Generalization of brick wall method in Kerr-Newman Black Hole for entropy calculation

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ABSTRACT

Using the brick wall method, we will calculate the entropy of Kerr-Newman black Hole and arrive at the already well established result. During the calculation, we will arrive at the generalized equation for F_{NSR} and F_{SR} mode of free energy, which can be used to evaluate the free energy of any black hole. And we will also show the detailed steps in our calculation to provide clarification about how we calculated it.

Introduction

Ever since bekenstein and hawking derived the expression for black hole entropy, the question of what it means has always been our priority. For the first time, it was t'Hooft who in his paper titled "On the quantum structure of a black hole", first tried to evaluate the entropy of free particles moving in the black hole background. Though the method may have some limitation, but here we will use the modified version of brick wall, i.e. we will consider the thin film. Our calculation will begin from first considering the klein-gordan equation for generalized metric and then we will calculate the free energy, which will be later used to evaluate **Entropy**.

Klein-Gordon Equation in Kerr-Newman

The line integral of kerr-newman black hole is given by this, for the rest of the calculation, we won't use it and in fact we will be working abstractly but towards the end the metric tensor mentioned here will be quite useful.

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\rho^2}\right)dt^2 + \rho^2 d\theta^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\Sigma\sin^2\theta}{\rho^2}d\psi^2 - \frac{2a\sin^2\theta(2GMr - Q^2)}{\rho^2}dtd\psi$$

where,

$$\begin{aligned}
\Delta &= a^2 + r^2 - 2GMr + Q^2 \\
\Sigma &= (r^2+a^2)^2 - a^2\Delta\sin^2\theta \\
\rho^2 &= r^2 + a^2\cos^2\theta = r^2 + a^2 - a^2\sin^2\theta \\
\sqrt{-g} &= \rho^2\sin\theta \\
\partial_\theta(\sqrt{-g}g^{\theta\theta}) &= \cos\theta \\
\partial_r(\sqrt{-g}g^{rr}) &= 2\sin\theta(r - GM)
\end{aligned}$$

And the respective inverse-metric in matrix form:

$$g^{ij} = \begin{vmatrix} \frac{g_{\psi\psi}}{g_{tt}g_{\psi\psi}-g_{t\psi}g_{t\psi}} & 0 & 0 & \frac{g_{t\psi}}{g_{t\psi}g_{t\psi}-g_{tt}g_{\psi\psi}} \\ 0 & \frac{1}{g_{rr}} & 0 & 0 \\ 0 & 0 & \frac{1}{g_{\theta\theta}} & 0 \\ \frac{g_{t\psi}}{g_{t\psi}g_{t\psi}-g_{tt}g_{\psi\psi}} & 0 & 0 & \frac{g_{tt}}{g_{tt}g_{\psi\psi}-g_{t\psi}g_{t\psi}} \end{vmatrix} = \begin{vmatrix} -\frac{\Sigma}{\rho^2\Delta} & 0 & 0 & -\frac{(2GMr-Q^2)a}{\rho^2\Delta} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ -\frac{(2GMr-Q^2)a}{\rho^2\Delta} & 0 & 0 & \left(\frac{\Delta - a^2\sin\theta}{\rho^2\Delta\sin\theta}\right) \end{vmatrix}$$

Using the above inverse metric for **Klein Gordan Equation** of massless scalar field.

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right)\phi = 0$$

Expanding it we get:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}\left(\sqrt{-g}g^{tt}\frac{\partial}{\partial t}\right)\phi + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\left(\sqrt{-g}g^{rr}\frac{\partial}{\partial r}\right)\phi + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial \psi}\left(\sqrt{-g}g^{\psi\psi}\frac{\partial}{\partial \psi}\right)\phi$$
$$+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial \theta}\left(\sqrt{-g}g^{\theta\theta}\frac{\partial}{\partial \theta}\right)\phi + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}\left(\sqrt{-g}g^{t\psi}\frac{\partial}{\partial \psi}\right)\phi + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial \psi}\left(\sqrt{-g}g^{\psi t}\frac{\partial}{\partial t}\right)\phi = 0$$

Since the metric doesn't depend on ψ and t, we can re-express it as:

$$\sqrt{-g}g^{tt}\frac{\partial^2}{\partial t^2}\phi + \frac{\partial}{\partial r}\left(\sqrt{-g}g^{rr}\frac{\partial}{\partial r}\right)\phi + \sqrt{-g}g^{\psi\psi}\frac{\partial^2}{\partial \psi^2}\phi + \frac{\partial}{\partial \theta}\left(\sqrt{-g}g^{\theta\theta}\frac{\partial}{\partial \theta}\right)\phi + 2\sqrt{-g}g^{t\psi}\frac{\partial^2}{\partial t\partial \psi}\phi = 0$$

