Progress in The Proof of The Conjecture $c < \text{rad}^2(abc)$ - Case : $c = a + 1$

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Received: date / Accepted: date

Abstract In this paper, we consider the $abc$ conjecture. We give some progress in the proof of the conjecture $c < \text{rad}^2(abc)$ in the case $c = a + 1$.

Keywords Elementary number theory · real functions of one variable · Number of solutions of elementary Diophantine equations.

Mathematics Subject Classification (2010) 11AXX · 26AXX

To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed
Mazen

1 Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$, noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i-1}$$

(1)

We note:

$$\mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a)$$

(2)

The $abc$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Esterlé of Pierre et Marie Curie University (Paris 6)) ([4]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $abc$ conjecture is given below:

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Conjecture 1 (abc Conjecture): Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then for each \( \epsilon > 0 \), there exists \( K(\epsilon) \) such that:

\[
c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}\tag{3}
\]

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) ([2]). A conjecture was proposed that \( c < \text{rad}^2(abc) \) ([1]). Here we will give a proof of it for the case \( c = a + 1 \).

Conjecture 2 Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2 \tag{4}
\]

This result, I think is the key to obtain the final proof of the veracity of the abc conjecture.

2 A Proof of the conjecture ([2]) case \( c = a + 1 \)

Let \( a, c \) positive integers, relatively prime, with \( c = a + 1 \) and \( R = \text{rad}(ac) \),

\[
c = \prod_{p \in J'} c_{j'}^{\beta_{j'}} \left( \beta_{j'} \geq 1 \right).
\]

If \( c < \text{rad}(ac) \) then we obtain:

\[
c < \text{rad}(ac) < \text{rad}^2(ac) \implies c < R^2
\]

and the condition (4) is verified.

If \( c = \text{rad}(ac) \), then \( a, c \) are not coprime, case to reject.

In the following, we suppose that \( c > \text{rad}(ac) \) and \( c \) and \( a \) are not prime numbers.

\[
c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac)\tag{6}
\]

2.1 \( \mu_a \neq 1, \mu_a \leq \text{rad}(a) \)

We obtain:

\[
c = a + 1 < 2\mu_a \text{rad}(a) \Rightarrow c < 2\text{rad}^2(a) \Rightarrow c < \text{rad}^2(ac) \implies c < R^2\tag{7}
\]

Then (6) is verified.
2.2 $\mu_c \neq 1$, $\mu_c \leq \text{rad}(c)$

We obtain:

$$c = \mu_c \text{rad}(c) \leq \text{rad}^2(c) < \text{rad}^2(ac) \implies c < R^2$$

(8)

and the condition (8) is verified.

2.3 $\mu_a > \text{rad}(a)$ and $\mu_c > \text{rad}(c)$

2.3.1 Case: $\mu_a = \text{rad}^p(a), q \geq 2$, $\mu_c = \text{rad}^p(c), p \geq 2$

In this case, we write $c = a + 1$ as $\text{rad}^{p+1}(c) - \text{rad}^{q+1}(a) = 1$. Then $\text{rad}(c), \text{rad}(a)$ are solutions of the Diophantine equation:

$$X^{p+1} - Y^{q+1} = 1 \text{ with } (p + 1)(q + 1) \geq 9$$

(9)

But the solutions of the equation (9) are: $X = \pm 3, p + 1 = 2, Y = \pm 2, q + 1 = 3$, we obtain $p = 1 < 2$, then $\text{rad}(c), \text{rad}(a)$ are not solutions of (9) and the case $\mu_a = \text{rad}^p(a), q \geq 2, \mu_c = \text{rad}^p(c), p \geq 2$ is to reject.

2.3.2 Case: $\text{rad}(c) < \mu_c < \text{rad}^2(c)$ and $\text{rad}(a) < \mu_a < \text{rad}^2(a)$:

We can write:

$$\mu_c < \text{rad}^2(c) \implies c < \text{rad}^3(c)$$

$$\mu_a < \text{rad}^2(a) \implies a < \text{rad}^3(a)$$

$$\implies ac < R^3 \implies a^2 < ac < R^3 \implies a < R\sqrt{R} < R^2 \implies c = a + 1 < R^2$$

(10)

2.3.3 Case: $\mu_a > \text{rad}^2(c) \text{ or } \mu_a > \text{rad}^2(a)$

I- We suppose that $\mu_c > \text{rad}^2(c)$ and $\text{rad}(a) < \mu_a \leq \text{rad}^2(a)$:

I-1- Case $\text{rad}(a) < \text{rad}(c)$: In this case $a = \mu_a, \text{rad}(a) \leq \text{rad}^2(a).\text{rad}(a) < \text{rad}^2(a)\text{rad}(c) < \text{rad}^2(ac) \implies a < R^2 \implies c < R^2$.

