Entropy Calculation using brick wall method in Kerr-Newman AdS Black Hole

Chandra Prakash
chandra.prakash7295@gmail.com

ABSTRACT

Using the brick wall method, we will calculate the entropy of Kerr-Newman AdS black Hole and arrive at the already well established result. During the calculation, we will be using the generalized equation for F_{NSR} and F_{SR} mode of free energy.

Introduction

Ever since Bekenstein and Hawking derived the expression for black hole entropy, the question of what it means has always been our priority. For the first time, it was t'Hooft who in his paper titled "On the quantum structure of a black hole", first tried to evaluate the entropy of free particles using quantum field theory in curved spacetime and statistical physics. Though the method has some limitations, but what we will consider here is a generalized method which will apply for any black hole. In the first section, we will consider the classic klein gordan equation and then after that calculate the free energy of excitations in quantum field. This free energy will then be used to calculate the total entropy of the thermal gas of quantum field excitations outside the event horizon in a thin film.

=

Klein-Gordon Equation in Kerr-Newman in AdS/CFT

The line element for the Kerr-Newman in AdS/CFT is given by

$$ds^2 = -\left(\frac{-a^2\sin^2\theta\Delta_\theta + \Delta_r}{\rho^2\Xi^2}\right)dt^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Sigma\sin^2\theta}{\rho^2\Xi^2}d\phi^2 - \frac{2a\sin^2\theta\{\Delta_r + (r^2+a^2)\Delta_\theta\}}{\rho^2\Xi^2}dtd\phi$$

where,

$$\Delta_r = (a^2+r^2)\left(1-\frac{r^2}{l^2}\right) + Q^2 - 2GMr - \alpha r^{1-3\omega}$$

$$\Delta_\theta = 1 + \frac{a^2\cos^2\theta}{l^2}$$

$$\Xi = 1 + \frac{a^2}{l^2}$$

$$\Sigma = \Delta_\theta(r^2+a^2)^2 - a^2\sin^2\theta\Delta_r$$

$$\rho^2 = r^2 + a^2\cos^2\theta = r^2 + a^2 - a^2\sin^2\theta$$

$$l^2 = \frac{3}{\Lambda}$$

The inverse metric of above line integral can be given via:

$$g^{\mu\nu} = \begin{vmatrix} \frac{g_{\phi\phi}}{\mathcal{D}} & 0 & 0 & \frac{g_{t\phi}}{-\mathcal{D}} \\ 0 & \frac{1}{g_{rr}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{\theta\theta}} & 0 \\ \frac{g_{t\phi}}{-\mathcal{D}} & 0 & 0 & \frac{g_{tt}}{\mathcal{D}} \end{vmatrix}$$

The Kerr-Newman has four coordinate singularity given by:

$$\Delta_r = (r - r_H)(r - r_-)(r - r_q)(r - r_c) = 0$$

where, r_H is the event horizon, r_- is the cauchy horizon and r_q and r_c are the cosmological horizon.

Substituting the KNAdS inverse-metric into the **Klein Gordan Equation** of massless scalar field .

$$\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right)\psi = 0$$

Expanding it and solving we get:

$$\frac{\partial}{\partial t}\left(\sqrt{-g}g^{tt}\frac{\partial}{\partial t}\right)\psi + \frac{\partial}{\partial r}\left(\sqrt{-g}g^{rr}\frac{\partial}{\partial r}\right)\psi + \frac{\partial}{\partial \phi}\left(\sqrt{-g}g^{\phi\phi}\frac{\partial}{\partial \phi}\right)\psi$$

$$+ \frac{\partial}{\partial \theta}\left(\sqrt{-g}g^{\theta\theta}\frac{\partial}{\partial \theta}\right)\psi + \frac{\partial}{\partial t}\left(\sqrt{-g}g^{t\phi}\frac{\partial}{\partial \phi}\right)\psi + \frac{\partial}{\partial \phi}\left(\sqrt{-g}g^{\phi t}\frac{\partial}{\partial t}\right)\psi = 0$$