Performing the separation of variable with WKB approximation $\phi(t,r,\theta,\psi) = e^{iR(r)}e^{iS(\theta)}e^{im\psi}e^{-iEt}$

$$\sqrt{-g}g^{tt}i^2E^2e^{-iEt}e^{iR(r)}e^{iS(\theta)}e^{im\psi} + i\frac{\partial}{\partial r}(\sqrt{-g}g^{rr})\frac{\partial R(r)}{\partial r}e^{iR(r)}e^{iS(\theta)}e^{im\psi}e^{-iEt} + \sqrt{-g}g^{rr}\frac{\partial^2}{\partial r^2}e^{iR(r)}e^{iS(\theta)}e^{im\psi}e^{-iEt}$$
$$+\sqrt{-g}g^{\psi\psi}\frac{\partial^2}{\partial \psi^2}e^{im\psi}e^{iS(\theta)}e^{-iEt} + i\frac{\partial}{\partial \theta}\left(\sqrt{-g}g^{\theta\theta}\right)\frac{\partial S(\theta)}{\partial \theta}e^{i\theta}e^{im\psi}e^{iR(r)}e^{-iEt} + \sqrt{-g}g^{\theta\theta}\frac{\partial^2}{\partial \theta^2}e^{i\theta}e^{im\psi}e^{iR(r)}e^{-iEt}$$
$$-2i^2mE\sqrt{-g}g^{t\psi}e^{im\psi}e^{i\theta}e^{iR(r)}e^{-iEt} = 0$$

Assuming the individual parts of the equation to be real and comparing the real and imaginary parts, we get:

$$-\sqrt{-g}g^{tt}E^2e^{-iEt}e^{iR(r)}e^{iS(\theta)}e^{im\psi} - \sqrt{-g}g^{rr}e^{iR(r)}e^{iS(\theta)}e^{im\psi}e^{-iEt}\left(\frac{\partial R(r)}{\partial r}\right)^2 - m^2\sqrt{-g}g^{\psi\psi}e^{im\psi}e^{iR(r)}e^{iS(\theta)}e^{-iEt}$$
$$-\sqrt{-g}g^{\theta\theta}e^{i\theta}e^{im\psi}e^{iR(r)}e^{-iEt}\left(\frac{\partial S(\theta)}{\partial \theta}\right)^2 + 2mE\sqrt{-g}g^{t\psi}e^{im\psi}e^{i\theta}e^{iR(r)}e^{-iEt} = 0$$

Simplifying this equation further:

$$-\sqrt{-g}g^{tt}E^2 - \sqrt{-g}g^{rr}\left(\frac{\partial R(r)}{\partial r}\right)^2 - m^2\sqrt{-g}g^{\psi\psi} - \sqrt{-g}g^{\theta\theta}\left(\frac{\partial S(\theta)}{\partial \theta}\right)^2 + 2mE\sqrt{-g}g^{t\psi} = 0$$

$$\left(\frac{\partial R(r)}{\partial r}\right)^2 = \frac{-g^{tt}E^2 - m^2g^{\psi\psi} - g^{\theta\theta}\left(\frac{\partial S(\theta)}{\partial \theta}\right)^2 + 2mEg^{t\psi}}{g^{rr}}$$

$$(k_r)^2 = \frac{1}{g^{rr}}\left\{-g^{tt}E^2 - m^2g^{\psi\psi} - g^{\theta\theta}(k_\theta)^2 + 2mEg^{t\psi}\right\}$$

(using standard notation)

Free Energy and Entropy

According to the theory of canonical ensemble and using semi-classical approximation, the free energy of scalar free particles within a shell of width "$L-(r_H+\epsilon)$" in Kerr-Newman background as:

$$\beta F = \sum_{E,m} \ln[1-e^{-\beta(E-m\Omega_H)}]$$

$$= \int dm \int_0^\infty d\Gamma(E) \ln[1-e^{-\beta(E-m\Omega_H)}]$$

$$= \int dm \left(\Gamma(E) \ln[1-e^{-\beta(E-m\Omega_H)}]_0^\infty - \int_0^\infty \Gamma(E) \frac{e^{-\beta(E-m\Omega_H)}}{1-e^{-\beta(E-m\Omega_H)}} \beta dE \right)$$

$$= -\beta \int dm \int_0^\infty \Gamma(E) \frac{1}{e^{\beta(E-m\Omega_H)}-1} dE$$