I-2- Case $\text{rad}(c) < \text{rad}(a) < \text{rad}^2(c)$: As $a \leq \text{rad}^2(a).\text{rad}(a) < \text{rad}^2(a).\text{rad}^2(c) \implies a < R^2 \implies c < R^2$.

Example: $2^{30}.5^2.127.353^2 = 3^7.5^3.13^2.17.1831 + 1$, $\text{rad}(c) = 25.127.353 = 448310$, $\text{rad}^2(c) = 200981856100$.

$\mu_c = 2^{29}.5.353 = 947577159680 \implies \text{rad}^2(c) < \mu_c < \text{rad}^3(c)$,

$\text{rad}(a) = 3.5.13.17.1831 = 6069765, \text{rad}^2(a) = 36842047155225$,

$\mu_a = 3^6.5^4.13^4 = 13013105625 < \text{rad}^2(a)$. It is the case: $\text{rad}(c) < \mu_c < \text{rad}^2(c)$ and $\text{rad}(a) < \mu_a \leq \text{rad}^2(a)$ with $\text{rad}(c) = 448310 < \text{rad}(a) =
6069 765 < \text{rad}^2(c) = 200981 856 100.

I-3- Case \text{rad}^2(c) < \text{rad}(a):

I-3-1- We suppose that \( c \leq \text{rad}^6(c) \), we obtain:

\[
\begin{align*}
&c \leq \text{rad}^6(c) \implies c \leq \text{rad}^2(c) \cdot \text{rad}^4(c) \implies c < \text{rad}^2(c) \cdot (\text{rad}(a))^2 = R^2 \implies c < R^2
\end{align*}
\]

Example: \( 5^8 \cdot 7^2 = 2^4 \cdot 3^7 \cdot 547 + 1 \implies 19140625 = 19140624 + 1, \text{rad}(c) = 5.7 = 35, \text{rad}(a) = 2.3547 = 3282 \implies \text{rad}(a) > \text{rad}^2(c) \), we obtain \( c = 19140625 > \text{rad}^6(c) = 42875 \) and \( c < \text{rad}^6(c) = 1838265625 \) and \( 3282 = \text{rad}(a) < \mu_a = 5832 < \text{rad}^2(a) = 10771524 \implies a < \text{rad}^3(a) = 35352141768. \)

I-3-2- We suppose that \( c > \text{rad}^6(c) \implies \mu_c > \text{rad}^5(c) \), we suppose \( \mu_a = \text{rad}^2(a) \implies a = \text{rad}^2(a) \). Then we obtain that \( x = \text{rad}(a) \) is a solution in positive integers of the equation:

\[
X^3 + 1 = c = \mu_c \cdot \text{rad}(c) \quad (11)
\]

If \( c = \text{rad}^n(c) \) with \( n \geq 7 \), we obtain an equation like (9) that gives a contradiction. In the following, we will study the cases \( \mu_c = A \cdot \text{rad}^n(c) \) with \( \text{rad}(c) \nmid A, n \geq 0 \). The above equation (11) can be written as:

\[
(X + 1)(X^2 - X + 1) = c
\]

Let \( \delta \) any divisor of \( c \), then:

\[
X + 1 = \delta \quad (13)
\]

\[
X^2 - X + 1 = \frac{c}{\delta} = \delta' = \delta^2 - 3X \quad (14)
\]

We recall that \( \text{rad}(a) > \text{rad}^2(c) \), it follows that \( \delta \) must verifies \( \delta - 1 > \text{rad}^2(c) \implies \delta > \text{rad}^2(c) + 1. \)

I-3-2-1- We suppose that \( \delta = l \cdot \text{rad}(c) \implies l \cdot \text{rad}(c) > \text{rad}^2(c) + 1 \implies l > \frac{\text{rad}^2(c) + 1}{\text{rad}(c)} \). We obtain \( l \geq \text{rad}(c) + 2 \) so \( \text{rad}(c) \) and \( l \) have the same parity.

