The congruence speed formula

Marco Ripà
sPIqr Society, World Intelligence Network
Rome, Italy
e-mail: marcokrt1984@yahoo.it

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Abstract: We solve a few open problems related to a peculiar property of the integer tetration \( b^a \), which is the constancy of its congruence speed for any sufficiently large \( b = b(a) \). Assuming radix-10 (the well known decimal numeral system), we provide an explicit formula for the congruence speed \( V(a) \in \mathbb{N}_0 \) of any \( a \in \mathbb{N}_1 \) that is not a multiple of 10, hypothesizing also a strict condition on \( b(a) \) to guarantee that \( V(a, b) = V(a) \) always holds. In particular, for any given \( n \in \mathbb{N} \), we prove to be true Ripà’s conjecture on the smallest \( a \) such that \( V(a) = n \).

Keywords: Tetration, Decadic number, Exponentiation, Integer sequence, Congruence speed, Conjecture, Modular arithmetic, Stable digit, Radix-10, Periodicity, General solution, Formula.

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1 Introduction

The aim of this paper is to give a general formula for the “congruence speed” of tetration [11, 15], affirmatively answering the final conjecture stated in [14]. The properties that arise from our study [17] are valid for many different numeral systems [1, 16], but (from here on out) we assume radix-10.

First of all, let we introduce the constancy of the congruence speed of the integer tetration \( b^a \).

**Definition 1.** Let \( a \in \mathbb{N} - \{0, 1\} \) not be a multiple of 10. Let \( d \in \mathbb{N} \). The power tower of height \( b \in \mathbb{N} - \{0\} \) represents the integer tetration \( b^a := a^{(b-1)a} \). Given \( b^{-1}a \equiv b^a \mod 10^d \) \( \land \) \( b^{-1}a \not\equiv b^{a+b} \mod 10^{d+1} \), \( \forall b > a \geq 2 \), \( V(a, b) = V(a) \) returns the strictly positive integer such that \( b^a \equiv b^{a+b} \mod 10^{d+V(a)} \) \( \land \) \( b^a \not\equiv b^{a+b} \mod 10^{d+V(a)+1} \), and we define \( V(a) \) as the “constant congruence speed” of the base \( a \).

Now, let we assume \( a \in \mathbb{N} : a \not\equiv 0 \mod 10 \) in the rest of the paper. Since it is known [14] that \( b - 1 \geq a \geq 2 \) is a sufficient but not necessary condition for \( V(a, b) = V(a) \), let \( b > a \geq 2 \) unless differently specified.
2 A formula for the constant congruence speed of $a$

In the present Section we study $V(a)$, taking into account every $a \not\equiv 0 \pmod{10}$ [11]. In the first subsection, for any given $V(a) = n \in \mathbb{N} - \{0, 1\}$, we show which are the smallest bases $a_1, a_2, \ldots, a_9$ whose residues in modulo 10 are 1, 2, ..., 9, respectively. The second subsection is devoted to provide a general formula which maps any $a$ whose constant congruence speed is given, for any $V(a, b) = V(a) \in \mathbb{N}$.

2.1 Finding bases with arbitrarily large $V(a)$ in the ring of the decadic integers

In order to describe the structure of $V(a, b) = V(a) \in \mathbb{N} - \{0\}$ in radix-10, for any sufficiently large base $a \not\equiv 0 \pmod{10}$, it can be useful to move the problem on $\mathbb{Z}_{10}$, the ring of the 10-adic integers.

Proposition 1. The 10-adic integers form a commutative ring, and we indicate it as $\mathbb{Z}_{10}$ [2].

Proposition 2. Any positive integer can be represented as a 10-adic integer $\alpha$. $\alpha$ can be written as an infinitely long string of digits going from right to left of a fixed digit. The aforementioned fixed digit, that we indicate as $s_1$, is the one which defines the congruence class (AKA residue modulo 10) of the corresponding base of the tetration $b^a$.

In particular, for any $n = 1, 2, 3, \ldots$ , let us consider $s_{r-n}s_{(n-1)}\ldots s_2s_1 \in \mathbb{Z}_{10}$, the residues modulo $10^n$ which satisfy the congruence relation $s_{n+1} \equiv s_0 \pmod{10^n}$. Now, let $s_{j+2} \in \{0, 1, 2, \ldots, 9\}$ and $s_1 \in \{1, 2, 3, \ldots, 9\}$. We have that $a_{s_1} := \sum_{j=0}^{n-1} s_{j+1} \cdot 10^j \equiv a_{s_1} \equiv s_1 \pmod{10}$. Thus, $a_{s_1}$ is a $n$-digits long decimal integer that has $s_1$ as its least significant digit.

