A magic formula $z^{\frac{1}{2}} = \pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}$ & "New" Way To Solve Quadratic Equations

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Abstract

In this paper we will give the formula of second roots (square root) of a complex number z=a+ib, which is very important and useful especially when we cannot write the complex number a+ib in the exponential or the geometric form. And, we will propose a precise direct method and fast to solve the equations of the second degree in the general case (which means with complex coefficients). We will also propose an algorithm that will make our calculations more easy and resolve fastly all type of equations of the second degree .

Keywords: Complex, Equations of the second degree, Algorithm, The second root.

1 Introduction:

We suppose that $z \in \mathbb{C} \backslash \mathbb{R}$ and we cannot write z in the exponential or the geometric form.

Hence, $\exists (a, b) \in \mathbb{R}$ / $z = a + ib$ (which means z is a complex not real with $b \neq 0$).

Thereafter, we write $z^{\frac{1}{2}}$ in the algebraic form $z^{\frac{1}{2}} = x + iy$ and we try to find a couple $(x, y) \in \mathbb{R}^2$ such that $(x + iy)^2 = a + ib$.

This step is a bit long, especially with the distressing calculations that repeats every time when practical work, specially when it comes to teach students how to find $z^{\frac{1}{2}}$ by hand, mostly when we are solving an equation of second degree with complex coefficients or a second degree differential equations.

This article aims to find once and for all $z^{\frac{1}{2}}$ in function of z (and so $\Delta^{\frac{1}{2}}$ in function of Δ) and therefore result in a direct algorithm, exact and very fast for us to determine the solutions of equation of second degree with complex coefficients.

2 The second root:

2.1 $z \in \mathbb{R}$

$$z^{\frac{1}{2}} = \pm i^{\frac{1-signe(z)}{2}} \sqrt{|z|}$$

where

if $z \geq 0$ then $signe(z) = +1$and $|z| = z$

if $z \leq 0$ then $signe(z) = -1$and $|z| = -z$

2.2 $z \notin \mathbb{R}$

Which means that $z \in \mathbb{C} \backslash \mathbb{R}$

$$z^{\frac{1}{2}} = \pm \frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}$$

See [1], [2]

where

$|z| = \sqrt{Re(z)^2 + Im(z)^2}$ (z's module) or $|z| = \sqrt{z\overline{z}}$

$Re(z)$ (The real part of z)

$Im(z)$ (the imaginary part of z).

Proof.

$$\left(\pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}\right)^2 = \frac{|z|^2+2\,|z|\,z+z^2}{2\,(|z|+Re(z))} = \frac{z\overline{z}+2\,|z|\,z+z^2}{2\,(|z|+Re(z))}$$

$$= \frac{z}{2}\left(\frac{\overline{z}+2\,|z|+z}{|z|+Re(z)}\right) = \frac{z}{2}\left(\frac{2\,|z|+2Re(z)}{|z|+Re(z)}\right)$$
$$= z$$

because $\overline{z}+z=2Re(z)$ and $z\overline{z}=|z|^2$. ■

Claim 1

$$\boxed{\begin{array}{c} \textit{If } z\in\mathbb{R} \quad \textit{then} \quad z^{\frac{1}{2}} = \pm i^{\frac{1-signe(z)}{2}}\sqrt{|z|} \\ \textit{If } z\notin\mathbb{R} \quad \textit{then } z^{\frac{1}{2}} = \pm\frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}} \end{array}}$$

3 The new way to solve an equation of second degre:

Let the equation:

$$az^2+bz+c=0$$

$$a\neq 0 \Longrightarrow z^2+\frac{b}{a}z+\frac{c}{a}=0$$

$$\Longleftrightarrow z^2+2\frac{b}{2a}z+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}=0$$
$$\Longleftrightarrow \left(z+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Let $\Delta\in\mathbb{C}$ such that $\Delta=b^2-4ac$.

$\left(z+\frac{b}{2a}\right)^2=\frac{\Delta}{4a^2}=\left(\frac{\Delta^{\frac{1}{2}}}{2a}\right)^2$

with $\Delta^{\frac{1}{2}}$ a second root of Δ (in $\mathbb{C}^*$ we have alwayse two second root $\pm\Delta^{\frac{1}{2}}$).

so

$$z=\frac{-b\pm\Delta^{\frac{1}{2}}}{2a}$$

We have two cases:

1. First case: If $\Delta \in \mathbb{R}$

$$\Delta^{\frac{1}{2}} = \pm i^{\frac{1-signe(\Delta)}{2}} \sqrt{|\Delta|}$$

so we can generalize:

$$z = \frac{-b \pm i^{\frac{1-signe(\Delta)}{2}} \sqrt{|\Delta|}}{2a} \qquad \text{if } \Delta \in \mathbb{R}$$

