Triple Cosines Lemma and $\pi$-sums of Arccosines

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Abstract. We obtain a relationship between cosines of two independent angles and cosine of the angle that depends on them in 3D space and then we use that relationship to obtain $\pi$-sums of Arccosines.

1. Triple Cosines Lemma

In a Cartesian coordinate system for a three-dimensional space of an ordered triplet of axes: OX, OY, OZ that go through the origin O, let the angle $\alpha$, the angle $\beta$. See Figure 1.

Figure 1. Triple Cosines Lemma
Let us find the angle $AOB = \gamma$.

**Lemma 1.**  
$\cos \gamma = \cos \alpha \cos \beta$

**Proof.** Let OM be a unit vector in the direction of OA, let OL be a unit vector in the direction of OB. OM = $(\cos \alpha, \sin \alpha, 0)$, OL = $(\cos \beta, 0, \sin \beta)$. Since the dot product of vectors OM and OL is: $OM \cdot OL = |OM| |OL| \cos \gamma = \cos \gamma$, finally we have: $\cos \gamma = \cos \alpha \cos \beta$.  

2. **Triple Arccosines Theorem**

**Theorem 1.**  
For any triangle cross section of a cube with a plane, where two sides of said cross section, meeting at first vertex of said cross section, said first vertex is located on first edge of the cube, said two sides and said first edge intersect at angles $\alpha_1$ and $\beta_3$, two sides of said cross section, meeting at second vertex of said cross section, said second vertex is located on second edge of the cube, said two sides and said second edge intersect at angles $\alpha_2$ and $\beta_1$, two sides of said cross section, meeting at third vertex of said cross section, said third vertex is located on third edge of the cube, said two sides and said third edge intersect at angles $\alpha_3$ and $\beta_2$, we have:

$$\alpha_1 + \beta_1 = \pi/2, \quad \alpha_2 + \beta_2 = \pi/2, \quad \alpha_3 + \beta_3 = \pi/2,$$

$$\arccos(\cos \alpha_1 \cos \beta_3) + \arccos(\cos \alpha_2 \cos \beta_1) + \arccos(\cos \alpha_3 \cos \beta_2) = \pi$$

**Proof.** Let ABCDA₁B₁C₁D₁ be a cube (see Figure 2).

Let $M \in [AB]$, $N \in [BB_1]$, $L \in [BC]$.

Let $\alpha_1 = \angle NMB$, $\alpha_2 = \angle BNL$, $\alpha_3 = \angle BLM$,

$\beta_1 = \angle MNB$, $\beta_2 = \angle NLB$, $\beta_3 = \angle BML$,

Then, considering the triangles MNB, NBL and MBL we have:

$$\alpha_1 + \beta_1 = \pi/2, \quad \alpha_2 + \beta_2 = \pi/2, \quad \alpha_3 + \beta_3 = \pi/2.$$
By applying the Triple Cosines Lemma 1 to the triangle MNL, since
\[ \cos \angle LMN = \cos \alpha_1 \cos \beta_3, \cos \angle MNL = \cos \alpha_2 \cos \beta_1, \]
\[ \cos \angle NLM = \cos \alpha_3 \cos \beta_2, \] and \( \angle LMN + \angle MNL + \angle NLM = \pi \), we prove the Theorem 1.

\[ \square \]

Figure 2. Triple Arccosines Theorem

**Remark 1.** Note that we could generalize Theorem 1 for the case of 4-sided, 5-sided and 6-sided cross section polygons: intersections of plane and cube,
instead of the triangle cross section as well as for intersections of n-dimen-
onal hypercubes(n >3).

