ABSTRACT. The Collatz Conjecture is one of the Mathematical conjectures. The conjecture is as follows: take any positive integer number \( n \). If \( n \) is even, divide it by 2. If \( n \) is odd, multiply it by 3 and add 1 to get \( (3n + 1) \). Repeat this process again and again. The Collatz Conjecture says that no matter what the number \( n \) is taken, the process will always eventually reach 1.

This paper consists of one chapter. In the following chapter, Collatz Conjecture is proved for all positive integer number \( n \).

1. Method

Suppose that \( n \) is a positive integer number. It is well known that there exists a positive integer \( k \) such that \( n \) is one of the elements in the following set:

\[
\{2^k, 2^k + 1, 2^k + 2, 2^k + 3, \ldots, 2^k + 2^k - 1\}.
\]

If Collatz Conjecture is proved for all the elements of the above set for arbitrary \( k \), the Conjecture is proved for all the positive integer numbers. The following proof is based on Mathematical induction. It is easy to see that the conjecture is true for \( k = 1, 2 \).

**Induction Hypothesis:** Suppose that \( k \geq 3 \) and the conjecture is proved for all elements of the set \( \{2^t, 2^t + 1, 2^t + 2, 2^t + 3, \ldots, 2^t + 2^t - 1\} \), where \( t \) is a positive integer number such that \( t \leq k - 1 \).

**Goal:** Prove that the conjecture is true for all the elements in the set \( \{2^k, 2^k + 1, 2^k + 2, \ldots, 2^k + 2^k - 1\} \).

**Proof:** Suppose that \( n = 2^k \). By dividing \( n \) by 2, \( k \)-times, we reach 1. So the conjecture is proved for \( n = 2^k \). Suppose that \( n = 2^k + m \). If \( m \) is an even integer number, then

\[
2^{-1}n = 2^{-1}2^k + 2^{-1}m = 2^{k-1} + 2^{-1}m.
\]
By the induction hypothesis the conjecture is true for $2^{-1}n$, so the conjecture is proved for $n$.

Now, suppose that $n = 2^k + 1$. Obviously, $n$ is an odd number. So

$$3(2^k + 1) + 1 = 3.2^k + 3 + 1$$

$$= 3.2^k + 4$$

$$= 2^2(3.2^{k-2} + 1) \quad (*)$$

In the last line of $(*)$, I consider $k \geq 2$, which is correct by induction hypothesis. Now, I just need to prove the conjecture for $3.2^{k-2} + 1$. It is easy to see that

$$3.2^{k-2} + 1 = 2.2^{k-2} + 2^{k-2} + 1$$

$$= 2^{k-1} + 2^{k-2} + 1 \quad (**$$)

By induction hypothesis the conjecture is true for $2^{k-1} + 2^{k-2} + 1$. So by the last line of $(**)$ the conjecture is proved for $n = 2^k + 1$.

Assume that $2^k + m$ is an odd integer number such that $m > 1$. So there exist a positive integer $i$ and odd integer number $l$ such that $(3m + 1 = 2^i l)$. If $i \geq 2$ we have:

$$3(2^k + m) + 1 = 3.2^k + 3m + 1$$

$$= 3.2^k + 2^i l$$

$$= 2.2^k + 2^k + 2^i l$$

$$= 2^{k+1} + 2^k + 2^i l$$

$$= 2^2(2^{k-1} + 2^{k-2} + 2^{i-2} l) \quad (*)$$

In the last line of $(*)$, I consider $k \geq 2$, which is correct by induction hypothesis. Now, by induction hypothesis the conjecture is true for $(2^{k-1} + 2^{k-2} + 2^{i-2} l)$. So the conjecture is proved for $n$. Until now I show that, the conjecture is true for $n = 2^k + m$ with $3m + 1 = 2^i l \quad (i \geq 2)$.

Now, consider an odd number $n = 2^k + m$. Since $m$ is an odd number there exists an odd integer number $l_1$ such that $(3m + 1 = 2. l_1)$. So we have:

$$3(2^k + m) + 1 = 3.2^k + 3m + 1$$

$$= 3.2^k + 2. l_1$$

$$= 2.2^k + 2^k + 2. l_1$$

$$= 2^{k+1} + 2^k + 2. l_1$$

$$= 2(2^k + 2^{k-1} + l_1)$. 

Now, put $m_1 := 2^{k-1} + l_1$. Since $m_1$ is an odd integer number there exists an odd integer number $l_2$ such that $(3m_1 + 1 = 2^il_2)$. If $i \geq 2$ then the conjecture is true for $2^k + m_1$ (as I proved in the last page). So the conjecture is true for $2^k + m$. Otherwise $i = 1$ and we have:

$$3(2^k + m_1) + 1 = 3.2^k + 3m_1 + 1$$
$$= 3.2^k + 2l_2$$
$$= 2.2^k + 2^k + 2l_2$$
$$= 2^{k+1} + 2^k + 2l_2$$
$$= 2(2^k + 2^{k-1} + l_2).$$

Again, put $m_2 := 2^{k-1} + l_2$. Since $m_2$ is an odd integer number there exists an odd integer number $l_3$ such that $3m_2 + 1 = 2^il_3$. If $i \geq 2$, the conjecture is true for $2^k + m_2$, so it is true for $2^k + m$. Otherwise $i = 1$ and we can continue the process. So by this process we get some $2^k + m_i$'s such that $m_i$'s are odd numbers. It is completely obvious that by this process $2^k + m_i \neq 2^k + m_j$ when $i \neq j$. In fact

$$m < l_1 < m_1 < l_2 < m_2 < l_3 < ...$$

Since $2^k + m_i$ are different for each $i$, after some finite steps there exists $j$ such that $(3m_j + 1 = 2^il_{j+1})$ $(i \geq 2)$. So the conjecture is true for $2^k + m_j$, so is true for $2^k + m$. 