To proceed further we need to know the expression for $\Gamma(E) = \int g(E)dE$, which describes the total number of modes with energy less than E and a fixed m, assuming $\phi(r,\theta,\psi,t) \neq 0$ for $r_H+\epsilon \leq r \leq L$ and $\phi = 0$ outside this shell, is obtained by integrating over the volume of phase space:

$$\Gamma(E,m) = \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\pi} \int dk_\theta k_r$$

$$= \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\pi} \int dk_\theta \frac{1}{\sqrt{g^{rr}}} \left(-g^{tt}E^2 - m^2 g^{\psi\psi} - g^{\theta\theta} (k_\theta)^2 + 2mEg^{t\psi} \right)^{1/2}$$

We will solve this integral, by assuming:

$$-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi} = a^2$$

$$\sqrt{g^{\theta\theta}} k_\theta = x \implies dk_\theta = \frac{1}{\sqrt{g^{\theta\theta}}} dx$$

Our integral becomes, with appropriate limit for $k_r^2 \geq 0$:

$$\Gamma(E) = \frac{1}{\pi} \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}}} \int_0^a \frac{1}{\sqrt{g^{\theta\theta}}} dx \sqrt{(a^2-x^2)}$$

$$= \frac{1}{\pi} \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr} g^{\theta\theta}}} \left| \frac{1}{2} \left(x\sqrt{a^2-x^2} + a^2 \sin^{-1} \frac{x}{a} \right) \right|_0^a$$

$$= \frac{1}{\pi} \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr} g^{\theta\theta}}} \quad \frac{\pi}{2} \left(-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi} \right)$$

Substituting it back to the original equation and simplifying that further:

$$F = -\frac{1}{2} \int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr} g^{\theta\theta}}} \int_0^\infty dE \frac{\left(-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi} \right)}{e^{\beta(E-m\Omega_H)}-1}$$

This integral diverges at $E = m\Omega_H$, which is why we need to split it into two parts, one where $0 \leq E \leq m\Omega_H$ and other where, $\Omega_H < E < \infty$.

$$F = -\frac{1}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{0}^{m\Omega_H} dE \frac{\left(-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi}\right)}{e^{\beta(E-m\Omega_H)} - 1}$$

$$-\frac{1}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{m\Omega_H}^{\infty} dE \frac{\left(-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi}\right)}{e^{\beta(E-m\Omega_H)} - 1}$$

Using the change of variable $E' = E - m\Omega_H$, our integral becomes:

$$F_{NSR} = -\frac{1}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{m\Omega_H}^{\infty} dE \frac{\left(-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi}\right)}{e^{\beta(E-m\Omega_H)} - 1}$$

$$= -\frac{1}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{0}^{\infty} dE' \frac{-g^{tt}(E'+m\Omega_H)^2 - m^2 g^{\psi\psi} + 2m(E'+m\Omega_H)g^{t\psi}}{e^{\beta E'} - 1}$$

$$= -\frac{1}{2}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{0}^{\infty} dE' \frac{1}{e^{\beta E'} - 1} \int_{b_+}^{b_-} dm \left[-g^{tt}(E'+m\Omega_H)^2 - m^2 g^{\psi\psi} + 2m(E'+m\Omega_H)g^{t\psi}\right]$$

($b_\pm$ are root of $g^{tt}(E'+m\Omega_H)^2 + m^2 g^{\psi\psi} - 2m(E'+m\Omega_H)g^{t\psi}$)

$$= \frac{1}{2}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{0}^{\infty} dE' \frac{1}{e^{\beta E'} - 1} \int_{b_-}^{b_+} dm \left[(g^{tt}\Omega_H^2 + g^{\psi\psi} - 2\Omega_H g^{t\psi})m^2 + 2E'(g^{tt}\Omega_H - g^{t\psi})m + g^{tt}E'^2\right]$$

$$= \frac{1}{2}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{0}^{\infty} dE' \frac{1}{e^{\beta E'} - 1} \int_{b_-}^{b_+} dm \left[\left(\frac{1}{g_{\psi\psi}}\right) m^2 + \frac{g_{\psi\psi}}{\mathcal{D}} E'^2\right]$$

Integrating this result with respect to m and substituting the limit we arrive at:

$$F_{NSR} = \frac{1}{2}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_{0}^{\infty} dE' \frac{1}{e^{\beta E'} - 1} \left[-\frac{4}{3} \frac{E'^3}{\{(-\mathcal{D})\}^{3/2}} (g_{\psi\psi})^2\right]$$

$$= -\frac{2}{3}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \frac{(g_{\psi\psi})^2}{\{(-\mathcal{D})\}^{3/2}} \int_{0}^{\infty} dE' \frac{E'^3}{e^{\beta E'} - 1}$$

$$= -\frac{2\Gamma(4)\zeta(4)}{3\beta^4} \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \frac{(g_{\psi\psi})^2}{\{(-\mathcal{D})\}^{3/2}}$$

$$= -\frac{2\Gamma(4)\zeta(4)}{3\beta^4} \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{(g_{rr}g_{\theta\theta})^{1/2}}{\{(-\mathcal{D})\}^{3/2}} (g_{\psi\psi})^2$$