On the other hand, we know that, $\forall a_{s_1}, \exists \alpha \in \mathbb{Z}_{10}$ such that $\alpha := \sum_{j=0}^{\infty} s_{j+1} \cdot 10^j \rightarrow a_{s_1}(n) = \sum_{j=0}^{n} s_{j+1} \cdot 10^j \equiv \sum_{j=0}^{\infty} s_{j+1} \cdot 10^j \pmod{10^n}$. This can be an efficient approach to solve the problem of finding the smallest $d_{1,3,7,9} \equiv \{1, 3, 7, 9\}(\pmod{10})$ whose constant congruence speed is equal to any given $n \in \mathbb{N} - \{0\}$.

Proposition 3. Let us consider the standard decimal numeral system (radix-10). It follows that the corresponding $g$-adic ring that we have to take into account is the decadic one ($g = 10$) [3], but 10 is not a prime number or a power of a prime (since 10 = 2 · 5 = $p_1 \cdot p_2$). Thus, for every odd $s_1$ (as defined in Proposition 2), we can find more than one polymorphic $\alpha \rightarrow a_{s_1}$ which arises when we solve in $\mathbb{Z}_{10} := \lim_{c \to \infty} \frac{\mathbb{Z}}{10^n \mathbb{Z}}$ (i.e., the set of formal series $\sum_{j=0}^{\infty} s_{j+1} \cdot 10^j, s_{j+1} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) the fundamental equation $y^t = y$. Therefore, assuming $s_{(n+1)} \not\equiv 0, \forall s_1 \in \{1, 3, 5, 7, 9\}$, we can find two order-$n$ residues of as many polymorphic integers (i.e., $\alpha' \neq \alpha''$ such that $\alpha' \equiv \alpha''(\pmod{10})$) whose expansions modulo 10 are always characterized by a constant congruence speed equal to $n$ (e.g., $s_1 = 7 \Rightarrow \alpha'_{s_1} = \ldots66295807$ and $\alpha''_{s_1} = \ldots92077057$ both satisfy $y^5 = y$, and $n = 7$ implies that $V(\alpha'(\pmod{10^7})) = V(66295807) = V(\alpha''(\pmod{10^7})) = V(2077057) = 7$).

Proposition 4. The constant congruence speed of $a$ is well defined if and only if $10 \nmid a$ [14]. In particular, $V(a, b) = V(a) \geq 1 \Rightarrow a \geq 2$ for any sufficiently large $b \in \mathbb{N} - \{0\}$, and in this regard we point out that $b \geq a + 1$ represents a sufficient, but not a necessary, condition. Our hypothesis is that $b \geq \text{len}(a) + 2$ it is enough to ensure that $V(a, b) = V(a)$ holds for any $a$ as above.
Conjecture 1. Let \( \text{len}(a) \in \mathbb{N} - \{0\} : 10^{\text{len}(a)-1} < a < 10^{\text{len}(a)} \) denote the number of digits of the base \( a \). \( \forall b \geq \text{len}(a) + 2, \ V(a, b) = V(a) \) (e.g., \( V(407922943, 2 \leq b \leq 10) = 9 \neq V(407922943, b \geq \text{len}(407922943) + 2) \)).

Remark 1. The statement of Conjecture 1 is certainly true for any \( a \equiv \{2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23 \}(\text{mod } 25) \), since (by References [13, 14]) \( \forall b \geq 3 \), we know that \( V(a) = V(a, b + 1) \leq V(a, b) \geq 1 \). Thus, \( V(a) = 1 \) implies that, \( \forall b \geq 3, \ V(a) = V(a, b) = 1 \geq V(a, b + 1) \geq 1 \Rightarrow V(a, b + 1) = V(a, b) = V(a) = 1 \) (e.g., \( V(2, b \geq 3) = V(2, b \geq \text{len}(2) + 2) = 1 \) is consistent with the expected result [15]).

