Pappus chain and division by zero calculus

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Abstract. We consider circles touching two of three circles forming arbeloi with division by zero and division by zero calculus.

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1. Introduction

Let $C$ be a point on the segment $AB$ such that $|BC| = 2a$ and $|CA| = 2b$ (see Figure 1). For an arbelos configuration formed by the three circles $\alpha$, $\beta$ and $\gamma$ with diameters $BC$, $CA$ and $AB$, respectively, we consider circles touching two of the three circles by the definition of division by zero [3]:

(1) \[ z \frac{z}{0} = 0 \text{ for any real number } z, \]

and division by zero calculus [21]. We use a rectangular coordinates system with origin $C$ such that the point $B$ has coordinates $(2a, 0)$. We call the line $AB$ the baseline.

2. Circles Touching Two of $\alpha$, $\beta$ and $\gamma$

If a circle touches one of given two circles internally and the other externally, we say that the circle touches the two circles in the opposite sense, otherwise in the same sense. Let $c = a + b$ and $d = \sqrt{ab}/c$.

Theorem 1. The following statements hold.

(i) A circle touches the circles $\beta$ and $\gamma$ in the opposite sense if and only if its has radius $r^\alpha_z$ and center of coordinates $(x^\alpha_z, y^\alpha_z)$ given by

\[ r^\alpha_z = \frac{abc}{a^2z^2 + bc} \text{ and } (x^\alpha_z, y^\alpha_z) = \left( -2b + \frac{b + c}{a} r^\alpha_z, 2zr^\alpha_z \right) \text{ for a real number } z. \]
Let the following statements hold.

(i) A circle touches the circles \( \alpha \) and \( \beta \) in the opposite sense if and only if its has radius \( r_z^\alpha \) and center of coordinates \( (x_z^\alpha, y_z^\alpha) \) given by

\[
    r_z^\alpha = \frac{abc}{b^2z^2 + ca} \quad \text{and} \quad (x_z^\alpha, y_z^\alpha) = \left(2a - \frac{c + a}{b}r_z^\beta, 2zr_z^\beta\right)
\]

for a real number \( z \).

(ii) A circle touches the circles \( \gamma \) and \( \alpha \) in the opposite sense if and only if its has radius \( r_z^\gamma \) and center of coordinates \( (x_z^\gamma, y_z^\gamma) \) given by

\[
    r_z^\gamma = |q_z^\gamma| \quad \text{and} \quad (x_z^\gamma, y_z^\gamma) = \left(\frac{b - a}{c}q_z^\gamma, 2zq_z^\gamma\right), \quad \text{where} \quad q_z^\gamma = \frac{abc}{c^2z^2 - ab}
\]

for a real number \( z \neq \pm d \).

**Proof.** Let \( \delta_z \) be the circle of radius and center described in (iii). Then we have

\[
    (x_z^\gamma - a)^2 + (y_z^\gamma)^2 = (a + q_z^\gamma)^2.
\]

Therefore \( \delta_z \) and \( \alpha \) touch internally or externally according as \( q_z^\gamma < 0 \) or \( q_z^\gamma > 0 \). Similarly \( \delta_z \) and \( \beta \) touch internally or externally according as \( q_z^\gamma < 0 \) or \( q_z^\gamma > 0 \). Hence \( \delta_z \) touches \( \alpha \) and \( \beta \) in the same sense. Conversely we assume that a circle \( \delta' \) of radius \( r \) touches \( \alpha \) and \( \beta \) in the same sense. Then there is a real numbers \( z \) such that \( r_{z^z} = r \). Therefore we have \( \delta' = \delta_z \) or \( \delta' = \delta_{-z} \). This proves (iii). The rest of the theorem can be proved similarly. \( \square \)

Essentially the same formulas as Theorem 1 can be found in [22], not so simple though. Simpler expression in the case \( z \) being an integer can be found in [4, 5] and cited in [1] and [12].

We denote the circle of radius \( r_z^\alpha \) and center of coordinates \( (x_z^\alpha, y_z^\alpha) \) by \( \alpha_z \). The circles \( \beta_z \) and \( \gamma_z \) are defined similarly. Notice that \( \alpha_{-1} = \beta_1 = \gamma_1 \) (resp. \( \alpha_{-1} = \beta_{-1} = \gamma_{-1} \)) is the incircle of the arbelos in the region \( y \geq 0 \) (resp. \( y \leq 0 \)).

The circle \( \gamma_z \) touches \( \alpha \) and \( \beta \) internally (resp. externally) if and only if \(|z| < d \) (resp. \(|z| > d \)). The external common tangents of \( \alpha \) and \( \beta \) have following equations [19, 20]:

\[
    (a - b)x \mp 2\sqrt{ab}y + 2ab = 0,
\]

which are denoted by \( \gamma_{\pm d} \).