We have \( \delta = l \cdot \text{rad}(c) < c = \mu_c \cdot \text{rad}(c) \implies l < \mu_c \). As \( \delta \) is a divisor of \( c \), then \( l \) is a divisor of \( \mu_c \), we write \( \mu_c = l \cdot m. \) From \( \mu_c = l(\delta^2 - 3X) \), we obtain:

\[
m = l^2 \cdot \text{rad}^2(c) - 3 \cdot \text{rad}(a) \implies 3 \cdot \text{rad}(a) = l^2 \cdot \text{rad}^2(c) - m
\]

A- Case \( 3|m \implies m = 3m', m' > 1 \): As \( \mu_c = ml = 3m' \implies 3|\text{rad}(c) \) and \( (\text{rad}(c), m') \) not coprime. We obtain:

\[
\text{rad}(a) = l^2 \cdot \text{rad}(c) \cdot \frac{\text{rad}(c)}{3} - m'
\]
It follows that \( a, c \) are not coprime, then the contradiction.

B - Case \( m = 3 \Rightarrow \mu_c = 3! \Rightarrow c = 3\text{rad}(c) = 3\delta = \delta(\delta^2 - 3X) \Rightarrow \delta^2 = 3(1 + X) = 3\delta \Rightarrow \delta = l\text{rad}(c) = 3 \), then the contradiction.

I-3-2-2 - We suppose that \( \delta = l\text{rad}^2(c), l \geq 2 \). In this case \( \text{rad}(a) = l\text{rad}^2(c) - 1 \) verifies \( \text{rad}(a) > \text{rad}^2(c) \). If \( l\text{rad}(c) \nmid \mu_c \) then the case to reject. We suppose that \( l\text{rad}(c)|\mu_c \Rightarrow \mu_c = m\text{rad}(c) \), then \( \frac{\delta}{\delta} = m = \delta^2 - 3\text{rad}(a) \).

C - Case \( m = 1 = c/\delta \Rightarrow \delta^2 - 3\text{rad}(a) = 1 \Rightarrow (\delta - 1)(\delta + 1) = 3\text{rad}(a) = \text{rad}(a)(\delta + 1) \Rightarrow \delta = 2 = l\text{rad}^2(c) \), then the contradiction.

D - Case \( m = 3 \), we obtain \( 3(1 + \text{rad}(a)) = \delta^2 = 3\delta \Rightarrow \delta = 3 = l\text{rad}^2(c) \). Then the contradiction.

E - Case \( m \neq 1, 3 \), we obtain: \( 3\text{rad}(a) = l^2\text{rad}^4(c) - m \Rightarrow \text{rad}(a) \) and \( \text{rad}(c) \) are not coprime. Then the contradiction.

I-3-2-3 - We suppose that \( \delta = l\text{rad}^n(c), l \geq 2 \) with \( n \geq 3 \). From \( c = \mu_c\text{rad}(c) = l\text{rad}^n(c)(\delta^2 - 3\text{rad}(a)) \), let \( m = \delta^2 - 3\text{rad}(a) \).

F - As seen above (paragraphs C,D), the cases \( m = 1 \) and \( m = 3 \) give contradictions, it follows the reject of these cases.

G - Case \( m \neq 1, 3 \). Let \( q \) a prime that divides \( m \), it follows \( q|\mu_c \Rightarrow q = c_j^\beta_j \Rightarrow c_j^\beta_j|\delta^2 \Rightarrow c_j^\beta_j|3\text{rad}(a) \). Then \( \text{rad}(a) \) and \( \text{rad}(c) \) are not coprime. It follows the contradiction.

I-3-2-4 - We suppose that \( \delta = \prod_{j \in J_1} c_j^\beta_j, \beta_j \geq 1 \) with at least one \( j_0 \in J_1 \) with \( \beta_{j_0} \geq 2 \), \( \text{rad}(c) \nmid \delta \) and \( \delta - 1 = \prod_{j \in J_1} c_j^\beta_j - 1 > \text{rad}^2(c) = \prod_{j' \in J_1} c_{j'}^2, J_1 \subset J' \). We can write:

\[
\delta = \mu_\delta\text{rad}(\delta), \quad \text{rad}(c) = m\text{rad}(\delta)
\]

Then we obtain:

\[
c = \mu_c\text{rad}(c) = \mu_c m\text{rad}(\delta) = \delta(\delta^2 - 3X) = \mu_\delta\text{rad}(\delta)(\delta^2 - 3X) \Rightarrow m, \mu_c = \mu_\delta(\delta^2 - 3X)
\]

