A Direct Proof of the Riemann Hypothesis

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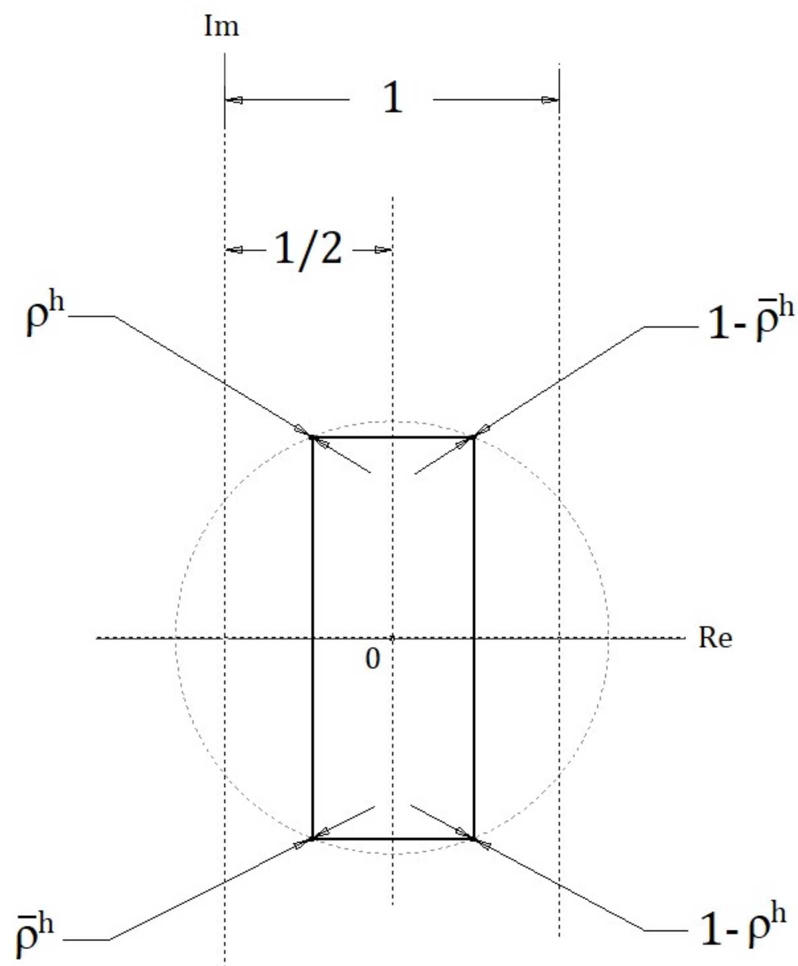
ABSTRACT. A function $\upsilon(s)$ is derived that shares all the non-trivial zeros of Riemann's zeta function $\zeta(s)$, and a novel representation of $\zeta(s)$ is presented that relates the two. From this the zeros of $\zeta(s)$ may be grouped according to two types: $\upsilon(s) = 0$ and $\upsilon(s) \neq 0$. A direct algebraic proof of the Riemann hypothesis is obtained by setting both functions to zero and solving for two general solutions for all the non-trivial zeros.

Introduction. It is well known by B. Riemann's functional equation that any non-trivial zeros of the zeta function $\zeta(s)$ that do not have a real part one-half must exist within the critical strip $0 < \Re(s) < 1$ at the vertices of rectangles, symmetric across the critical line $\Re(s) = 1/2$ and symmetric across the real axis. [1][2][3] This implies that for any two hypothetical non-trivial zeros ρ^h and $1 - \bar{\rho}^h$ symmetric across the critical line from one another ("critically symmetric"), where $\bar{\rho}^h$ is the complex conjugate of ρ^h, $\left|\Re\left(\rho^h\right) - 1/2\right| = \left|\Re\left(1 - \bar{\rho}^h\right) - 1/2\right|$ and $\Im\left(\rho^h\right) = \Im\left(1 - \bar{\rho}^h\right)$, which this paper will refer to as "Riemann's symmetric vertices property".



Graphic definition of Riemann's symmetric vertices property (not to scale).