Using **WKB approximation** i.e. $\psi(t, r, \theta, \psi) = e^{iR(r)}e^{iS(\theta)}e^{im\phi}e^{-iEt}$:

$$\sqrt{-g}g^{tt}i^2E^2e^{-iEt}e^{iR(r)}e^{iS(\theta)}e^{im\phi} + i\frac{\partial}{\partial r}(\sqrt{-g}g^{rr})\frac{\partial R(r)}{\partial r}e^{iR(r)}e^{iS(\theta)}e^{im\phi}e^{-iEt} + \sqrt{-g}g^{rr}\frac{\partial^2}{\partial r^2}e^{iR(r)}e^{iS(\theta)}e^{im\phi}e^{-iEt}$$

$$+\sqrt{-g}g^{\phi\phi}\frac{\partial^2}{\partial \phi^2}e^{im\phi}e^{iS(\theta)}e^{-iEt} + i\frac{\partial}{\partial \theta}\left(\sqrt{-g}g^{\theta\theta}\right)\frac{\partial S(\theta)}{\partial \theta}e^{i\theta}e^{im\phi}e^{iR(r)}e^{-iEt} + \sqrt{-g}g^{\theta\theta}\frac{\partial^2}{\partial \theta^2}e^{i\theta}e^{im\phi}e^{iR(r)}e^{-iEt}$$

$$-2i^2mE\sqrt{-g}g^{t\phi}e^{im\phi}e^{i\theta}e^{iR(r)}e^{-iEt} = 0$$

Comparing real and imaginary parts of the equation, we arrive at:

$$\left(\frac{\partial R(r)}{\partial r}\right)^2 = \frac{1}{g^{rr}}\left[-g^{tt}E^2 - m^2g^{\phi\phi} - g^{\theta\theta}\left(\frac{\partial S(\theta)}{\partial \theta}\right)^2 + 2mEg^{t\phi}\right]$$

$$k_r^2 = \frac{1}{g^{rr}}\left[-g^{tt}E^2 - m^2g^{\phi\phi} - g^{\theta\theta}(k_\theta)^2 + 2mEg^{t\phi}\right]$$

(using standard notation)

Free Energy and Entropy

According to the theory of canonical ensemble and using semi-classical approximation, the free energy of scalar free particles within a shell of width $L-(r_H+\epsilon)$ in Kerr-Newman AdS background is:

$$\beta F = \sum_{E,m} \ln[1-e^{-\beta(E-m\Omega_H)}]$$

$$= \int dm \int_0^\infty d\Gamma(E)\ln[1-e^{-\beta(E-m\Omega_H)}]$$

$$= \int dm \left(\Gamma(E)\ln[1-e^{-\beta(E-m\Omega_H)}]_0^\infty - \int_0^\infty \Gamma(E)\frac{e^{-\beta(E-m\Omega_H)}}{1-e^{-\beta(E-m\Omega_H)}}\beta dE\right)$$

$$= -\beta \int dm \int_0^\infty \Gamma(E)\frac{1}{e^{\beta(E-m\Omega_H)}-1}dE$$

To proceed further we need to know the expression for $\Gamma(E) = \int g(E)dE$, which describes the total number of modes with energy less than E and a fixed m, assuming $\phi(r,\theta,\psi,t) \neq 0$ for $r_H+\epsilon \leq r \leq L$ and $\phi = 0$ outside this shell, is obtained by integrating over the volume of phase space:

$$\Gamma(E,m) = \int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{1}{\pi}\int dk_\theta k_r$$

$$= \int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{1}{\pi}\int dk_\theta \frac{1}{\sqrt{g^{rr}}}\left(-g^{tt}E^2 - m^2 g^{\phi\phi} - g^{\theta\theta}(k_\theta)^2 + 2mEg^{t\psi}\right)^{1/2}$$

Our integral becomes, with appropriate limit for $k_r^2 \geq 0$:

$$\Gamma(E) = \frac{\pi}{2}\int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}}\left(-g^{tt}E^2 - m^2 g^{\phi\phi} + 2mEg^{t\phi}\right)$$

