Zone Theorem for Arrangements in three dimensions

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Abstract

In this note, a simple description of the zone theorem in three dimensions is given. Arrangements in three dimensions are useful for constructing higher-order Voronoi diagrams in plane. An elementary and very intuitive treatment of this result is also given.

Keywords: Computational Geometry, Zone Theorem, Arrangements, k-Nearest Neighbours

1 Introduction

Zone theorem is important in analysing incremental algorithms for constructing arrangements. Most popular text books of Computational Geometry (see e.g. [1, 8]) describe zone theorem for Arrangements in two dimensions. Specialised books like [2, 6] describe zone theorem for hyperplanes in d-dimensions. Three dimensional arrangements are useful for constructing higher order Voronoi diagrams in plane [2, 6, 7, 8]. An elementary and very intuitive treatment of this is given in Section 4. Proofs of Zone theorem in higher dimensions use Euler's relation: $\sum_{i=0}^{d}(-1)^i F_i \geq 0$ [4, 5, 6]. As most students of Computational Geometry are not familiar with these result, only zone theorem in two dimensions is taught in most Computational Geometry courses. In this note, a proof of zone theorem in three dimensions which can be easily taught in Computational Geometry courses is described. The proof uses zone theorem in two dimensions [1, 4, 6, 8, 9]. The proof is a essentially a simplified version of proof given by Edelsbrunner, Seidel and Sharir[4].

2 Definitions and basic properties

Arrangement in two dimensions is basically a set of n lines (infinite lines and not segments), and in three dimensions of n planes. We will assume that the lines in two dimensions and the planes in three dimensions are in general position. Thus, in two dimension no two lines are parallel and no three lines meet in a single point [1, 4, 8]. Similarly, in three dimensions we will assume that

- No two planes are parallel. Thus, each pair of planes meet in a line. And any three planes in a point.
- No three planes intersect in a common line and no four planes in a (common) point.

Set of lines, in two dimensions, will partition the plane into regions called faces. And set of planes in three dimensions will partition the space into regions, which we will call cells.

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Let C be any bounded cell. As planes are in general position, each point or vertex v in the arrangement is determined by three planes. Thus, there will be three edges of C incident at any vertex v of C. Moreover, each edge of C is determined by two vertices of C. If $|V_C|$ is the number of vertices of C and $|E_C|$ is the number of edges of C, then $2|E_C| = 3|V_C|$.

As C is on one side of each plane, C will be a convex polytope (3-dimensional analogue of polygon). The set of edges, vertices and faces on boundary of C will form a planar graph (take a point O inside C and draw a sphere with centre as O enclosing the polytope, if x is a point on C, then line Ox will intersect sphere at some point x', point x is mapped to x'). In a planar graph, if $|V|$ is the number of vertices, $|E|$ the number of edges and $|F|$ the number of faces, then by Euler's formula $|E| - |V| + 2 = |F|$. If $|F_C|$ is the number of faces of C then $|F_C| = |E_C| - |V_C| + 2 = |E_C| - \frac{2}{3}|E_C| + 2 = \frac{1}{3}|E_C| + 2$. Thus, $|E_C| < 3|F_C|$ and hence $|V_C| = \frac{2}{3}|E_C| < 2|F_C|$. Or, $|V_C| = O(|F_C|)$ and $|E_C| = O(|F_C|)$.

In two dimensions, let S be a line different from n given lines (also in general position). Then S will intersect (cut) some faces of the arrangement. Zone(S) is defined as the set of faces through which line S passes. If $C \in \text{zone}(S)$, is a face which is cut, then let $|C|$ be the number of edges on the boundary of face C in the (original) arrangement, then the size of zone(S), $|\text{zone}(S)| = \sum_{C \in \text{zone}(S)} |C|$. It is known that $|\text{zone}(S)| = O(n)$ (see [1, 4, 6, 8, 9]).

Similarly, in three dimensions, let S be a plane different from n given planes (also in general position). Then S will intersect (cut) some cells of the arrangement. Zone(S) is defined as the set of cells which the plane S intersects. If $C \in \text{zone}(S)$, is a cell which is cut, then let $|F_C|$ be the number of faces on the boundary of cell C in the (original) arrangement, then the size of zone(S), $|\text{zone}(S)| = \sum_{C \in \text{zone}(S)} |F_C|$.

Remark Normally the size of zone is defined as the sum of number of edges, vertices and faces of all cells in the zone, but as the number of vertices and edges in a cell are $O(|F_i|)$, the two definitions are equivalent up to multiplicative constants. Moreover, as each face is on boundary of two cells, the total number of cells in a zone will also be bounded by $O(\sum_C |F_C|)$.

3 Zone Theorem in 3-dimensions

We will assume that all planes of the arrangement (together with S) are in general position, as size of zone is not smaller in this case [2, 4, 6].