This final integral for F_{NSR} is quite complicated, which is why we will taylor expand it at $r = r_H$ like this and neglect all other contributions:

$$\frac{1}{(r-r_H)^2}F(r,\theta) = \frac{F(r_{_H},\theta)}{(r-r_H)^2} + \frac{F'(r_{_H},\theta)}{(r-r_H)} + \mathcal{O}(r-r_H)$$

where

$$F(r,\theta) = \frac{(g_{rr}g_{\theta\theta})^{1/2}}{\{(-\mathcal{D})\}^{3/2}}(g_{\psi\psi})^2(r-r_{_H})^2$$

Taylor Expanding and approximating:

$$\frac{1}{(r-r_{_H})^2}\left[\frac{(g_{rr}g_{\theta\theta})^{1/2}(g_{\psi\psi})^2}{(-\mathcal{D})^{3/2}}(r-r_{_H})^2\right] \approx \left|\frac{\sqrt{\frac{\rho^4}{\Delta}}\frac{\sin^4\theta(r^2+a^2)^4}{\rho^4}}{\left\{\Delta\sin^2\theta\right\}^{3/2}}(r-r_H)^2\right|_{r=r_H}\frac{1}{(r-r_H)^2}$$

$$= \frac{\sin^4\theta(r_H^2+a^2)^4}{\rho_H^2\left\{(r_H-r_-)^2\sin^3\theta\right\}}\frac{1}{(r-r_H)^2}$$

$$= \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{(r_H-r_-)\right\}^2}\frac{1}{(r-r_H)^2}$$

$$= \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{2(r_H-GM)\right\}^2}\frac{1}{(r-r_H)^2}$$

At the last step of approximation, we used:

$$r_+ - r_- = \frac{-b+\sqrt{D}}{2a} - \frac{-b-\sqrt{D}}{2a}$$

$$= \frac{\sqrt{D}}{a} = \frac{2ar_+ + b}{a} = 2r_+ + \frac{b}{a}$$

Using these result our integral for F_{NSR} mode now becomes:

$$F_{NSR} \approx -\frac{2\Gamma(4)\zeta(4)}{3\beta^4}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{2(r_H-GM)\right\}^2}\frac{1}{(r-r_H)^2}$$

$$= -\frac{2\Gamma(4)\zeta(4)}{3\beta^4}\int d\theta d\psi \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{2(r_H-GM)\right\}^2}\int_{r_H+\epsilon}^{L} dr \frac{1}{(r-r_H)^2}$$

$$= \frac{\Gamma(4)\zeta(4)}{6\beta^4}\int d\theta d\psi \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{(r_H-GM)\right\}^2}\left(\frac{1}{r-r_H}\right)_{r_H+\epsilon}^{L}$$

$$= -\frac{\zeta(4)}{\beta^4}\int d\theta d\psi \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{(r_H-GM)\right\}^2}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right] \qquad \text{(using } L = r_H+\epsilon+\delta\text{)}$$

Before I move on, we need to discuss about the limits of m, we required that, $k_r^2 \geq 0$, it asserted that, $0 \leq g^{\theta\theta} k_\theta^2 \leq -g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi}$, but there is no restriction on latter other than that $-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi} \geq 0$ or $g^{tt}E^2 + m^2 g^{\psi\psi} - 2mEg^{t\psi} \leq 0$ and from that it follows:

$$g^{tt}E^2 + m^2 g^{\psi\psi} - 2mEg^{t\psi} \leq 0$$

$$g^{tt}(E' + m\Omega_H)^2 + m^2 g^{\psi\psi} - 2m(E' + m\Omega_H)g^{t\psi} \leq 0$$

$$(g^{tt}\Omega_H^2 + g^{\psi\psi} - 2\Omega_H g^{t\psi})m^2 + 2E'(g^{tt}\Omega_H - g^{t\psi})m + g^{tt}E'^2 \leq 0$$

$$\left(\frac{\mathcal{D}}{\mathcal{D}g_{\psi\psi}}\right)m^2 + 2E'\left(-\frac{g_{\psi\psi}}{\mathcal{D}}\frac{g_{t\psi}}{g_{\psi\psi}} + \frac{g_{t\psi}}{\mathcal{D}}\right)m + \frac{g_{\psi\psi}}{\mathcal{D}}E'^2 \leq 0$$

$$\left(\frac{1}{g_{\psi\psi}}\right)m^2 + \frac{g_{\psi\psi}}{\mathcal{D}}E'^2 \leq 0$$

$$-\sqrt{-\frac{E'^2}{\mathcal{D}}g_{\psi\psi}} \leq m \leq \sqrt{-\frac{E'^2}{\mathcal{D}}g_{\psi\psi}}$$