Proposition 5. \( g = 10 = 2 \cdot 5 = p_1 \cdot p_2 \Rightarrow \gcd(p_1, p_2) = 1 \) (see Proposition 3). Since in \( \mathbb{Z}_{10} \) (which is not an integral domain) \( \exists h \neq 0 \land r \neq 0 \) such that \( h \cdot r = 0 \), it follows that, for every \( n \in \mathbb{N} \), \( 5^{2^n} \cdot 2^{5^n} \equiv 0(\text{mod } 10^n) \) by the ring homomorphism \( \phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10^n}^\times \). Since the sequence \( \{5^{2^n}\}_n := 5^{2^0}, 5^{2^1}, 5^{2^2}, \ldots \) converges 5-adically to 0 and 2-adically to 1 and \( \{2^{5^n}\}_\infty = 1 - \{5^{2^n}\}_\infty \) the above is the unique pair which induces the decomposition of \( \mathbb{Z}_{10} \). Thus, \( \mathbb{Z}_{10} \cong \mathbb{Z}_5 \oplus \mathbb{Z}_2 \) (where \( \oplus \) indicates the direct sum) since, for \( p \) prime, the complete ring \( \mathbb{Z}_p \) contains only the two idempotents elements 0 and 1, and the 5-adically plus 2-adically convergence implies the 10-adically convergence (by Cauchy’s convergence criterion). Hence, assume \( h(n) \equiv 5^{2^n} \) and \( r(n) \equiv 2^{5^n} \) in order to solve the fundamental equation \( y^5 = y, \) introduced by Proposition 3.

More formally: let \( \lim_{n \to \infty} \frac{\mathbb{Z}}{10^n \mathbb{Z}} \) indicate the subring of the cartesian product \( \prod_{n=1}^{\infty} \frac{\mathbb{Z}}{10^n \mathbb{Z}} \) of discrete topological spaces \( \frac{\mathbb{Z}}{10^n \mathbb{Z}} \) originated by all the sequences \((a_1, a_2, a_3, \ldots)\) such that, for every \( n \geq 1, \ a_{n+1} \equiv a_n(\text{mod } 10^n) \); we have the map \( \phi : \mathbb{Z}_{10} \to \mathbb{Z}_5 \oplus \mathbb{Z}_2, \ \phi(\alpha) \mapsto (\sum_{n=0}^{\infty} 2^n \cdot a_{n+1})_{5^n}, \sum_{n=0}^{\infty} 5^n \cdot a_{n+1})_{2^n} \) such that \( \alpha := \sum_{n=0}^{\infty} 5^n \cdot a_{n+1} \cdot 10^n \) in \( \mathbb{Z}_{10} \Rightarrow \alpha \mapsto 2^{5^n} \cdot a_{5^n}, \) and \( \alpha \mapsto 5^{2^n} \cdot a_{2^n} \) (let \( h_{m+1} := \sum_{n=0}^{m} v_{n+1} \cdot 5^n \) and \( h_{2m+1} := \sum_{n=0}^{m} w_{n+1} \cdot 2^n \), since \( \sum_{n=0}^{\infty} (5^{2^n} \cdot 2^{5^n}) = \alpha \) (mod \( 2^{5^{m+1}} \)).

Given \( s_1 = 5, \) if \( h_n = 5^{2^n}(\text{mod } 10^n), \) then \( h_n = \cdots 92256259918212890625 \). [2]

Similarly, for \( s_1 = 2, r_n = 2^{5^n}(\text{mod } 10^n) \Rightarrow \lim_{n \to \infty} r_n = \cdots 804103263499879186432 \).

Now, let \( y_i(t) \) represent the \( i \) solutions in \( \mathbb{Z}_{10} \) of the equation \( y^t = y \). If \( t = 2, \) then \( \exists t : y(t) \in \{0, 1\} \iff y(2) \in \{h, 1 - r\} \), so let \( y(2) = h \) and \( y(2) = 1 - r \).

Following the path above, it is possible to verify that all the solutions of \( y^t = y \) belong to the set \( y_i(5) \) [4]. Thus, for every given \( i \) such that \( y_i(t) \in \{0, 1\}, \ y_i(5) \to a(n) = s_{n-5}(n-1) \ldots s_{2n-1} \Rightarrow V(a(n)) = k \geq n \), where \( k = n \) if and only if \( s_{(n+1)} \neq 0 \) (since \( s_{n+1} \ldots s_{2n-1} = s_{(n+1)} \ldots s_{2n-1} \text{mod } 10^n \) \( \land s_n \ldots s_2 \cdot s_1 \neq s_{(n+1)} \ldots s_{2n-1} \text{mod } 10^n \)) is a sufficient and necessary condition for \( V(a(n)) = n \).