- If \( \mu_c = \mu_\delta \Rightarrow m = \delta^2 - 3X = (\mu_\delta\text{rad}(\delta))^2 - 3X \). As \( \delta < \delta^2 - 3X \Rightarrow m > \delta \Rightarrow \text{rad}(c) > m > \mu_c\text{rad}(\delta) > \text{rad}^2(c) \) because \( \mu_c > \text{rad}^2(c) \), it follows \( \text{rad}(c) > \text{rad}^2(c) \). Then the contradiction.
- We suppose that $\mu_c < \mu_\delta$. As $rad(a) = \mu_\delta rad(\delta) - 1$, we obtain:

$$rad(a) > \mu_c rad(\delta) - 1 > 0 \implies R > c rad(\delta) - rad(c) > 0 \implies$$
$$c > R > c rad(\delta) - rad(c) > 0 \implies 1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2$$

$\implies$ The contradiction (16)

- We suppose that $\mu_\delta < \mu_c$. In this case, from the equation (25) and as $(m, \mu_\delta) = 1$, it follows that we can write:

$$\mu_c = \mu_1 \mu_2, \quad \mu_1, \mu_2 > 1 \quad (17)$$

so that

$$m \mu_1 = \delta^2 - 3 X, \quad \mu_2 = \mu_\delta \quad (18)$$

But:

$$rad(a) = \delta - 1 = \mu_\delta rad(\delta) > rad^2(c) \implies 0 > m^2 rad^2(\delta) - \mu_2 rad(\delta) + 1$$

Let $P(Z)$ the polynomial:

$$P(Z) = m^2 Z - \mu_2 Z + 1 \implies P(rad(\delta)) < 0 \quad (19)$$

The discriminant of $P(Z)$ is:

$$\Delta = \mu_2^2 - 4 m^2 \quad (20)$$

- $\Delta = 0 \implies \mu_2 = 2m$, but $(m, \mu_2) = 1$, then the contradiction. Case to reject.

- $\Delta < 0 \implies P(Z)$ has no real roots. From (19) it follows that $P(Z) > 0, \forall Z \in \mathbb{R}$. Then the contradiction with $P(rad(\delta)) < 0$. Case to reject.

- $\Delta > 0 \implies \mu_2 > 2m \implies \frac{\mu_2}{m} > 2$. We denote $t = \sqrt{\Delta} > 0$. The roots of $P(Z) = 0$ are $Z_1, Z_2$ with $Z_1 < Z_2$, given by:

$$Z_1 = \frac{\mu_2 - t}{2m^2}, \quad Z_2 = \frac{\mu_2 + t}{2m^2} \quad (21)$$

We approximate $t$ by $\tilde{t}$:

$$t = \sqrt{\mu_2^2 - 4m^2} = \mu_2 \left(1 - \frac{4m^2}{\mu_2^2}\right)^{\frac{1}{2}} \implies \tilde{t} = \mu_2 - \frac{2m^2}{\mu_2} > 0$$

Then, we obtain $\tilde{Z}_1, \tilde{Z}_2$ as :

$$\tilde{Z}_1 = \frac{\mu_2 - \tilde{t}}{2m^2} = \frac{1}{\mu_2}, \quad \tilde{Z}_2 = \frac{\mu_2 + \tilde{t}}{2m^2} = \frac{\mu_2}{m^2} - \frac{1}{\mu_2} \quad (22)$$
As $\mu_2^2 - 4m^2 > 0 \implies \mu_2^2 - m^2 > 3m^2 > 0 \implies \frac{\mu_2^2}{m^2} - 1 > 0$, we will give below the proof that $\text{rad}(\delta) > \tilde{Z}_2 \implies P(\text{rad}(\delta)) > 0$, then the contradiction with $P(\text{rad}(\delta)) < 0$; we write:

$$\text{rad}(\delta) > \frac{\mu_2^2}{m^2} - \frac{1}{\mu_2}, \quad \mu_2 > 0 \implies$$

$$\mu_2 \cdot \text{rad}(\delta) > \frac{\mu_2^2}{m^2} - 1$$

$$\delta > \frac{\mu_2^2 - m^2}{m^2} > \frac{3m^2}{m^2}$$

as $\delta > 3 \implies \delta > \frac{\mu_2^2}{m^2} - 1 > 3 \implies \text{rad}(\delta) > \frac{\mu_2}{m^2} > \frac{1}{\mu_2} > \frac{3}{\mu_2}$  \hspace{1cm} (23)

If follows $P(\text{rad}(\delta)) > 0$ and the contradiction with the conclusion of the equation \[19\].

It follows that the case $c > \text{rad}^6(c)$ and $a = \text{rad}^3(a)$ is impossible.