In his paper [3] Riemann writes that the "symmetrical form" [2]

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s),$$

of his functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{1}$$

induced him to introduce the integral $\Gamma(s/2)$ in place of $\Gamma(s)$ in order to define the ξ function as

$$\xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

This is an entire function that satisfies

$$\xi(s) = \xi(1-s),$$

which reveals the symmetry between $\xi(s)$ and $\xi(1-s)$.

Because $\zeta(s)$ is a multiplicative factor of $\xi(s)$, and because s and $1-s$ are reflections of each other through the real point one-half, by definition of $\xi(s)$ all non-trivial zeros of the Riemann zeta function must comply with Riemann's symmetric vertices property. [1][2][3] Riemann's symmetric vertices property is necessary for there to exist any non-trivial zeros ρ_n off the critical line. And this paper proposes that one can prove the non-existence of any hypothetical zeros off the critical line algebraically (that the Riemann hypothesis is correct) by putting the Riemann zeta function in the form

$$a + 2\,b\,\upsilon\,\omega + c\,\omega^2 = \zeta,$$

and solving for the general solution of ζ's zeros directly.

Motivation for this form. Given

$$a + b + c = 0,$$

there are only two types of solutions:

Type 1. Two terms negate each other and the third is zero, which has the geometric representation of a line ("Type 1 linear solution"),

and

Type 2. Two terms negate the third, which has the geometric representation of a plane ("Type 2 planar solution").

This holds true for this paper's novel form as well. Given a, b, c having no roots,

$$a + 2\,b\,\upsilon\,\omega + c\,\omega^2 = 0,$$

implies

$$\upsilon = \frac{-a - c\,\omega^2}{2\,b\,\omega},$$

where the Type 1 linear solution is

$$a = -c\,\omega^2, \qquad \upsilon = 0,$$

and the Type 2 planar solution is

$$\upsilon = \frac{-a - c\,\omega^2}{2\,b\,\omega}, \qquad \upsilon \neq 0.$$

These two solutions may be considered grouped simply by $\upsilon = 0$ or $\upsilon \neq 0$. Furthermore, υ need not be analytic (however it may be defined), as these solutions are purely algebraic. It is from this form then that the following claim may now be made.

Claim. Let ρ be a non-trivial zero of the Riemann zeta function and ρ^h a hypothetical non-trivial zero off the critical line. Given only two types of zeros of the Riemann zeta function, such that the first type contains all the critical zeros

$$\Re(\rho) = \frac{1}{2},$$

and the second type does not comply with Riemann's symmetric vertices property

$$\left|\Re(\rho^h) - \frac{1}{2}\right| \neq \left|\Re(1 - \bar{\rho}^h) - \frac{1}{2}\right|,$$

the Riemann hypothesis is necessarily correct

$$\forall\, \rho \left\{\exists\ \frac{1}{2}\left\{RealPartOf\left\{\frac{1}{2}, \rho\right\}\right\}\right\}.$$

See the appendix for an outline of this proof.

Proof of the Claim. Begin by bringing all of the irrational properties of Riemann's functional equation (1), including $\zeta(1-s\,)$, into a single function $\upsilon(s)$ so that $\upsilon(s)$ relates to $\zeta(s)$ only by rational functions. First multiply both sides of (1) by $(s-1)^3$, subtract one, then multiply by i. This gives

$$i(\zeta(s)(s-1)^3 - 1)\ = i\ \left(2^s \pi^{s-1} (s-1)^3 \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) - 1\right).$$

Then add $-2\,\mathrm{Im}(s)$ to both sides and multiply by $\sin(\arg(s))$.

$$\frac{\Im(s)\big(i((s-1)^3\zeta(s)-1)-2\,\Im(s)\big)}{|s|}$$
$$= \sin\big(\arg(s)\big)\Bigg(-2\,\mathrm{Im}(s) + i\,\left(2^s\pi^{s-1}(s-1)^3\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)-1\right)\Bigg).$$