Substituting it back to the original equation and simplifying that further, we arrive at:

$$F = -\frac{1}{2}\int dm \int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}}\int_0^\infty dE \frac{\left(-g^{tt}E^2 - m^2 g^{\phi\phi} + 2mEg^{t\phi}\right)}{e^{\beta(E-m\Omega_H)}-1}$$

A quick look at the integrand and we observe that this integral diverges at $E = m\Omega_H$, which is why we need to split it into two parts, one where $0 \leq E \leq m\Omega_H$ and other where, $\Omega_H < E < \infty$.

$$F = -\frac{1}{2}\int dm \int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}}\int_0^{m\Omega_H} dE \frac{\left(-g^{tt}E^2 - m^2 g^{\phi\phi} + 2mEg^{t\phi}\right)}{e^{\beta(E-m\Omega_H)}-1}$$

$$-\frac{1}{2}\int dm \int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}}\int_{m\Omega_H}^{\infty} dE \frac{\left(-g^{tt}E^2 - m^2 g^{\phi\phi} + 2mEg^{t\phi}\right)}{e^{\beta(E-m\Omega_H)}-1}$$

$$= F_{NSR} + F_{SR}$$

Evaluating F_{NSR}, we get this final expression:

$$F_{NSR} = -\frac{2\Gamma(4)\zeta(4)}{3\beta^4}\int d\theta d\phi \int_{r_H+\epsilon}^{L} dr \frac{(g_{rr}g_{\theta\theta})^{1/2}}{\{(-\mathcal{D})\}^{3/2}}(g_{\phi\phi})^2$$

$$\text{(where } -\mathcal{D} = g_{t\phi}^2 - g_{tt}g_{\phi\phi}\text{)}$$

This particular integral is quite complicated to solve which is why we wil taylor expand it at $r = r_H$ like this and neglect all other contributions:

$$\frac{(g_{rr}g_{\theta\theta})^{1/2}}{\{(-\mathcal{D})\}^{3/2}}(g_{\phi\phi})^2 = \frac{1}{(r-r_H)^2}F(r,\theta) = \frac{F(r_H,\theta)}{(r-r_H)^2} + \frac{F'(r_H,\theta)}{(r-r_H)} + \mathcal{O}(r-r_H)$$

where

$$F(r,\theta) = \frac{(g_{rr}g_{\theta\theta})^{1/2}}{\{(-\mathcal{D})\}^{3/2}}(g_{\phi\phi})^2(r-r_H)^2$$

Using the metric and performing simplification to the result we arrive at:

$$\frac{(g_{rr}g_{\theta\theta})^{1/2}}{(-\mathcal{D})^{3/2}}(g_{\phi\phi})^2 \approx f(r_H,\theta)\frac{1}{(r-r_H)^2}$$

where

$$f(r_H,\theta) = \frac{(r_H^2+a^2)}{4\ \Xi}\frac{4(r_H^2+a^2)^2}{[(r_H-r_-)(r_H-r_q)(r_H-r_c)]^2}\frac{[\Delta_\theta^2(r_H^2+a^2)](r_H^2+a^2\cos^2\theta)^2\Xi^3\sin\theta}{\Delta_\theta^2\left[(a^2+r_H^2)^2+2a^2\sin^2\theta(a^2+r_H^2)\right]^{3/2}}$$

Using this result our integral now becomes:

$$F_{NSR} \approx -\frac{\zeta(4)}{\beta^4}\int d\theta d\phi\ f(r_H,\theta)\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right] \qquad \text{(using } L = r_H+\epsilon+\delta\text{)}$$

$$F_{SR} \approx -\frac{\zeta(4)}{2\beta^4}\int d\theta d\phi f(r_H,\theta)\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right] \qquad \text{(using } L = r_H+\epsilon+\delta\text{)}$$

Using the above results we can easily evaluate the total free energy in the thin film as:

$$F = F_{NSR} + F_{SR}$$

$$= -\frac{3\zeta(4)}{2\beta^4}\int d\theta d\phi\ f(r_H,\theta)\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