3. \(\pi\)-sum of 6 Arccosines Theorem

Let us consider a tetrahedron SABC, having a triangle ABC as a base, a h-
eight SO, wherein the point O is located inside ABC. Since SO is perpendic-
ular to the base ABC, then OAS, OBS and OCS are perpendicular to the ba-
se ABC as well (see Figure 3).

\[
\begin{align*}
\alpha_1 &= \angle SAO, \\
\beta_{11} &= \angle OAC, \\
\beta_{12} &= \angle OAB, \\
\gamma_1 &= \angle SAC, \\
\delta_1 &= \angle SAB, \\
\alpha_2 &= \angle SBO, \\
\beta_{21} &= \angle ABO, \\
\beta_{22} &= \angle OBC, \\
\gamma_2 &= \angle SBA, \\
\delta_2 &= \angle SBC,
\end{align*}
\]

Figure 3. \(\pi\)-sum of 6 Arccosines Theorem
\[\alpha_3 = \angle SCO, \ \beta_{31} = \angle BCO, \ \beta_{32} = \angle OCA, \ \gamma_3 = \angle BCS, \ \delta_3 = \angle SCA,\]

**Theorem 2.** For any tetrahedron, where a projection of the apex of the tetrahedron on the base is located inside the base and for the first vertex of the base: the angle between the edge, connecting the apex with said first vertex and a segment, connecting said first vertex with said projection is \(\alpha_1\), the angles between a the edge, connecting the apex with said first vertex and two base's edges, meeting at said first vertex are \(\gamma_1\) and \(\delta_1\), for the second vertex of the base: the angle between the edge, connecting the apex with said second vertex and a segment, connecting said second vertex with said projection is \(\alpha_2\), the angles between the edge, connecting the apex with said second vertex and two base's edges, meeting at said second vertex are \(\gamma_2\) and \(\delta_2\), for the third vertex of the base: the angle between the edge, connecting the apex with said third vertex and a segment, connecting said third vertex with said projection is \(\alpha_3\), the angles between the edge, connecting the apex with said third vertex and two base's edges, meeting at said third vertex are \(\gamma_3\) and \(\delta_3\), we have:

\[
\arccos(\cos \gamma_1 / \cos \alpha_1) + \arccos(\cos \delta_1 / \cos \alpha_1) + \\
\arccos(\cos \gamma_2 / \cos \alpha_2) + \arccos(\cos \delta_2 / \cos \alpha_2) + \\
\arccos(\cos \gamma_3 / \cos \alpha_3) + \arccos(\cos \delta_3 / \cos \alpha_3) = \pi
\]

**Proof.** By applying the Triple Cosines Lemma 1 to the angles at the vertex A of tetrahedron SABC, we have:

\[
\cos \gamma_1 = \cos \alpha_1 \cos \beta_{11}, \ \cos \delta_1 = \cos \alpha_1 \cos \beta_{12}.
\]

Thus, \(\beta_{11} = \arccos(\cos \gamma_1 / \cos \alpha_1)\), \(\beta_{12} = \arccos(\cos \delta_1 / \cos \alpha_1)\), So, \(\angle BAC = \beta_{11} + \beta_{12} = \arccos(\cos \gamma_1 / \cos \alpha_1) + \arccos(\cos \delta_1 / \cos \alpha_1)\). Similarly, \(\angle ABC = \arccos(\cos \gamma_2 / \cos \alpha_2) + \arccos(\cos \delta_2 / \cos \alpha_2)\) and \(\angle BCA = \arccos(\cos \gamma_3 / \cos \alpha_3) + \arccos(\cos \delta_3 / \cos \alpha_3)\).

Since \(\angle BAC + \angle ABC + \angle BCA = \pi\), we prove the Theorem 2. \(\square\)

**Remark 2.** Note that Theorem 2 can be generalized for the pyramid, having
a convex n-sided polygon $A_1 \ldots A_n$ as a base, a height $SO$, which is perpendicular to the base $A_1 \ldots A_n$ of the pyramid $SA_1 \ldots A_n$ and $O$ locates inside the base $A_1 \ldots A_n$. Similarly to the Theorem 2, considering the triangles $SOA_1$, $\ldots$, $SOA_n$, that are perpendicular to the base $A_1 \ldots A_n$, we can prove that the sum of $2n$ Arccosines, (two Arccosines per each vertex of the base $A_1 \ldots A_n$) is equal to the sum of interior angles of the polygon $A_1 \ldots A_n$: $(n - 2) \pi$.

References


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