The $|s|$ in the denominator suggests that neither side of the equation may meet the conditions necessary for complex differentiation going forward. Because this is an algebraic proof there will be no need to apply the Cauchy-Riemann equations for the above or anything that follows. Therefore, continue by adding $2\,\mathrm{Im}(s)\,(2\,\mathrm{Im}(s)+i)\cos\big(\arg(s)\big)$ to both sides, and dividing both sides by $\sin\big(\arg(s)\big)\,\bar{s}$

$$\frac{i\,\left((s^*)^2+(s-1)^2\big((s-1)\zeta(s)-1\big)\right)}{\bar{s}}$$
$$= \frac{1}{\sin\big(\arg(s)\big)\,\bar{s}}\Bigg(\sin\big(\arg(s)\big)\bigg(-2\,\mathrm{Im}(s) + i\,\left(2^s\pi^{s-1}(s-1)^3\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)-1\right)\bigg) + 2\,\mathrm{Im}(s)(2\,\mathrm{Im}(s)+i)\cos\big(\arg(s)\big)\Bigg)$$

Divide both sides by $2(s-1)$ and let the right hand side be $\upsilon(s)$, such that

$$\frac{i\,\left((\bar{s})^2+(s-1)^2\big((s-1)\zeta(s)-1\big)\right)}{2\,(s-1)\,\bar{s}} = \upsilon(s), \qquad s\neq 0. \tag{2}$$

Solve back for the Riemann zeta function from (2).

$$\zeta(s) = \frac{(s-1)^2 - 2\,i\,(s-1)\,\upsilon(s)\,\bar{s} - (\bar{s})^2}{(s-1)^3}, \qquad s\neq 0.$$

Now expand the right hand side to

$$\zeta(s) = \frac{1}{s-1} - \frac{2\,i\,\upsilon(s)\,\bar{s}}{(s-1)^2} - \frac{(\bar{s})^2}{(s-1)^3}, \qquad s\neq 0$$

in order to define the rational functions in the terms above. Add the second two terms on the right hand side to both sides and let the right hand side be a_s, such that

$$
\begin{aligned}
\zeta(s) + \frac{2\,i\,v(s)\,\bar{s}}{(s-1)^2} + \frac{(\bar{s})^2}{(s-1)^3} &= \frac{1}{s-1}. \\
&= a_s
\end{aligned}
$$

Divide both sides by $1 - s$ and let the right hand side be b_s, such that

$$
\begin{aligned}
\frac{\zeta(s)}{1-s} + \frac{2\,i\,v(s)\,\bar{s}}{(1-s)(s-1)^2} + \frac{(\bar{s})^2}{(1-s)(s-1)^3} &= \frac{a_s}{1-s} \\
-a_s\,\zeta(s)\ - a_s\frac{2\,i\,v(s)\,\bar{s}}{(s-1)^2} - a_s\frac{(\bar{s})^2}{(s-1)^3} &= -\frac{1}{(s-1)^2}. \\
&= b_s
\end{aligned}
$$

Divide both sides once more by $1 - s$ and let the right hand side be c_s, such that

$$
\begin{aligned}
-a_s\frac{\zeta(s)}{1-s} - a_s\frac{2\,i\,v(s)\,\bar{s}}{(1-s)(s-1)^2} - a_s\frac{(\bar{s})^2}{(1-s)(s-1)^3} &= \frac{b_s}{1-s} \\
-b_s\,\zeta(s)\ - b_s\frac{2\,i\,v(s)\,\bar{s}}{(s-1)^2} - b_s\frac{(\bar{s})^2}{(s-1)^3} &= \frac{1}{(s-1)^3}. \\
&= c_s
\end{aligned}
$$

Multiply both sides by $i\,(s\ -\ 1)^3\bar{s}$ and let the right hand side be ω_s, such that

$$
\begin{aligned}
-i\,(s\ -\ 1)^3\,\bar{s}\,\zeta(s)\,b_s + 2\,b_s\,(s\ -\ 1)\,v(s)\,(\bar{s})^2 - i\,b_s\,(\bar{s})^3 &= i\,(s\ -\ 1)^3\bar{s}\,c_s \\
i\frac{\zeta(s)\,\bar{s}}{a_s} - 2\,b_s\,(1\ -\ s)\,v(s)\,(\bar{s})^2 - i\,c_s\,(1\ -\ s)(\bar{s})^3 &= i\,\bar{s} \quad . \\
&= \omega_s
\end{aligned}
$$