**Corollary 1.** The following statements hold.

(i) The distance between the center of the circle \( \alpha_z \) and the baseline equals \( 2|z|r_{z}^\alpha \).
(ii) The distance between the center of the circle \( \beta_z \) and the baseline equals \( 2|z|r_{z}^\beta \).
(ii) The distance between the center of the circle \( \gamma_z \) and the baseline equals \( 2|z|r_{z}^\gamma \).

**Corollary 2.** The following statements hold.

(i) The ratio between the distance from the center of \( \alpha_z \) to the perpendicular to the baseline at \( A \) and the radius of \( \alpha_z \) is constant and equal to \((b + c)/a)r_{z}^\alpha \).
(ii) The ratio between the distance from the center of \( \beta_z \) to the perpendicular to the baseline at \( B \) and the radius of \( \beta_z \) is constant and equal to \((c + a)/b)r_{z}^\beta \).
(ii) The ratio between the distance from the center of \( \gamma_z \) to the perpendicular to the baseline at \( C \) and the radius of \( \gamma_z \) is constant and equal to \((a - b)/c)r_{z}^\gamma \) for \( z \neq \pm d \).
3. Division by zero

The circle \( \alpha_z \) has an equation \((x - x_z^\alpha)^2 + (y - y_z^\alpha)^2 = (r_z^\alpha)^2\), which is arranged as
\[
\alpha_z(x, y) = \frac{bc((x - a)^2 + y^2 - a^2) - 4abcy + a^2((x + 2b)^2 + y^2)z^2}{a^2z^2 + bc} = 0.
\]
Therefore we get \((x - a)^2 + y^2 = a^2, y = 0\) and \((x + 2b)^2 + y^2 = 0\) in the case \(z = 0\) from \(\alpha_z(x, y) = 0\), \(\alpha_z(x, y)/z = 0\), and \(\alpha_z(x, y)/z^2 = 0\), respectively by (1).

They represent the circle \(\alpha = \alpha_0\), the baseline and the point circle \(A\), respectively. We denote the point circle \(A\) and the baseline by \(\alpha_\infty\) and \(\alpha_{\infty}\), respectively, and consider that they also touch \(\alpha\) and \(\gamma\) (see Figure 2). Someone may consider that \(\alpha_\infty\) is orthogonal to \(\alpha\) and \(\gamma\) and does not touch them. But (1) implies \(\tan(\pi/2) = 0\). Therefore we can consider that \(\alpha_\infty\) still touches \(\alpha\) and \(\gamma\). We also consider that \(\alpha_\infty\) and \(\alpha_{\infty}\) touch.

![Figure 2](image-url)

We have \(\beta_0 = \beta\), and denote the point \(B\) and the baseline by \(\beta_\infty\) and \(\beta_{\infty}\), respectively. We also have \(\gamma_0 = \gamma\), and denote the point \(C\) and the baseline by \(\gamma_\infty\) and \(\gamma_{\infty}\), respectively.

4. Pappus chain

Let \(r_A = ab/(a + b)\). Circles of radius \(r_A\) are said to be Archimedean.

**Theorem 2.** We assume that \(a \neq b\), and \(w\) and \(z\) are real numbers. The two circle of each of the three pairs \(\alpha_z, \alpha_w; \beta_z, \beta_w; \gamma_z, \gamma_w\) touch if and only if \(|w - z| = 1\).

**Proof.** If \(|w| > d\) and \(|z| > d\), we get \(r_w^\gamma = q_w^\gamma\) and \(r_z^\gamma = q_z^\gamma\), and
\[
(x_w^\gamma - x_z^\gamma)^2 + (y_w^\gamma - y_z^\gamma)^2 - (r_w^\gamma + r_z^\gamma)^2 = \frac{4a^2b^2c^2((w - z)^2 - 1)}{(c^2w^2 - ab)(c^2z^2 - ab)}.
\]
Hence the theorem holds. If \(|w| < d\) and \(|z| > d\), we get \(r_w^\gamma = -q_w^\gamma\) and \(r_z^\gamma = q_z^\gamma\). Then \(\gamma_w\) and \(\gamma_z\) touch if and only if \(\gamma_z\) touches \(\gamma_w\) from inside of \(\gamma_w\). While we have
\[
(x_w^\gamma - x_z^\gamma)^2 + (y_w^\gamma - y_z^\gamma)^2 - (r_w^\gamma - r_z^\gamma)^2 = \frac{4a^2b^2c^2((w - z)^2 - 1)}{(c^2w^2 - ab)(c^2z^2 - ab)}.
\]
Hence the theorem holds for $\gamma_w$ and $\gamma_z$. If $|w| < d$ and $|z| < d$, both $\gamma_w$ and $\gamma_z$ touch $\alpha$ and $\beta$ internally. Therefore they do not touch. If $w = d$, $z = d \pm 1$ and $a \neq b$, then $\gamma_w$ and $\gamma_z$ have only one point in common, whose coordinates equal

\begin{equation}
\begin{pmatrix}
-2ra\sqrt{a} \mp \sqrt{b} \\
\sqrt{a} \pm \sqrt{b}
\end{pmatrix}.
\end{equation}

Therefore they touch. Since the figure is symmetric in the baseline, $\gamma_{-d}$ and $\gamma_{-d+1}$ also touch. Now the theorem is proved for the circles $\gamma_w$ and $\gamma_z$. The rest of the theorem is proved in a similar way.