In particular, we should note that if \( y_j(5) \) is coprime to 10 (where \( y_j(5) \) indicates a pentamorphic integer belonging to \( y_i(5) \)), then \( y_j(5) \to a_{1,3,7,9}(\text{mod } 10^n) \) is enough to find the smallest \( a \equiv \{1, 3, 7, 9\}(\text{mod } 10) \) such that \( V(a_{1,3,7,9}) \) is at least equal to \( n \) (see Proposition 6), for every \( n \geq 1 \). Hence, considering any of the aforementioned four congruence classes, the smallest base \( a(n) := a_{\min}(n) \) such that \( V(a) \geq n \) is given by
\[ \bar{a}_{1,3,7,9}(n) = \begin{cases} 
(1 - 2 \cdot 5^n)(\text{mod } 10^n) & \text{iff } a \equiv 1(\text{mod } 10) \\
\min \left((5^n - 2^5)(\text{mod } 10^n), -\left(5^n + 2^5\right)(\text{mod } 10^n)\right) & \text{iff } a \equiv 3(\text{mod } 10) \\
\min \left((5^n + 2^5)(\text{mod } 10^n), (2^5 - 5^n)(\text{mod } 10^n)\right) & \text{iff } a \equiv 7(\text{mod } 10) \\
(2 \cdot 5^n - 1)(\text{mod } 10^n) & \text{iff } a \equiv 9(\text{mod } 10) 
\end{cases} \]

**Proposition 6.** Let \( h(n) = 5^2 \) and \( r(n) = 2^5 \), as usual. Assume \( t \geq 5 \) and let \( y_t(t) \) represent all the solutions in \( \mathbb{Z}_{10} \) of the equation \( y^t = y \) (i.e., \( t \in \{1, 2, 3, \ldots, 14, 15\} \)). Let \( \alpha'_{s_1} \cup \alpha''_{s_1} \) (if any) denote the subset of all the \( y_t(t) \equiv s_1(\text{mod } 10) \) which are not congruent to \( \{0, 1\} \) modulo 10^2. It follows that \( y_t(5) \ni \{\alpha'_{1}, \alpha'_{2}, \alpha'_{3}, \alpha'_{4}, \alpha'_{5}, \alpha''_{5}, \alpha'_{6}, \alpha''_{7}, \alpha'_{8}, \alpha''_{9}, \alpha''_{10}\} \), since \( y_{14}(t) : 0^t = 0 \) and \( y_{15}(t) : 1^t = 1 \) show the existence of two solutions of \( y^2 = y \) which are not included in the aforementioned subset. In order to understand how the remaining \( y_t(t) \) anticipate the recurrence rules stated in Section 2.2, it can be helpful to preliminary observe that the \( y_t(t) \) follow from
\[ \lim_{n \to \infty} \frac{5^n}{5^n + 2^5} = \frac{1 + \sqrt{5}}{2} \Rightarrow y = \lim_{n \to \infty} 5^n = \lim_{n \to \infty} 5^n + 2^n = y^2 \Rightarrow y_{j_1}(2) = y_{(12,14,15)}(t) = (\alpha'_{1}, \alpha'_{9}, 0, 1) = \{-\sqrt{5}, \sqrt{5}, 1, -1\}, \text{ and we can easily verify that } \alpha'_{9} = -\alpha'_{1} = \sqrt{5} = \lim_{n \to \infty} \frac{5^n - 2^n}{5^n} \quad [5, 6]. \]