Because

$$
i\frac{\zeta(s)\,\bar{s}}{a_s} - 2\,b_s\,(1\ -\ s)\,v(s)\,(\bar{s})^2 - i\,c_s\,(1\ -\ s)(\bar{s})^3 = \frac{\omega_s}{a_s}\ \ \zeta(s)\ + 2\,a_s\,v(s)\,\omega_s^2 + b_s{\omega_s}^3,
$$

solving for $\zeta(s)$ from

$$
\frac{\omega_s}{a_s}\ \ \zeta(s)\ + 2\,a_s\,v(s)\,\omega_s^2 + b_s{\omega_s}^3 = \omega_s
$$

gives the desired form

$$
\begin{aligned}
\zeta(s) &= -a_s\,(2\,a_s\,v(s)\,\omega_s\ +\ b_s\,\omega_s^2 -\ 1) \\
&= a_s + 2\,b_s\,v(s)\,\omega_s\ \ + c_s\,{\omega_s}^2
\end{aligned}
\quad . \tag{3}
$$

Because a_s, b_s and c_s are multiplicative inverses of $(s-1)$ to some power, and zero has no reciprocal, a_s, b_s and c_s have no roots. And because $(s - 1)\zeta(s) - 1 = -1, \zeta(s) = 0$ in the numerator of (2), (2) reduces to

$$\begin{aligned} \upsilon(s) &= \frac{-a_s - c_s\omega_s^2}{2\, b_s\, \omega_s} \\ &= \frac{i\,((s^*)^2 - (s - 1)^2)}{2\,(s - 1)\,\bar{s}} \end{aligned} \tag{4}$$

for all the zeros (trivial and non-trivial) of $\zeta(s)$. (3) is not only an alternative form of (1), but also contains just two types of solutions for $\zeta(s) = 0$ that may be grouped simply according to the zeros of $\upsilon(s)$, which are the Type 1 linear solution and the Type 2 planar solution. Solve first for the linear type $\upsilon(s) = 0$, which gives

$$\begin{aligned} 0 &= \frac{-a_s - c_s\,\omega_s^2}{2\, b_s\, \omega_s} \\ &= \frac{i\,((\bar{s})^2 - (s - 1)^2)}{2\,(s - 1)\,\bar{s}} \\ &= \frac{i\left(1 - 2\,i\,\Im(s)\right)(2\,\Re(s) - 1)}{2\,(s - 1)\,\bar{s}} \Longrightarrow \Re(s) = \frac{1}{2} : a_s = -c_s\,\omega_s^2 \end{aligned}.$$

Because $2\,\Re(s) - 1 = 0$ implies $\Re(s) = 1/2$, and no other part of (4) could equal zero by the definitions of complex arithmetic, the solution $\upsilon(s) = 0$ is linear on the critical line.

$$\Re(s) = \frac{1}{2}, \qquad \upsilon(s) = 0. \tag{5}$$

Solve next for the real part of s from the Type 2 planar solution

$$\upsilon(s) = \frac{-a_s - c_s\,\omega_s^2}{2\, b_s\, \omega_s}, \qquad \upsilon(s) \neq 0, \tag{6}$$

given the quadratic formula applied to (6). One gets

$$\Re(s) = \frac{\pm\sqrt{-(2\,\Im(s) + i)^2(\upsilon(s)^2 - 1)} + 2\,\Im(s) + \upsilon(s) + i}{2\,\upsilon(s)}, \qquad \upsilon(s) \neq 0, \tag{7}$$

where all the zeros consist of pairs across the critical line from each other. The trivial zeros are applicable to (7) as positive solutions, but because any counterpart to these could not exist symmetrically across the critical line inside the critical strip, much less on the real line, their negative solution counterparts are extraneous.