The above expression for free energy can be simplified further by using surface gravity:

$$\kappa = \frac{2\pi}{\beta}$$

$$= \frac{1}{2(r_H^2+a^2)}(r_H-r_-)(r_H-r_q)(r_H-r_c) \qquad \text{(evaluated at } r = r_H\text{)}$$

and:

$$d\mathcal{A} = \sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta d\phi = \sqrt{\frac{\rho^2}{\Delta_\theta}\frac{\Sigma\sin^2\theta}{\rho^2\Xi^2}}d\theta d\psi$$

$$= \frac{(r_H^2+a^2)}{\Xi}\sin\theta d\theta d\psi$$

Using these two above results in our simplication, we arrive at:

$$F = -\frac{3\zeta(4)}{2\beta^4}\int d\theta d\phi \frac{(r_H^2+a^2)\sin\theta}{4\ \Xi}\frac{4(r_H^2+a^2)^2}{[(r_H-r_-)(r_H-r_q)(r_H-r_c)]^2}\frac{[\Delta_\theta^2(r_H^2+a^2)](r_H^2+a^2\cos^2\theta)^2\Xi^3}{\Delta_\theta^2\left[(a^2+r_H^2)^2+2a^2\sin^2\theta(a^2+r_H^2)\right]^{3/2}}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= -\frac{3\zeta(4)}{8\pi^2\beta^2}\int d\mathcal{A}\frac{(r_H^2+a^2)(r_H^2+a^2\cos^2\theta)^2\Xi^3}{4\left[(a^2+r_H^2)^2+2a^2\sin^2\theta(a^2+r_H^2)\right]^{3/2}}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

Then finally the entropy becomes:

$$S = \beta^2\frac{\partial F}{\partial\beta}$$

$$= -\beta^2\frac{\partial}{\partial\beta}\left[\frac{3\zeta(4)}{8\pi^2\beta^2}\int d\mathcal{A}\frac{(r_H^2+a^2)(r_H^2+a^2\cos^2\theta)^2\Xi^3}{4\left[(a^2+r_H^2)^2+2a^2\sin^2\theta(a^2+r_H^2)\right]^{3/2}}\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= \frac{3\zeta(4)}{4\pi^2\beta}\int d\mathcal{A}\frac{(r_H^2+a^2)(r_H^2+a^2\cos^2\theta)^2\Xi^3}{4\left[(a^2+r_H^2)^2+2a^2\sin^2\theta(a^2+r_H^2)\right]^{3/2}}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= \frac{\int d\mathcal{A}}{4} \qquad \left(\text{using } \frac{3\zeta(4)}{\pi^2\beta}\frac{(r_H^2+a^2)(r_H^2+a^2\cos^2\theta)^2\Xi^3}{\left[(a^2+r_H^2)^2+2a^2\sin^2\theta(a^2+r_H^2)\right]^{3/2}}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right] = 1\right)$$

Conclusions and Discussions

The result we finally got, tells us that the entropy of hawking particles in thin shell near the event horizon is actually proportional to area of event horizon! All the steps in the calculation remain same for any black hole.

Citations

[1] J. D. Bekenstein, Phys. Rev. D7, 2333 (1973); 9, 3292 (1974).

[2] S. W. Hawking, Nature (London) 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[3] J.-W. Ho, W. T. Kim, Y.-J. Park and H.-J. Shin, Class. Quantum Grav. 14, 2617

[4] Z. Xu, J. Wang, Kerr-Newman-AdS Black Hole In Quintessential Dark Energy (1997)

[5] M. H. Lee and J. K. Kim, Phys. Rev. D54, 3904 (1996)

[6] S. W. Kim, W. T. Kim, Y. -J. Park and H. Shin, Phys. Lett. B392, 311 (1997).

[7] C. Prakash; Generalization of brick wall method in Kerr-Newman Black Hole for entropy calculation

[8] X. H. Ge and Y. G. Shen, Class. Quant. Grav. **20**, 3593-3602 (2003)