Considering \( t = 5 \), we find in a similar way all the other roots (e.g., see References [7-10] for \( \alpha'_{3}, \alpha''_{3}, \alpha'_{7}, \) and \( \alpha''_{7} \), so it is possible to conclude that \( y_{t \leq 13}(t) = 5 \) are such that \( \alpha'_{1} = -\alpha'_{3}, \alpha'_{2} = -\alpha'_{8}, \alpha'_{3} = -\alpha'_{7}, \alpha''_{3} = -\alpha''_{7}, \alpha'_{4} = -\alpha'_{6}, \alpha'_{5} = -\alpha'_{5}, \) and \( \alpha''_{9} = 1. \) Furthermore, for any \( n, \)
\[ r(n)^2 + 1 = h(n) 
\]
In general, as clearly explained by Michon in Reference [4], we have
\[ y_{j_1}(t) = \begin{cases} 
(1 - 2 \cdot 5^n)(\text{mod } 10^n) & \text{iff } i = 1 \\\n\alpha'_{2} = h - r = \ldots 455303245144122416553040789004103623499879186432 & \text{iff } i = 2 \\
\alpha'_{3} = -h - r = \ldots 528709779454848385762121375881529641333704193 & \text{iff } i = 3 \\
\alpha'_{4} = h - 1 = \ldots 57423423203896109040106619977392256259918212890624 & \text{iff } i = 4 \\
\alpha'_{5} = h = \ldots 57423423203896109040106619977392256259918212890625 & \text{iff } i = 5 \\
\alpha'_{6} = h - 1 = \ldots 4257657676910389998998933800022607743740081787109375 & \text{iff } i = 6 \\
\alpha'_{7} = -h - r = \ldots 4712960922054115161423787624184700358666295807 & \text{iff } i = 7 \\
\alpha'_{8} = h + r = \ldots 619764556823373316963702781719635923418092077057 & \text{iff } i = 8 \\
\alpha'_{9} = -r = \ldots 9544967548558775834469541260198596736500120813568 & \text{iff } i = 9 \\
\alpha'_{10} = 2 - h - 1 = \ldots 1484684646179221800821323995478451251936425781249 & \text{iff } i = 10 \\
\alpha'_{9} = -1 = \ldots 99999999999999999999999999 & \text{iff } i = 11 
\end{cases} \]
\[ \text{Since } \phi: \mathbb{Z}_{10} \to \frac{z}{10^n \mathbb{Z}} \text{ it follows that } \alpha = a(\text{mod } 10^n) \Rightarrow \alpha'_{s_1}(n) \equiv \alpha''_{s_1}(n) = 0. \]

**Proposition 7.** Let \( s_1 \in \{2, 4, 6, 8\} \) and assume \( s_{(n+1)} \neq 0. \) Let \( \alpha'_{s_1}(n) := \alpha'_{s_1}(mod \ 10^n) \). Since,
\[ \forall n \geq 1, s_{(n+1)} \neq 0 \Rightarrow V\left(\alpha'_{s_1}(mod \ 10^n)\right) = n, \] we only need to compute the residues modulo \( P\left(V\left(\alpha'_{s_1}(n)\right)\right) \) (see [14], Section 5) of \( \alpha'_{s_1} \), in order to find the smallest bases \( \bar{a}_{s_1}(n) \) of the integer tetration \( b^a \) such that \( V\left(\alpha'_{s_1}(n)\right) = V\left(\bar{a}_{s_1}(n)\right) = n \) (e.g., if \( s_1 = 2 \) and \( n = 4 \), which implies that
\[ s_{(4+1)} = 8 \neq 0, \text{ then } V(\alpha'_{2}(4)) = V(6432) = 4 \Rightarrow V(6432(\text{mod } 5^4)) = V(182) = V(\alpha_{2}(4)) \Rightarrow \alpha_{2}(4) = 182. \]

Sometimes, \(\alpha'_{s_{1}}(n) \pmod{5^n} \) returns residues \(\tilde{a}_{[2,4,6,8]}(n)\) which are not congruent modulo 10 to \(s_{1}\) (e.g., \(V(\alpha'_{8}(9)) = V(120813568) = 9 \Rightarrow 120813568 \equiv 1672943(\text{mod } 5^9)\) would suggest that \(\tilde{a}_{8}(9)\) is equal to 1672943, but clearly 1672943 \(\not\equiv 8(\text{mod } 10)\), and (if this is the case) we can check that \(\hat{a}_{[2,4,6,8]}(n) + k \cdot 5^n = \hat{a}_{[2,4,6,8]}(n)\) still holds for some \(k \in \mathbb{N}\) (referring to the example above, we verify that \(k = 3\) holds because \(\tilde{a}_{8}(9) + 3 \cdot 5^9 = 7532318 = \tilde{a}_{8}(9)\)).

In particular, if \(s_{1} = 6\), the condition \(\hat{a}_{6}(n) = 5^n + 1\) trivially comes from the exclusion of the residue 1 (furthermore, \(V(1) = 0 [14]\)). Thus, \(s_{1} = 6 \Rightarrow k = 1\) for any \(n\), while \(s_{1} = 4 \Rightarrow k = 0\).

This concludes the proof that, for every \(n \geq 1\), \(\exists! k \in \mathbb{N}_{0} : \alpha'_{[2,4,6,8]}(n) - k \cdot 5^n = \hat{a}_{[2,4,6,8]}(n)\).