Upon examination of the square root in (7), because

$$-(2\,\Im(s) + i)^2(\upsilon(s)^2 - 1) = (\upsilon(s)^2 - 1)(\bar{s} - s + 1)^2,$$

one can also express (7) as

$$\Re(s) = \frac{\pm\sqrt{(v(s)^2 - 1)(\bar{s} - s + 1)^2} + 2\,\Im(s) + v(s) + i}{2\,v(s)},$$

which also provides a pair of $v(s)$'s, given by

$$v(s) = \frac{\pm\sqrt{(2\,\mathrm{Im}(s) + i)^2}\,|1 - 2\,\mathrm{Re}(s)| + (\bar{s} + s - 1)(2\,\mathrm{Im}(s) + i)}{4\,(|(s)^2| - \bar{s})}.$$

This gives a total of four possible hypothetical zeros (two sets of pairs) across the real and critical lines from each other, as were graphically defined at the beginning of this paper, and as implied by (1). Now one can ask the question, given any hypothetical non-trivial zero ρ^h off the critical line, *is it possible for any* $1 - \bar{\rho}^h$ *to be critically symmetric to* ρ^h*?* That is; for any $\Re(\rho^h) \neq 1/2$, *is it possible given the two solutions in* (6) *to have a* $\Re(1 - \bar{\rho}^h) \neq 1/2$ *equidistant to* ρ^h *from the critical line, considering what has been presented so far?*

This is elementary to verify. Because

$$r = \sqrt{(v - x)^2 + (w - y)^2}$$

gives the distance r between any two points $v + i\,w$ and $x + i\,y$ on the complex plane, the distance r_{cr} from any point s to the nearest point $1/2 + i\,\Im(\mathrm{s})$ on the critical line is given by

$$r_{cr} = \left|\Re(s) - \frac{1}{2}\right|.$$

One can then check if critical symmetry between the positive and negative solutions of (7) is mathematically possible. Setting the two distances equal to each other

$$\left|\Re(\rho^h) - \frac{1}{2}\right| = \left|\Re(1 - \bar{\rho}^h) - \frac{1}{2}\right|,$$

where ρ^h is either the positive or negative solution to (7) and $1 - \bar{\rho}^h$ is either the positive or negative as well, one gets

$$\begin{aligned}
&\left|\frac{\pm\sqrt{-(2\,\Im(\rho^h) + i)^2(v(\rho^h)^2 - 1)} + 2\,\Im(\rho^h) + v(\rho^h) + i}{2\,v(\rho^h)} - \frac{1}{2}\right| \\
&= \left|\frac{\pm\sqrt{-(2\,\Im(\rho^h) + i)^2(v(1 - \bar{\rho}^h)^2 - 1)} + 2\,\Im(\rho^h) + v(1 - \bar{\rho}^h) + i}{2\,v(1 - \bar{\rho}^h)} - \frac{1}{2}\right|, \\
&\rho^h = 1 - \bar{\rho}^h \quad \vee \quad v(\rho^h) = \pm 1 \quad \vee \quad v(1 - \bar{\rho}^h) = \pm 1
\end{aligned} \tag{8}$$

The imaginary part of ρ^h is on both sides of the equation because Riemann's symmetric vertices property not only requires $|\Re(\rho^h) - 1/2| = |\Re(1 - \bar{\rho}^h) - 1/2|$, but also $\Im(\rho^h) = \Im(1 - \bar{\rho}^h)$. Now one can evaluate the solutions of (8). The value inside the square root reduces to zero for $v(s) = \pm 1$, but because

$$\frac{i\,((\bar{s})^2 - (s - 1)^2)}{2\,(s - 1)\,\bar{s}} = \pm 1 \Rightarrow s \pm i\,\bar{s} = 1$$

from (5) is false, this solution is extraneous. The only other possible solution $\rho^h = 1 - \bar{\rho}^h$ in (7) is meaningless, as the only arguments that could apply would be the critical zeros, which would give $\upsilon(\rho^h) = 0$ for $\Re(\rho^h) = 1/2$, leaving (7) undefined, which implies that the Type 2 planar solution cannot contain any arguments with a real part one-half. Since there is only one other type of solution for the Riemann zeta function zeros, it may not only be stated that a zero is critical if and only if $\upsilon(s) = 0$, but also that

$$\left|\Re(\rho^h) - \frac{1}{2}\right| \neq \left|\Re(1 - \bar{\rho}) - \frac{1}{2}\right|, \qquad \upsilon(\rho^h) \neq 0$$