Further, let us enclose the arrangement in a "bounding box"[1, 6] by having six planes $x = \pm A, y = \pm A, z = \pm A$— basically we compute coordinates of all $\binom{n}{3}$ vertices of arrangement (by taking every possible set of three planes) and choosing A to be larger than the (absolute value of) largest coordinate. Thus, all cells inside the bounding box will be bounded.

Let Q be any plane of the arrangement. Then as all planes are in general position, each of them will intersect Q in a line. All these lines will lie in the plane Q and form a two-dimensional (planar) arrangement of lines (say $\mathcal{L}_Q$).

Let us remove plane Q from the arrangement $\mathcal{A}$ and let the resulting arrangement be called $\mathcal{A} - \mathcal{Q}$.

Let C be a cell in the arrangement $\mathcal{A} - \mathcal{Q}$. If the plane Q does not cut (intersect) cell C, then cell C and all its faces will be present (unchanged) in arrangement $\mathcal{A}$.

If the plane Q cuts (intersects) cell C, then the cell C gets divided into two parts— say the part of C above the plane Q and the part of C below Q (or if Q is horizontal then left and right of Q). Let us call the two parts as C_1 and C_2. Part of C intersected by Q will lie in plane Q (definition of intersection) and hence will be a face (say f_Q) in the two dimensional arrangement $\mathcal{L}_\mathcal{Q}$.

If face f (of C in $\mathcal{A} - \mathcal{Q}$ is not intersected by Q, then face f will be present (unchanged) in either C_1 or C_2.

If face f is intersected by Q, then f will get split into two parts one above Q and the other below Q (or one on left and the other on right). Let the part in C_1 be called f_1 and part in C_2 be called f_2. Let the boundary (part common to both) be called e_Q. As e_Q is (also) in plane Q, e_Q will be an edge in two dimensional arrangement $\mathcal{L}_\mathcal{Q}$. Edge e_Q is in face f_Q.

To prove the zone-theorem we need following intermediate result

Claim 1 *Assume $\mathcal{A}$ is an arrangement of n planes, Q is a plane in $\mathcal{A}$, and S is a plane not in $\mathcal{A}$. Let C be a cell in zone(S). Let f be a face of cell C not lying in plane Q. Then total number of such pairs (f, C) (of face f and cell C) is at most the sum of*

1. *size of zone(S) in arrangement $\mathcal{A} - \mathcal{Q}$ and*
2. *size of zone(S) in two dimensional arrangement $\mathcal{L}_\mathcal{Q}$*

Remark The first size is count of faces (along with their multiplicities) and second of edges (along with their multiplicities).

Proof: Assume that cell C is in zone(S) (of arrangement $\mathcal{A} - \mathcal{Q}$) and f is a face of C, not lying in (part of) plane Q. As C is in zone(S), plane S passes through cell C.

If Q does not cut cell C, then cell C (along with all its faces) will be unchanged in arrangement $\mathcal{A}$ (and as S passes through C), cell C will also be in zone(S) in arrangement $\mathcal{A}$. In this case $f' = f$ and $C' = C$. Or the same pair is present in both $\mathcal{A}$ and $\mathcal{A} - \mathcal{Q}$.

Let us assume that Q cuts C. Then cell C gets divided into two parts (say) C_1 and C_2. If the plane Q does not cut face f, then face f will remain intact in one part (say) C_i (for $i = 1$ or 2). If part C_i contains f, then there is one-to-one correspondence between the pair (f, C) and the pair (f, C_i) (i.e., pair (f, C) corresponds to pair (f, C_i) and conversely). Note that right hand side will be larger if C_i is not in the zone (see below).

Since S passes through C, it will pass through either C_1 or C_2 or both. If S passes through only one part (say) C_i (for $i = 1$ or 2), then only C_i will be in the zone(S) in arrangement $\mathcal{A}$. In this case, for the pair (f, C) we will have the corresponding pair (f, C_i) and conversely (or in case f is intersected by Q, then the pair (f_i, C_i) where f_i is the part of f in C_i).

We are left with the case when face f is also cut by Q and both C_1 and C_2 are in the zone(S).

If S passes through both C_1 and C_2, then both C_1 and C_2 will be in the zone(S) in arrangement $\mathcal{A}$. As S passes through both C_1 and C_2, it will also intersect the common boundary of C_1 and C_2. But as Q passes through the common boundary of C_1 and C_2, the common part will be a face (say f_Q) in the two dimensional arrangement $\mathcal{L}_\mathcal{Q}$. And as S intersects f_Q, face f_Q will be in the two dimensional zone($S \bigcap Q$).