After sucessfully evaluating the F_{NSR} we will use the change of variable $E = m\Omega_H x$ for F_{SR} to arrive at:

$$F_{SR} = -\frac{1}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_0^{m\Omega_H} dE \frac{\left(-g^{tt}E^2 - m^2 g^{\psi\psi} + 2mEg^{t\psi}\right)}{e^{\beta(E - m\Omega_H)} - 1}$$

$$= -\frac{1}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_0^{1} m\Omega_H dx \frac{-g^{tt}(m\Omega_H x)^2 - m^2 g^{\psi\psi} + 2m(m\Omega_H x)g^{t\psi}}{e^{\beta m\Omega_H(x-1)} - 1}$$

$$= -\frac{\Omega_H}{2}\int dm \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int_0^{1} m^3 dx \frac{-g^{tt}(\Omega_H x)^2 - g^{\psi\psi} + 2(\Omega_H x)g^{t\psi}}{e^{\beta m\Omega_H(x-1)} - 1}$$

$$= -\frac{\Omega_H}{2}\int_0^1 dx \int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}} \int dm \frac{\Omega^2 m^3}{e^{\beta m\Omega_H(x-1)} - 1}$$

(using $\Omega^2 = -g^{tt}(\Omega_H x)^2 - g^{\psi\psi} + 2(\Omega_H x)g^{t\psi}$)

The limit for integral:

$$-g^{tt}(m\Omega_H x)^2 - m^2 g^{\psi\psi} + 2m(m\Omega_H x)g^{t\psi} \geq 0$$

$$m^2\{-g^{tt}(\Omega_H x)^2 - g^{\psi\psi} + 2(\Omega_H x)g^{t\psi}\} \geq 0 \qquad \text{(this suggests } m^2 \geq 0\text{)}$$

$$(g^{tt}\Omega_H^2)x^2 - (2\Omega_H g^{t\psi})x + g^{\psi\psi} \leq 0$$

$$\frac{1}{\mathcal{D}}\left[(g_{\psi\psi}\Omega_H^2)x^2 + (2\Omega_H g_{t\psi})x + g_{tt}\right] \leq 0 \qquad (\mathcal{D} = -(g_{t\psi})^2 + g_{tt}g_{\psi\psi})$$

As it is self explanatory, in the region where $-\mathcal{D} > 0$ our "x" will have real solutions and from that we could easily deduce that,

$$\left[(g_{\psi\psi}\Omega_H^2)x^2 + (2\Omega_H g_{t\psi})x + g_{tt}\right] \geq 0$$

$$\implies 0 \leq x \leq \frac{-2g_{t\psi}\Omega_H - \sqrt{(2g_{t\psi}\Omega_H)^2 - 4g_{tt}g_{\psi\psi}\Omega_H^2}}{2g_{\psi\psi}\Omega_H^2}$$

$$0 \leq x \leq -\frac{(g_{t\psi} + \sqrt{-\mathcal{D}})}{g_{\psi\psi}\Omega_H} \leq 1 \qquad \text{(using } -\mathcal{D} = (g_{t\psi})^2 - g_{tt}g_{\psi\psi} = \Delta\sin^2\theta)$$

$$0 \leq x \leq -\frac{(g_{t\psi} + \sqrt{\Delta}\sin\theta)}{g_{\psi\psi}\Omega_H} \leq 1$$

$$0 \leq x \leq 1 + \frac{\sqrt{-\mathcal{D}}}{g_{t\psi}} \qquad \text{(using } \Omega_H = -\tfrac{g_{t\psi}}{g_{\psi\psi}})$$

(inside event horizon where $g_{t\psi} < 0$ and $\Delta > 0$)