In other words, the Type 2 planar solution (7) is not critically symmetric and therefore does not comply with Riemann's symmetric vertices property. The negative solution is extraneous with the positive only applying to the trivial zeros. Because all non-trivial zeros must comply with Riemann's symmetric vertices property, no non-trivial zeros could exist as Type 2 planar solutions. And because there are only two types of solutions for $\zeta(s) = 0$, all the non-trivial zeros of the Riemann zeta function must then be restricted to the Type 1 linear solution. Since the Type 1 linear solution is on the critical line, all the non-trivial zeros of the Riemann zeta function ρ must have a real part one-half

$$\forall\, \rho \left\{\exists\, \frac{1}{2} \left\{RealPartOf\left\{\frac{1}{2}, \rho\right\}\right\}\right\},$$

and the Riemann hypothesis is correct. ■

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References

[1] E. C. Titchmarsh, *The Theory of the Riemann Zeta Function, Second revised (Heath-Brown) edition,* Oxford University Press, (1986).

[2] A. Ossicini, *An Alternative Form of the Functional Equation for Riemann's Zeta Function II*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 53, 2 (2014)

[3] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie*, (1859). Gesammelte Werke, Teubner, Leipzig, (1892). Reprinted by Dover, NY, (1953).

Appendix. The outline below contains notation that may be unfamiliar to some readers so the following explanation will be necessary for some. In the body of the paper these statements are written in plain English. But below,

$$\forall\, x \left\{\exists\, a \left\{Predicate\, \{a, x\}\right\}\right\}$$

is read "for every" $\forall$ "element" x "of some relevant set" {} "there exists" $\exists$ "an element" a "that relates to it" $\{a, x\}$ "in some meaningful way" $Predicate$. There may also be two elements that relate to x. For example,

$$\forall\ x\ \left\{\exists\ a \wedge b\left\{Predicate\{a \wedge b, x\}\right\}\right\}$$

is read "for every" $\forall$ "element" x "of some relevant set" {} "there exists" $\exists$ "two elements" a and b "that both relate to it" $\{a \wedge b, x\}$ "in some meaningful way" $Predicate$. Or in the case where x relates to either a or b,

$$\forall\ x\ \left\{\exists\ a \vee b\left\{Predicate\{a \vee b, x\}\right\}\right\}$$

is read "for every" $\forall$ "element" x "of some relevant set" {} "there exists" $\exists$ "one of two elements" a or b "that relates to it" $\{a \vee b, x\}$ "in some meaningful way" $Predicate$. Lastly, there may be two elements x and y that relate to a single element a, where

$$\forall\ x \wedge y\ \left\{\exists\ a\left\{Predicate\{a, x \wedge y\}\right\}\right\}$$

is read "for every" $\forall$ "element" x "and" y "of some relevant set" {} "there exists" $\exists$ "an element" a "that relates to it" $\{a, x \wedge y\}$ "in some meaningful way" $Predicate$.