Where

if $\Delta \geq 0$ then $signe(\Delta) = +1$and $|\Delta| = \Delta$

if $\Delta \leq 0$ then $signe(\Delta) = -1$and $|\Delta| = -\Delta$

2. Second case: If $\Delta \notin \mathbb{R}$ (means that $\Delta \in \mathbb{C}\backslash\mathbb{R}$)

$$\Delta^{\frac{1}{2}} = \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} \text{ and } z = \frac{-b \pm \Delta^{\frac{1}{2}}}{2a}$$

so

$$z = \frac{-b \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}}{2a}$$

with

$\|\Delta\| = \sqrt{Re(\Delta)^2 + Im(\Delta)^2}$ (Δ's module) or $\|\Delta\| = \sqrt{\Delta\overline{\Delta}}$

$Re(\Delta)$ (The real part of Δ)

$Im(\Delta)$ (the imaginary part of Δ).

Claim 2

$$\boxed{\begin{array}{l} \textit{First case: If } \Delta \in \mathbb{R} \textit{ then } z = \frac{-b\pm i^{\frac{1-signe(\Delta)}{2}} \sqrt{|\Delta|}}{2a} \\ \textit{Second case: If } \Delta \notin \mathbb{R} \textit{ then } z = \frac{-b\pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}}{2a} \end{array}}$$

4 Algorithm

$\rightarrow a, b, c$
$\rightarrow \Delta = b^2 - 4ac$

$$\left| \begin{array}{l} \rightarrow \text{ if } \Delta \in \mathbb{R} \text{ so } z = \frac{-b \pm i^{\frac{1-signe(\Delta)}{2}} \sqrt{|\Delta|}}{2a} \quad \text{end} \\ \rightarrow \text{ else } \left| \begin{array}{l} \rightarrow \delta = \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} \\ \rightarrow z = \frac{-b\pm\delta}{2a} \quad end \end{array} \right. \end{array} \right.$$

5 Example 1:

Calculate all second root of $z = 21 + 20i$

It is impossible to write it in the exponential or the geometric form.

By applying our formula:

$$z^{\frac{1}{2}} = \pm \frac{\frac{|z|+z}{2}}{\sqrt{\frac{|z|+Re(z)}{2}}}$$

we have
$|z| = \sqrt{(21)^2 + (20)^2} = \sqrt{441 + 400} = \sqrt{841}$

then

$|z| = 29$

therefore,

$$z^{\frac{1}{2}} = \pm \frac{\frac{29+21+20i}{2}}{\sqrt{\frac{29+21}{2}}} = \pm \frac{25 + 10i}{\sqrt{25}}$$

$\Longrightarrow$

$$z^{\frac{1}{2}} = \pm (5 + 2i)$$

6 Example 2:

To solve in $\mathbb{C}$ the equation:

$$z^2 - (2 + i)z + 2i = 0$$

We have $\Delta = (2+i)^2 - 4 \times 2i = 4 - 1 + 4i - 8i$
$\Longrightarrow$

$$\Delta = 3 - 4i$$

$\Delta \in \mathbb{C}/\mathbb{R} \Longrightarrow \Delta^{\frac{1}{2}} = \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$ and $z = \frac{-b \pm \Delta^{\frac{1}{2}}}{2a}$
$\|\Delta\| = \sqrt{9+16}$

$\|\Delta\| = 5$

so

$$\Delta^{\frac{1}{2}} = \pm \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}} = \pm \frac{\frac{5+3-4i}{2}}{\sqrt{\frac{5+3}{2}}}$$

$\Longrightarrow$

$$\Delta^{\frac{1}{2}} = \pm \frac{4-2i}{2} = \pm(2-i)$$

$\Longrightarrow$

$$z = \frac{(2+i) \pm (2-i)}{2}$$

$\Longrightarrow$

$$S = \{i, 2\}$$

We can verify that

$$(z-i)(z-2) = z^2 - (2+i)z + 2i$$

.

7 Conclusion

Ultimately,

1. We found the general formula that gives us the second roots (square root) of a complex number.

   So, let $\alpha \in \mathbb{C}\backslash\mathbb{R}$, then

$X^2 = \alpha \Longleftrightarrow X = \pm \frac{\frac{\|\alpha\|+\alpha}{2}}{\sqrt{\frac{\|\alpha\|+Re(\alpha)}{2}}}$

2. We have a new methode and new way to resolve an equation of second degree with complex coefficients.There are two case:

$\Delta \in \mathbb{R}$ or not (with $\Delta = b^2 - 4ac$).

And, we gave all the fast and efficient methods and formulas to solve an equation of second degree with complex coefficients.

Reference:

[1] A. Arbai: Principes d'algèbre et de Géométrie - 2ème édition 37 - 40 (2015).

[2] A. Arbai: A methode to solve the equations of the second degree in the general case (with complex coefficients) - International Journal of Scientific Engineering and Applied Science Volume-2, Issue-12 (2016).