I-3-3- We suppose $c > \text{rad}^6(c) \implies c = \text{rad}^6(c) + h$, $h > 0$ and $\mu_a < \text{rad}^2(a) \implies a + l = \text{rad}^3(a)$, $l > 0$. Then we obtain:

$$\text{rad}^6(c) + h = \text{rad}^3(a) - l + 1$$ \hspace{1cm} (24)

As $\text{rad}^2(c) < \text{rad}(a)$ (see I-3), we obtain the equation:

$$\text{rad}^3(a) - (\text{rad}^2(c))^3 = h + l - 1 = m > 0$$

Let $X = \text{rad}(a) - \text{rad}^2(c)$, then $X$ is an integer root of the polynomial $H(X)$ defined as:

$$H(X) = X^3 + 3R \cdot \text{rad}(c)X - m = 0$$ \hspace{1cm} (25)

To resolve the above equation, we note $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m, \quad u \cdot v = -R \cdot \text{rad}(c) < 0 \implies u^3 \cdot v^3 = -R^3 \text{rad}^3(c)$$

It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - R^3 \text{rad}^3(c) = 0$$ \hspace{1cm} (26)

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4R^3 \text{rad}^3(c) = \alpha^2, \quad \alpha > 0$$ \hspace{1cm} (27)

The two real roots of (26) are:

$$t_1 = u^3 = \frac{m + \alpha}{2}$$ \hspace{1cm} (28)

$$t_2 = v^3 = \frac{m - \alpha}{2}$$ \hspace{1cm} (29)
As \( m = \text{rad}^2(a) - \text{rad}^6(c) > 0 \), we obtain that \( \alpha = \text{rad}^2(a) + \text{rad}^6(c) > 0 \), then from the equation (27), it follows that \((\alpha = x, m = y)\) is a solution of the Diophantine equation:

\[
x^2 - y^2 = N
\]

with \( N = 4R^3\text{rad}^3(c) > 0 \). From the equations (28-29), we remark that \( \alpha \) and \( m \) verify the following equations:

\[
x + y = 2u^3 = 2\text{rad}^3(a) \tag{31}
\]
\[
x - y = -2v^3 = 2\text{rad}^6(c) \tag{32}
\]
\[
\quad \text{then } x^2 - y^2 = N = 4R^3\text{rad}^3(c) \tag{33}
\]

Let \( Q(N) \) be the number of the solutions of (30) and \( \tau(N) \) is the number of suitable factorization of \( N \), then we announce the following result concerning the solutions of the Diophantine equation (30) (see theorem 27.3 in [3]):

- If \( N \equiv 2(\text{mod } 4) \), then \( Q(N) = 0 \).
- If \( N \equiv 1 \) or \( N \equiv 3(\text{mod } 4) \), then \( Q(N) = [\tau(N)/2] \).
- If \( N \equiv 0(\text{mod } 4) \), then \( Q(N) = [\tau(N/4)/2] \).

\([x] \) is the integral part of \( x \) for which \( [x] \leq x < [x] + 1 \).

Let \( (\alpha', m') \), \( \alpha', m' \in \mathbb{N}^* \) be another pair, solution of the equation (30), then \( \alpha'^2 - m'^2 = x^2 - y^2 = N = 4R^3\text{rad}^3(c) \), but \( \alpha = x \) and \( m = y \) verify the equation (31) given by \( x + y = 2\text{rad}^3(a) \), it follows \( \alpha', m' \) verify also \( \alpha' + m' = 2\text{rad}^3(a) \), that gives \( \alpha' - m' = 2\text{rad}^6(c) \), then \( \alpha' = x = \alpha = \text{rad}^2(a) + \text{rad}^6(c) \) and \( m' = y = m = \text{rad}^3(a) - \text{rad}^6(c) \). We have given the proof of the uniqueness of the solutions of the equation (30) with the condition \( x + y = 2\text{rad}^3(a) \). As \( N = 4R^3\text{rad}^3(c) \equiv 0(\text{mod } 4) \) \( \Rightarrow Q(N) = [\tau(N/4)/2] = [\tau(\text{rad}^6(c).\text{rad}^3(a))/2] > 1 \). But \( Q(N) = 1 \), then the contradiction.

It follows that the case \( \mu_a \leq \text{rad}^2(a) \) and \( c > \text{rad}^6(a) \) is impossible.