The theorem holds for the two pairs $\alpha_z$, $\alpha_w$; $\beta_z$, $\beta_w$ in the case $a = b$. The theorem shows that any Pappus chain, whose members touch $\beta$ and $\gamma$, is expressed by $\cdots, \alpha_{z-2}, \alpha_{z-1}, \alpha_z, \alpha_{z+1}, \alpha_{z+2}, \cdots$ for a real number $z$. Also it shows that Corollary 1 is a generalization of Pappus chain theorem.

One of the circles $\gamma_{d+1}$ and $\gamma_{d-1}$ is the incircle of the curvilinear triangle made by $\alpha$, $\beta$ and $\gamma_d$, and the other touches the three in the region $y < 0$. Hence $y_{d+1}^2$ and $y_{d-1}^2$ have different signs. While $y_{d-1}^2 > 0$ shows that $\gamma_{d+1}$ is the incircle of the curvilinear triangle. Therefore the center of $\gamma_{d-1}$ lies in the region $y < 0$ (see Figure 3).

**Corollary 3.** If $a \neq b$ and $z = d$ or $z = -d$, then the two smallest circles passing through the point of tangency of $\gamma_z$ and $\gamma_{z\pm 1}$ and touching the baseline are Archimedean.

![Figure 3: The two circles in red are Archimedean.](image)

5. **Division by zero calculus**

If $f(z) = \cdots + C_{-2}(z-a)^{-2} + C_{-1}(z-a)^{-1} + C_0 + C_1(z-a) + C_2(z-a)^2 + \cdots$ is the Laurent expansion of a function $f(z)$ around $z = a$, the definition $f(a) = C_0$ is called the division by zero calculus [16], [21].

Let $g_z(x, y) = (x - x_z^0)^2 + (y - y_z^0)^2 - (r_z^0)^2$. Then $g_z(x, y) = 0$ is an equation of the circle $\gamma_z$ for $z \neq \pm d$. Let

$$g_z(x, y) = \cdots + C_{-2}(z-d)^{-2} + C_{-1}(z-d)^{-1} + C_0 + C_1(z-d) + \cdots$$

be the Laurent expansion of $g_z(x, y)$ around $z = d$, then we have

$$\cdots = C_{-4} = C_{-3} = C_{-2} = 0,$$
\[ C_{-1} = d((a - b)x - 2\sqrt{ab}y + 2ab), \]
\[ C_0 = \left( x - \frac{a - b}{4} \right)^2 + \left( y - \frac{\sqrt{ab}}{2} \right)^2 - \left( \frac{\sqrt{a^2 + 18ab + b^2}}{4} \right)^2, \]
\[ C_n = -\frac{1}{2} \left( \frac{-1}{2d} \right)^n ((a - b)x + 2\sqrt{ab}y + 2ab), \text{ for } n = 1, 2, 3, \ldots. \]

Therefore \( C_{-1} = 0 \) gives an equation of the line \( \gamma_d \), but \( C_0 = 0 \) does not. Also \( C_n = 0 \) gives an equation of the line \( \gamma_{-d} \) for \( n = 1, 2, 3, \ldots \).

Let \( \varepsilon \) be the circle given by the equation \( C_0 = 0 \). We have considered this circle in [19], which has the following properties (see Figure 4):

(i) The points, where \( \gamma_{d} \) touches \( \alpha \) and \( \beta \), lie on \( \varepsilon \).

(ii) The radical center of the three circles \( \alpha \), \( \beta \) and \( \varepsilon \) has coordinates \((0, -\sqrt{ab})\), and lies on the line \( \gamma_{-d} \).

We would like to state one more here:

(iii) The radical axis of the circles \( \varepsilon \) and \( \gamma \) passes though the points of coordinates \((0, 3\sqrt{ab})\) and \((2ab/(b - a), 0)\), where the latter coincides with the point of intersection of \( \gamma_{d} \) and \( \gamma_{-d} \).

The \( y \)-axis meets \( \gamma \) and \( \gamma_{\pm d} \) in the points of coordinates \((0, \pm 2\sqrt{ab})\) and \((0, \pm \sqrt{ab})\), respectively. Hence the six points, where the \( y \)-axis meets \( \gamma, \gamma_{\pm d} \), the baseline, the radical axis of \( \gamma \) and \( \varepsilon \), are evenly spaced. Those points are denoted in magenta in Figure 4. Reflecting the figure in the baseline, we also get similar results for the Laurent expansion of \( g_z(x, y) \) around \( z = -d \).

Figure 4: The green line denotes the radical axis of the circles \( \gamma \) and \( \varepsilon \).

For more applications of division by zero and division by zero calculus to circle geometry, see [2], [6], [7, 8, 9, 10, 11, 12, 13, 14, 15] [16, 17, 18, 19], and for an extensive reference see [21].
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