This proof follows the direct form $P_1 \wedge \ldots \wedge P_n \Rightarrow Q$ and depends on the definition D of Riemann's symmetric vertices property.

$$D: \left|\Re(\rho^h) - 1/2\right| = \left|\Re(1-\bar{\rho}^h) - 1/2\right| \ \wedge\ \Im(\rho^h) = \Im(1-\bar{\rho}^h) \Longleftarrow \exists\ \Re(\rho^h) \neq 1/2$$

$$\zeta(s) = a_s + 2\ b_s\ \upsilon(s)\ \omega_s\ + c_s\omega_s{}^2 : a_s = \tfrac{1}{s-1}, b_s = -\tfrac{1}{(s-1)^2}, c_s = \tfrac{1}{(s-1)^3}, \omega_s = i\ \bar{s} \qquad P_1$$

$$\Rightarrow \forall\ \zeta(s) = 0\left\{\exists\ \upsilon(s) = \frac{-a_s - c_s\ \omega_s{}^2}{2\ b_s\ \omega_s}\left\{ReductionBy\left\{\upsilon(s) = \frac{-a_s - c_s\ \omega_s{}^2}{2\ b_s\ \omega_s}, \zeta(s) = 0\right\}\right\}\right\} \qquad P_2$$

$$\Rightarrow \forall\ \zeta(s) = 0\left\{\exists\ \upsilon(s) = 0\ \vee\ \upsilon(s) \neq 0\left\{TwoTypesOf\{\upsilon(s) = 0\ \vee\ \upsilon(s) \neq 0, \zeta(s) = 0\}\right\}\right\} \qquad P_3$$

$$\Rightarrow \forall\ \upsilon(\rho) = 0\left\{\exists\ \Re(\rho) = \frac{1}{2}\left\{NecessaryConditionFor\left\{\Re(\rho) = \frac{1}{2}, \upsilon(\rho) = 0\right\}\right\}\right\} \qquad P_4$$

$$\Rightarrow \forall\ \upsilon(\rho^h) \neq 0\left\{\exists\ \rho^h \wedge 1 - \bar{\rho}^h\left\{SolutionsFor\{\rho^h \wedge 1 - \bar{\rho}^h, \upsilon(\rho^h) \neq 0\ \}\right\}\right\} \qquad P_5$$

$$\Rightarrow \forall\ \upsilon(\rho^h) \neq 0\left\{\nexists\ \Re(\rho^h) = \frac{1}{2}\left\{SolutionFor\left\{\Re(\rho^h) = \frac{1}{2}, \upsilon(\rho^h) \neq 0\right\}\right\}\right\} \qquad P_6$$

$$\Rightarrow \Re(\rho) = \frac{1}{2} \Leftrightarrow \upsilon(\rho) = 0 \qquad P_7$$

$$\Rightarrow \forall\ \Re(\rho^h) \neq \frac{1}{2}\left\{\nexists\ \Re(1-\rho^h) \neq \frac{1}{2}\left\{CriticallySymmetricTo\left\{\Re(1-\rho^h) \neq \frac{1}{2}, \Re(\rho^h) \neq \frac{1}{2}\right\}\right\}\right\} \qquad P_8$$

$$\Rightarrow \forall\ \rho\left\{\nexists\ \upsilon(\rho) \neq 0\left\{PossibleConditionOf\{\upsilon(\rho) \neq 0, \rho\}\right\}\right\} \qquad P_9$$

$$\Rightarrow \forall\ \rho\left\{\exists\ \upsilon(\rho) = 0\left\{NecessaryConditionFor\{\upsilon(\rho) = 0, \rho\}\right\}\right\} \qquad P_{10}$$

$$\Rightarrow \forall\ \rho\left\{\exists\ \tfrac{1}{2}\left\{RealPartOf\left\{\tfrac{1}{2}, \rho\right\}\right\}\right\} \qquad Q$$

P_1 Because of the definitions of algebra, trigonometry and complex arithmetic

P_2 Because of the implication of P_1 and because anything multiplied by zero is

	zero
P_3	Because of the implication of P_2 and because zero has no reciprocal
P_4	Because of the implication of P_3 and the definitions of complex arithmetic
P_5	Because of the implication of P_3 and the definition of the quadratic formula
P_6	Because of the implication of P_5 and the definitions of complex arithmetic
P_7	Because of the reverse implication of P_4 and P_6
P_8	Because of the implication of P_5 and the definitions of complex arithmetic
P_9	Because of the implication of P_8 and the definition of Riemann's symmetric vertices property
P_{10}	Because of the implication of P_3 and P_9
Q	Because of the implication of P_{10}