On a result in the sieve method *

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Abstract

In this note, we generalize a lemma of Heath-Brown in the sieve method.

Keywords: Möbius function, primes, induction.

1 Introduction

Let $\mu(d)$ be the Möbius function, $\Omega(n)=\sum\limits_{p|n}1,\ p$ is a prime variable.

In [2], Heath-Brown obtained the following result.

Lemma A. Let $z_1,\ z_2>1,\ and\ Q=\prod\limits_{p<z_1}p.$

Then

$$|\sum_{d|(n,Q)}\mu(d)-\sum_{d|(n,Q),\ d<z_2}\mu(d)|\leq\sum_{d|(n,Q),\ z_2\leq d<z_1z_2}1.$$

For applications of this lemma see [1] and [2].

We give a generalization to this lemma.

Write $\wp=\{p:p\ is\ prime\},\ z_1,\ z_2>1,\ \forall\mathcal{P}\subseteq\wp,\ Q=Q(\mathcal{P})=\prod\limits_{p<z_1,\ p\in\mathcal{P}}p,$

if $\mathcal{P}$ is empty set, that is $\mathcal{P}=\emptyset,$ then $Q(\emptyset)=1.$

Lemma B. Let $z_1,\ z_2>1,\ and\ \forall\mathcal{P}\subseteq\wp,\ Q=Q(\mathcal{P})=\prod\limits_{p<z_1,\ p\in\mathcal{P}}p.$

Then

$$|\sum_{d|(n,Q)}\mu(d)-\sum_{d|(n,Q),\ d<z_2}\mu(d)|\leq\sum_{d|(n,Q),\ z_2\leq d<z_1z_2}1.$$

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2 Proof of Lemma B

We use induction on $\Omega(n)$.

The proof is the same as that in [2].

If $n = 1$ *or* $\mathcal{P} = \emptyset$, the lemma holds.

We assume the result is true for n and for all $\mathcal{P} \subseteq \wp$.

We want to prove the lemma for pn and for all $\mathcal{P} \subseteq \wp$.

If $p \geq z_1$ or $p|n$ or $p \notin \mathcal{P}$, then $(np, Q) = (n, Q)$, the result follows by induction.

We suppose $p < z_1,\ p \nmid n$ and $p \in \mathcal{P}$.

$$\sum_{d|(n,Q)} \mu(d) - \sum_{d|(n,Q),\ d<z_2} \mu(d) = \sum_{d|(n,Q),\ d\geq z_2} \mu(d). \tag{2.1}$$

We want to prove

$$|\sum_{d|(pn,Q),\ d\geq z_2} \mu(d)| \leq \sum_{d|(pn,Q),\ z_2\leq d<z_1z_2} 1. \tag{2.2}$$

We have

$$\sum_{d|(pn,Q),\ d\geq z_2} \mu(d) = \sum_{d|(n,Q),\ d\geq z_2} \mu(d) + \sum_{pd|(pn,Q),\ pd\geq z_2} \mu(pd) = \sum\nolimits_1 + \sum\nolimits_2,\ say. \tag{2.3}$$

By the induction assumption, we have

$$|\sum\nolimits_1| \leq \sum_{d|(n,Q),\ z_2\leq d<z_1z_2} 1. \tag{2.4}$$

$\sum_2 = -\sum_{pd|(pn,Q),\ d\geq z_2/p} \mu(d) = -\sum_{d|(n,Q),\ d\geq z_2/p} \mu(d).$

If $p < z_2$, then by the induction assumption,

$|\sum_2| \leq \sum_{d|(n,Q),\ z_2/p\leq d<z_1z_2/p} 1.$

Hence

$$|\sum\nolimits_1 + \sum\nolimits_2| \leq \sum_{d|(n,Q),\ z_2\leq d<z_1z_2} 1 + \sum_{d|(n,Q),\ z_2/p\leq d<z_1z_2/p} 1 = \sum_{d|(pn,Q),\ z_2\leq d<z_1z_2} 1. \tag{2.5}$$

If $p \geq z_2$ and $(n, Q) > 1$, in this case $z_2 < z_1$, since $p \nmid n$ and $p \in \mathcal{P}$, then

$$|\sum_2| = |-\sum_{d|(n,Q)} \mu(d)| = 0 \leq \sum_{pd|(pn,Q),\ z_2 \leq pd < z_1 z_2} 1.$$

Thus, unless $p \geq z_2$ and $(n, Q) = 1$, we obtain

$$|\sum_1 + \sum_2| \leq \sum_{d|(pn,Q),\ z_2 \leq d < z_1 z_2} 1.$$

If $p \geq z_2$ and $(n, Q) = 1$, since $p < z_1$ and $p \in \mathcal{P}$, in this case $z_2 < z_1$, we have

$$\sum_{d|(pn,Q),\ d \geq z_2} \mu(d) = \sum_{d|(p,Q),\ d \geq z_2} \mu(d) = -1,$$

hence

$$|\sum_{d|(pn,Q),\ d \geq z_2} \mu(d)| \leq \sum_{d|(pn,Q),\ z_2 \leq d < z_1 z_2} 1.$$

The lemma follows.

References

[1] R. C. Baker, G. Harman and J. Pintz, The difference between consecutive primes, II, Proc. London Math. Soc. (3) 83(2001) 532-562.

[2] D. R. Heath-Brown, The number of primes in a short interval, J. Reine Angew. Math. 389(1988) 22-63.