With the limits obtained from above inequality our integral for free energy of super-radiant mode will become:

$$F_{SR} = -\frac{\Omega_H}{2}\int_0^\alpha dx \int d\theta d\psi \int_{r_H+\epsilon}^L dr \frac{\Omega^2}{\sqrt{g^{rr}g^{\theta\theta}}}\int_0^\infty dm \frac{m^3}{e^{\beta m\Omega_H(x-1)} - 1}$$

$$= -\frac{\Omega_H}{2}\int_0^\alpha dx \int d\theta d\psi \int_{r_H+\epsilon}^L dr \frac{\Omega^2}{\sqrt{g^{rr}g^{\theta\theta}}}\left[\frac{\Gamma(4)\zeta(4)}{\{\beta\Omega_H(x-1)\}^4}\right]$$

$$= -\frac{\Gamma(4)\zeta(4)}{2\beta^4\Omega_H^3}\int d\theta d\psi \int_{r_H+\epsilon}^L dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}}\int_0^\alpha dx \frac{\Omega^2}{(x-1)^4}$$

$$= \frac{\Gamma(4)\zeta(4)}{2\beta^4\Omega_H^3}\int d\theta d\psi \int_{r_H+\epsilon}^L dr \frac{1}{\sqrt{g^{rr}g^{\theta\theta}}}\int_0^\alpha dx \frac{g^{tt}\Omega_H^2 x^2 - 2\Omega_H g^{t\psi}x + g^{\psi\psi}}{(x-1)^4}$$

(using $\Omega^2 = -g^{tt}(\Omega_H x)^2 - g^{\psi\psi} + 2(\Omega_H x)g^{t\psi}$)

$$= \frac{\Gamma(4)\zeta(4)}{2\beta^4\Omega_H^3}\int d\theta d\psi \int_{r_H+\epsilon}^L dr \sqrt{g_{rr}g_{\theta\theta}}\int_0^\alpha dx \frac{g_{\psi\psi}\Omega_H^2 x^2 + 2\Omega_H g_{t\psi}x + g_{tt}}{\mathcal{D}(x-1)^4}$$

$$= \frac{\Gamma(4)\zeta(4)}{2\beta^4\Omega_H^3}\int d\theta d\psi \int_{r_H+\epsilon}^L dr \sqrt{g_{rr}g_{\theta\theta}}\int_0^\alpha dx\; g_{\psi\psi}\Omega_H^2\frac{x^2}{\mathcal{D}(x-1)^4} + 2\Omega_H g_{t\psi}\frac{x}{\mathcal{D}(x-1)^4} + g_{tt}\frac{1}{\mathcal{D}(x-1)^4}$$

$$= \frac{\Gamma(4)\zeta(4)}{2\beta^4\Omega_H^3}\int d\theta d\psi \int_{r_H+\epsilon}^L dr \frac{\sqrt{g_{rr}g_{\theta\theta}}}{\mathcal{D}}\left[\; g_{\psi\psi}\Omega_H^2\left|\frac{-3x^2+3x-1}{3(x-1)^3}\right|_0^\alpha + 2\Omega_H g_{t\psi}\left|\frac{1-3x}{6(x-1)^3}\right|_0^\alpha + g_{tt}\left|-\frac{1}{3(x-1)^3}\right|_0^\alpha\right]$$

Before we proceed ahead let's simplify the integrand:

$$I = g_{\psi\psi}\Omega_H^2 \frac{-3\alpha^2+3\alpha-1}{3(\alpha-1)^3} + 2\Omega_H g_{t\psi}\frac{1-3\alpha}{6(\alpha-1)^3} - g_{tt}\frac{1}{3(\alpha-1)^3} - \frac{g_{\psi\psi}\Omega_H^2}{3} + \frac{\Omega_H g_{t\psi}}{3} - \frac{g_{tt}}{3}$$

$$= g_{\psi\psi}\Omega_H^2 \frac{-3\alpha^2+3\alpha-1}{3(\alpha-1)^3} + \Omega_H g_{t\psi}\frac{1-3\alpha}{3(\alpha-1)^3} - g_{tt}\frac{1}{3(\alpha-1)^3} + \frac{\Omega_H g_{t\psi} - g_{\psi\psi}\Omega_H^2 - g_{tt}}{3}$$

$$= g_{\psi\psi}\Omega_H^2 \frac{-3(\alpha^2-\alpha)-1}{3(\alpha-1)^3} + \Omega_H g_{t\psi}\frac{(3-2)-3\alpha}{3(\alpha-1)^3} - g_{tt}\frac{1}{3(\alpha-1)^3} + \frac{\Omega_H g_{t\psi} - g_{\psi\psi}\Omega_H^2 - g_{tt}}{3}$$

$$= g_{\psi\psi}\Omega_H^2 \frac{-3(\alpha-1)^2-3(\alpha-1)-1}{3(\alpha-1)^3} + \Omega_H g_{t\psi}\frac{-3(\alpha-1)-2}{3(\alpha-1)^3} - g_{tt}\frac{1}{3(\alpha-1)^3} + \frac{\Omega_H g_{t\psi} - g_{\psi\psi}\Omega_H^2 - g_{tt}}{3}$$

