1} < \tau_{0,n} < \frac{\tau}{2} \).

\[ Z_I = \left[ \frac{6S_E[h_{ij}]}{\pi} e^{-S_E[h_{ij}]} \int_{\tau}^{\tau} d\tau_{0,1} \times \prod_{n=1}^{N} \int_{\tau_{0,n}}^{\tau} d\tau_{0,n+1} \right] \]  

(34)

This yields

\[ Z_N \equiv Z_0 \left( \frac{Z_I}{N!} \right)^N \]  

(35)

\( Z_0 \) is the determinant of the fluctuations about the true vacua. Furthermore

\[ Z_0(\pm \delta_{ij}, \pm \delta_{ij}) = N (\det \hat{F}_0[h_{ij}])^{1/2} = \frac{2\sqrt{R_0}}{\pi} e^{-\sqrt{R_0} \tau} \]  

(36)

and

\[ Z_I' = 2\sqrt{R_0} \left[ \frac{6S_E[h_{ij}]}{\pi} e^{-S_E[h_{ij}]} = 4 \left[ \frac{2R_0^{3/2} \delta_{ij}}{\pi} e^{-\sqrt{R_0} \delta_{ij}} \right] \right] \]  

(37)

The final expression is obtained by summing over all even \( N \). This is required due to the fact, that only for even \( N \) the instantons satisfy the boundary conditions (30).
\[ Z(\delta_{ij}, -\delta_{ij}) = Z_0 \sum_{N \text{ odd}} \left( \frac{Z_{i'}^N}{N!} \right) \]  

(39)

\[ = \sqrt{2\sqrt{R_0}} e^{-\sqrt{R_0} \tau} \left( e^{\tau z_{i'}} + e^{-\tau z_{i'}} \right) \]  

(40)

\[ = \sqrt{\frac{2\sqrt{R_0}}{\pi}} e^{-\sqrt{R_0} \tau} \cosh \left( 4\tau \frac{2R_0^2 \delta_{ij}}{\pi} e^{-\frac{4\sqrt{R_0} \tau}{3}} \right) \]  

(41)

This is the tunnelling-propagator. It can be understood as a propagator of the system

\[ \mathcal{H} \Psi = 0 \]  

(42)

which is called the Wheeler-DeWitt equation. Ref [5] [6, Chapter 1.2]. Classically \( h_{ij}(t - t_0) \) is a nontrivial family of trajectories in a double-well potential (5), representing contracting and expanding spacetimes with constant scalar curvature as well as inflationary periods. From a quantum point of view these solutions describe tunnelling processes between two Euclidean metrics and their quantum fluctuations through a potential well with height \( R_0 \). A universe delineated by this system is probably a cyclic one, tunnelling back and forth between the two, flat vacuum states.
Bibliography