As face f is also cut by Q, intersection of f and Q will be a line segment (say) e_Q. As e lies in plane Q, e_Q will be an edge in the two dimensional arrangement $\mathcal{L}_\mathcal{Q}$. Thus, for the two entries (f_1, C_1) and (f_2, C_2) on right hand side we have two entries: the pair (f, C) in three dimensional arrangement $\mathcal{A} - \mathcal{Q}$, and also have the pair (e_Q, f_Q) in the two dimensional arrangement $\mathcal{L}_\mathcal{Q}$. Thus, for the two entries (f_1, C_1) and (f_2, C_2) on the left hand side, we also have two entries (f, C) and (e_Q, f_Q) on the right hand side. ∎

As each face of a cell in $\mathcal{A}$ is in exactly one plane of the arrangement, it does not lie in remaining $n-1$ planes. Thus, if we take any pair (f, C) (for face f in cell C lying in zone(S)), it will not lie in $n-1$ planes. Or if we take each plane in turn as plane Q and add we get

$$(n-1)|\text{zone}(S)| \leq \sum_{Q \in \mathcal{A}} \left(|\text{zone}_{\mathcal{A}-\mathcal{Q}}(S)| + |\text{zone}_{\mathcal{L}_\mathcal{Q}}(S \bigcap Q)| \right)$$

To get the bounds, let $z(n)$ be the largest possible value of $|\text{zone}(S)|$ for arrangement of n-planes. Then, if we are considering this arrangement and this set S (for which the value of $|\text{zone}(S)|$ is the largest), then we have

$$(n-1)z(n) \leq \sum_{Q \in \mathcal{A}} \left(|\text{zone}_{\mathcal{A}-\mathcal{Q}}(S)| + |\text{zone}_{\mathcal{L}_\mathcal{Q}}(S \bigcap Q)| \right)$$

As $\mathcal{A} - \mathcal{Q}$ is an arrangement of $n-1$ planes (Q is excluded), $|\text{zone}_{\mathcal{A}-\mathcal{Q}}(S)| \leq z(n-1)$. Further, as two dimensional arrangement $\mathcal{L}_\mathcal{Q}$ is in plane Q and each line corresponds to one of the other plane, the number of lines in $\mathcal{L}_\mathcal{Q}$ is $n-1$. By the two dimensional zone theorem (see [1, 4, 6, 8, 9]), number of edges in zone will be linear. Hence, for some constant c, $|\text{zone}_{\mathcal{L}_\mathcal{Q}}(S \bigcap Q)| \leq c(n-1)$. Thus, our equation becomes

$$(n-1)z(n) \leq \sum_{Q \in \mathcal{A}} (z(n-1) + c(n-1)) = nz(n-1) + cn(n-1)$$

To solve this, we put $f(n) = z(n)/n$ or $z(n) = nf(n)$, the equation becomes

$$f(n) \leq f(n-1) + c$$

Or $f(n) = cn$, or $z(n) = nf(n) = cn^2$. Hence we get the Zone theorem:

Theorem 1 *Assume that we are given an arrangement $\mathcal{A}$ of n planes in three dimension. Let S be a plane different from the planes of the arrangement. Then size of zone(S),*

$$|zone(S)| = O(n^2)$$

4 k-Nearest Neighbours and Arrangements

Assume $S = \{(x_i, y_i\}_{i=1}^n$ is a set of n points in the plane. Assume $p = (h, k)$ and $q = (r, s)$ are two points of S.

A query (or test) point (x, y) will be closer to $p = (h, k)$ then to $q = (r, s)$ iff

$$\begin{array}{rcl} (x-h)^2 + (y-k)^2 & < & (x-r)^2 + (y-s)^2 \text{ or} \\ \cancel{x^2} - 2hx + h^2 + \cancel{y^2} - 2ky + k^2 & < & \cancel{x^2} - 2rx + r^2 + \cancel{y^2} - 2sy + s^2 \text{ or} \\ 2rx + 2sy - r^2 - s^2 & < & 2hx + 2ky - h^2 - k^2 \end{array}$$

Define $f(u, v) = 2ux + 2vy - u^2 - v^2$

Then above condition becomes, $f(h, k) > f(r, s)$.

As equation $z = f(u, v)$ is an equation of plane, the above condition is equivalent to saying that plane $z = 2hx + 2ky - h^2 - k^2$ is above the plane $z = 2rx + 2sy - r^2 - s^2$.

Hence, if we draw an arrangement of n-planes with plane $z = 2x_i x + 2y_i y - x_i^2 - y_i^2$ for the i^{th}point, then the nearest neighbour of query point (x, y) will be the topmost plane above (x, y) (the one visible from (x, y, ∞)). The second nearest neighbout will be the second top most plane and so on. Hence, to find k^{th}closest point, we need to only consider planes which have $(k-1)$ planes above them.

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