$$= -3\frac{g_{\psi\psi}\Omega_H^2}{3(\alpha-1)} - 3\frac{\Omega_H g_{t\psi} + g_{\psi\psi}\Omega_H^2}{3(\alpha-1)^2} - \frac{g_{\psi\psi}\Omega_H^2 + 2\Omega_H g_{t\psi} + g_{tt}}{3(\alpha-1)^3} + \frac{\Omega_H g_{t\psi} - g_{\psi\psi}\Omega_H^2 - g_{tt}}{3}$$

$$= I_1 + I_2 + I_3 + I_4$$

Now comes the time when we evaluate the α:

$$\alpha = -\frac{g_{t\psi} + \sqrt{(g_{t\psi})^2 - g_{tt}g_{\psi\psi}}}{g_{\psi\psi}\Omega_H}$$

$$= \frac{g_{t\psi} + \sqrt{-\mathcal{D}}}{g_{t\psi}}$$

$$\implies \alpha - 1 = \frac{\sqrt{-\mathcal{D}}}{g_{t\psi}}$$

Let's simplify I_1/Ω_H^3:

$$\frac{I_1}{\Omega_H^3} = -3\frac{g_{\psi\psi}\Omega_H^2}{3\Omega_H^3(\alpha-1)}$$

$$= -3\frac{g_{\psi\psi}}{3\Omega_H(\alpha-1)}$$

$$= \frac{3(g_{\psi\psi})^2}{3\sqrt{-\mathcal{D}}} = -3\frac{\mathcal{D}}{3(-\mathcal{D})^{3/2}}(g_{\psi\psi})^2 = -3\frac{I_3}{\Omega_H^3}$$

(using $\Omega_H = -\frac{g_{t\psi}}{g_{\psi\psi}}$)

Considering and simplifying I_2/Ω_H^3:

$$\frac{I_2}{\Omega_H^3} = -3\frac{\Omega_H g_{t\psi} + g_{\psi\psi}\Omega_H^2}{3\Omega_H^3(\alpha-1)^2}$$

$$= \frac{g_{t\psi} + g_{\psi\psi}\Omega_H}{\Omega_H^2(\alpha-1)^2}$$

$$= \frac{g_{t\psi} - g_{t\psi}}{\Omega_H^2(\alpha-1)^2}$$

$$= 0$$

Considering I_3/Ω_H^3:

$$\frac{I_3}{\Omega_H^3} = -\frac{g_{\psi\psi}\Omega_H^2 + 2\Omega_H g_{t\psi} + g_{tt}}{3\Omega_H^3(\alpha-1)^3}$$

$$= -\frac{(g_{t\psi})^2 - 2(g_{t\psi})^2 + g_{tt}g_{\psi\psi}}{3\Omega_H^3 g_{\psi\psi}(\sqrt{-\mathcal{D}})^3}(g_{t\psi})^3 \qquad \left(\text{using } \Omega_H = -\frac{g_{t\psi}}{g_{\psi\psi}}\right)$$

$$= \frac{\mathcal{D}}{3(-\mathcal{D})^{3/2}}(g_{\psi\psi})^2 \qquad (\mathcal{D} = -(g_{t\psi})^2 + g_{tt}g_{\psi\psi})$$

and I_4/Ω_H^3 becomes:

$$\frac{I_4}{\Omega_H^3} = \frac{\Omega_H g_{t\psi} - g_{\psi\psi}\Omega_H^2 - g_{tt}}{3\Omega_H^3}$$

$$= \frac{\Omega_H g_{t\psi} - g_{\psi\psi}\Omega_H^2 - g_{tt}}{3\Omega_H^3}$$

$$= \frac{-(g_{t\psi})^2 - (g_{t\psi})^2 - g_{tt}g_{\psi\psi}}{3g_{\psi\psi}\Omega_H^3}$$

$$= \frac{2(g_{t\psi})^2 + g_{tt}g_{\psi\psi}}{3(g_{t\psi})^3}(g_{\psi\psi})^2$$

$$= \frac{3(g_{t\psi})^2 + \mathcal{D}}{3(g_{t\psi})^3}(g_{\psi\psi})^2 = 3\frac{(g_{\psi\psi})^2}{3(g_{t\psi})} + \frac{\mathcal{D}}{3(g_{t\psi})^3}(g_{\psi\psi})^2$$