II- We suppose that \( \text{rad}(c) < \mu_c \leq \text{rad}^2(c) \) and \( \mu_a > \text{rad}^6(a) \):

II-1- Case \( \text{rad}(c) < \text{rad}(a) \): As \( c \leq \text{rad}^3(c) = \text{rad}^2(c).\text{rad}(c) \Rightarrow c < \text{rad}^2(c).\text{rad}(a) \Rightarrow c < R^2 \)

II-2- Case \( \text{rad}(a) < \text{rad}(a) < \text{rad}^2(a) \): As \( c \leq \text{rad}^3(c) = \text{rad}^2(c).\text{rad}(c) \Rightarrow c < \text{rad}^2(c).\text{rad}(a) \Rightarrow c < R^2 \)

II-3- Case \( \text{rad}^2(a) < \text{rad}(c) \):

II-3-1- We suppose que \( a \leq \text{rad}^6(a) \Rightarrow a \leq \text{rad}^2(a).\text{rad}^4(a) \Rightarrow a < \text{rad}^2(a).\text{rad}(c)^2 = R^2 \Rightarrow a < R^2 \Rightarrow 1 + a \leq R^2 \), but \((c, a) = 1\), it follows \( c < R^2 \).
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II-3-2. We suppose $a > \text{rad}^6(a)$ and $\mu_c \leq \text{rad}^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting $a,c$), we arrive at a contradiction. It follows that the case $\mu_c \leq \text{rad}^2(c)$ and $a > \text{rad}^6(a)$ is impossible.

2.3.4 III - Case $\mu_c > \text{rad}^2(c)$ and $\mu_a > \text{rad}^2(a)$

We can write $c > \text{rad}^3(c) \Rightarrow c = \text{rad}^3(c) + h$ and $a = \text{rad}^3(a) + l$ with $h,l > 0$ positive integers.

III-1. We suppose $\text{rad}^3(a) < \text{rad}^3(c)$. We obtain the equation:

$$\text{rad}^3(c) - \text{rad}^3(a) = l - h + 1 = m > 0$$ (34)

Let $X = \text{rad}(c) - \text{rad}(a)$, from the above equation, $X$ is a real root of the polynomial:

$$P(X) = X^3 + 3RX - m = 0$$ (35)

As above, to resolve (35), we put $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m$$ (36)
$$uv = -R < 0 \Rightarrow u^3 \cdot v^3 = -R^3$$ (37)

Then $u^3, v^3$ are the roots of the equation:

$$H(Z) = Z^2 - mZ - R^3 = 0$$ (38)

The discriminant of $H(Z)$ is:

$$\Delta = m^2 + 4R^3 = (\text{rad}^3(c) + \text{rad}^3(a))^2 = \alpha^2, \quad \text{taking} \quad \alpha > 0 \Rightarrow \alpha = \text{rad}^3(c) + \text{rad}^3(a)$$ (39)

From the equation (39), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N$$ (40)

with $N = 4R^3 > 0$ and $N \equiv 0 \pmod{4}$. Using the same method as in I-3-3-, we arrive to a contradiction.

III-2. We suppose $\text{rad}(c) < \text{rad}(a)$. We obtain the equation:

$$\text{rad}^3(a) - \text{rad}^3(c) = h - l - 1 = m > 0$$ (41)

Let $X = \text{rad}(a) - \text{rad}(c)$, from the above equation, $X$ is a real root of the polynomial:

$$P(X) = X^3 + 3RX - m = 0$$ (42)

As above, to resolve (42), we put $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m$$ (43)
$$uv = -R < 0 \Rightarrow u^3 \cdot v^3 = -R^3$$ (44)
Then $u^3, v^3$ are the roots of the equation:

$$H(Z) = Z^2 - mZ - R^3 = 0 \quad (45)$$

The discriminant of $H(Z)$ is:

$$\Delta = m^2 + 4R^3 = (\text{rad}^3(c) + \text{rad}^3(a))^2 = \alpha^2,$$

taking $\alpha > 0 \Rightarrow \alpha = \text{rad}^3(c) + \text{rad}^3(a) \quad (46)$

From the equation (46), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N \quad (47)$$

with $N = 4R^3 > 0$ and $N \equiv 0 \pmod{4}$. Using the same method as in I-3-3-, we arrive to a contradiction.

It follows that the case $\mu_c > \text{rad}^2(c)$ and $\mu_a > \text{rad}^2(a)$ is impossible.

We can announce the following theorem:

**Theorem 1 (Abdelmajid Ben Hadj Salem, 2020)** Let $a, c$ positive integers relatively prime with $c = a + 1$, then $c < \text{rad}^2(ac)$.

**References**