Our Integrand just after considering I_1,I_2 & I_3 becomes:

$$F_{SR} \approx -2\frac{\Gamma(4)\zeta(4)}{6\beta^4}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{\sqrt{g_{rr}g_{\theta\theta}}}{\mathcal{D}}\frac{\mathcal{D}}{(-\mathcal{D})^{3/2}}(g_{\psi\psi})^2$$

$$= -2\frac{\Gamma(4)\zeta(4)}{6\beta^4}\int d\theta d\psi \int_{r_H+\epsilon}^{L} dr \frac{\sqrt{g_{rr}g_{\theta\theta}}}{(-\mathcal{D})^{3/2}}(g_{\psi\psi})^2$$

$$\approx 2\frac{\Gamma(4)\zeta(4)}{6\beta^4}\int d\theta d\psi \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{2(r_H-GM)\right\}^2}\left(\frac{1}{r-r_H}\right)_{r_H+\epsilon}^{L}$$

$$= -\frac{\zeta(4)}{2\beta^4}\int d\theta d\psi \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{(r_H-GM)\right\}^2}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right] \qquad (\text{using } L = r_H+\epsilon+\delta)$$

Since now we know both super-radiant and non-superradiant mode of free energy, our total free energy finally becomes:

$$F = F_{NSR} + F_{SR}$$

$$= -\frac{3\zeta(4)}{2\beta^4}\int d\theta d\psi \frac{(r_H^2+a^2)^4\sin\theta}{(r_H^2+a^2\cos^2\theta)^2\left\{(r_H-GM)\right\}^2}\left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

Since we know:

$$\kappa = \frac{2\pi}{\beta} = \frac{r_+ - r_-}{2(r_H^2+a^2)}$$

$$= \frac{r_H - GM}{r_H^2+a^2}$$

also:

$$d\mathcal{A} = \sqrt{g_{\theta\theta}g_{\psi\psi}}\,d\theta d\psi = \sqrt{\frac{\Sigma\sin^2\theta}{\rho^2}\rho^2}\,d\theta d\psi$$

$$= (r_H^2+a^2)\sin\theta d\theta d\psi$$

We can use these two results to simplify our equation even further:

$$F = -\frac{3\zeta(4)}{2\beta^4} \int d\theta d\psi \frac{(r_H^2 + a^2)^4 \sin\theta}{(r_H^2 + a^2\cos^2\theta)^2 \left\{(r_H - GM)\right\}^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= -\frac{3\zeta(4)}{2\beta^4} \int d\theta d\psi (r_H^2 + a^2)\sin\theta \frac{(r_H^2 + a^2)^2}{(r_H - GM)^2} \frac{(r_H^2 + a^2)}{(r_H^2 + a^2\cos^2\theta)^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= -\frac{3\zeta(4)}{2\beta^4} \frac{\beta^2}{4\pi^2} \int d\mathcal{A} \frac{(r_H^2 + a^2)}{(r_H^2 + a^2\cos^2\theta)^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= -\frac{3\zeta(4)}{8\pi^2\beta^2} \int d\mathcal{A} \frac{(r_H^2 + a^2)}{(r_H^2 + a^2\cos^2\theta)^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

Then finally the entropy becomes:

$$S = \beta^2 \frac{\partial F}{\partial \beta}$$

$$= -\beta^2 \frac{\partial}{\partial\beta} \left[\frac{3\zeta(4)}{8\pi^2\beta^2} \int d\mathcal{A} \frac{(r_H^2 + a^2)}{(r_H^2 + a^2\cos^2\theta)^2} \frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= 2\beta^2 \frac{3\zeta(4)}{8\pi^2\beta^3} \int d\mathcal{A} \frac{(r_H^2 + a^2)}{(r_H^2 + a^2\cos^2\theta)^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= \frac{3\zeta(4)}{4\pi^2\beta} \int d\mathcal{A} \frac{(r_H^2 + a^2)}{(r_H^2 + a^2\cos^2\theta)^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right]$$

$$= \frac{\int d\mathcal{A}}{4} \qquad \left(\text{using } \frac{3\zeta(4)}{\pi^2\beta} \frac{(r_H^2+a^2)}{(r_H^2+a^2\cos^2\theta)^2} \left[\frac{\delta}{\epsilon(\epsilon+\delta)}\right] = 1\right)$$

Conclusions and Discussions

This is the result we finally got, it tells us that the entropy of hawking particles in thin shell near the event horizon is actually proportional to area of event horizon, which was our expected result. From the above calculation, where we derived the final expression of metric expressed purely in terms of metric, which can be used to evaluate the entropy of any black hole via this modified thin film brick wall method. Though through our explicit calculation, both F_{SR} and F_{NSR} seem negative, unlike what other authors have been trying to show. Just before we substituted the value of α in F_{SR}, our equation matched exactly to the equation (17) of *Entropy in the Kerr-Newman Black Hole*. But somehow their final result seem to be positive, which suggests, either they missed the $-$ sign when they substituted the value of Ω_H or they took the wrong value of α.

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