Finally, if \(s_{1} = 5\), we can find bases \(\alpha'_{5}(n) < \alpha'_{5}(n)\) and \(\alpha''_{5}(n) < \alpha''_{5}(n)\) with a constant congruence speed at least equal to \(n\), by simply taking into account that \(P'(\alpha'_{5}(n)) = P'(\alpha''_{5}(n)) = 5 \cdot 2^{n+1}\) (see [14], Section 5), and introducing the additional condition \(n > 2\).

Thus,
\[
V\left(\alpha'_{5}(n) \pmod{10 \cdot 2^n}\right) \geq n \land V\left(\alpha''_{5}(n) \pmod{10 \cdot 2^n}\right) \geq n,
\]
(3)

and Equation (3) let us confirm the validity of Equation (5) (e.g., if \(n = 20\), then \(\alpha'_{5}(20) = 92256259918212890625\) is congruent modulo \(10 \cdot 2^{20}\) to 9437185 and \(V(9437185) = 20\), while \(V\left(\alpha''_{5}(20) \pmod{10 \cdot 2^{20}}\right) = V(6291455) = 21 > n\)).

### 2.2 Bases \(a_{s_{1}} \equiv s_{1}(\text{mod } 10)\) characterized by a given \(V(\alpha_{s_{1}}) = n, \forall n \in \mathbb{N}\)

Let \(\tilde{a}(n) := \min_{\alpha} \{a : V(a) = n\}, \forall n \in \mathbb{N} \setminus \{0\}\). Let \(a_{s_{1}}(n) \equiv s_{1}(\text{mod } 10)\) for every \(s_{1} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\). Consequently, \(\forall n \geq 1, \tilde{a}_{s_{1}}(n) = \min_{\alpha} (a_{s_{1}} : V(a_{s_{1}}) = n)\).

We show that Equations (4) is true for any \(n \geq 2\) (i.e., \(n \geq 2 \Rightarrow a_{5}(n) = \tilde{a}(n)\)).

\[
\tilde{a}(n) = \min \left(2^{n} \cdot 2 \cdot \cos \left(\frac{\pi (n-1)}{2}\right) - 4 \cdot \sin \left(\frac{\pi (n-1)}{2}\right) + 5\right) + 1,
\]
(4)

\[
2^{n} \cdot \left(4 \cdot \sin \left(\frac{\pi (n-1)}{2}\right) - 2 \cdot \cos \left(\frac{\pi (n-1)}{2}\right) + 5\right) - 1.
\]

Hence,
\[
\tilde{a}(n) = \begin{cases}
2^{n} \cdot \left(5 + 2 \cdot \sin \left(\frac{\pi n}{2}\right) + 4 \cdot \cos \left(\frac{\pi n}{2}\right)\right) + 1 & \text{ iff } n \equiv \{2, 3\}(\text{mod } 4) \\
2^{n} \cdot \left(5 - 2 \cdot \sin \left(\frac{\pi n}{2}\right) - 4 \cdot \cos \left(\frac{\pi n}{2}\right)\right) - 1 & \text{ iff } n \equiv \{0, 1\}(\text{mod } 4)
\end{cases}
\]
(5)

Now, assume \(b > a \geq 2\) (as usual), even if we are strongly persuaded that also \(b \geq \text{len}(a) + 2\) represents a sufficient condition for \(V(a, b) = V(a)\), as predicted by Conjecture 1 [1, 16]. Then, for any given \(n \in \mathbb{N} \setminus \{0, 1\}\), \(V(a_{s_{1}}(n)) = n\), \(\forall s_{1} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}\), if and only if Equations (6), (7), (8), (10), (11), (14), (15), (16), and (17) are satisfied.

\[
a_{1}(n) = \begin{cases}
\left(2^{4 \cdot 5^{n+1} + 1} - 1\right)(\text{mod } 10^{n}) + j_{n} \cdot 10^{n}, \forall j_{n} \neq \left(\frac{2^{4 \cdot 5^{n+1} + 1} - 1}{\frac{10^{n}}{10}}\right) & \text{mod } 10^{n} \\
10^{n} + 1 + k \cdot 10^{n}, \forall k \equiv \{0, 1, 2, 3, 4, 5, 6, 7, 8\}(\text{mod } 10)
\end{cases}
\]
(6)

Since \((k + 1) \cdot 10^{n} + 1 > \left(2^{4 \cdot 5^{n+1} + 1} - 1\right)(\text{mod } 10^{n})\) is always true, Equation (6) implies that if \(n : \left(2^{4 \cdot 5^{n+1} + 1} - 1\right)(\text{mod } 10^{n}) \not\equiv \left(2^{4 \cdot 5^{n+1} + 1} - 1\right)(\text{mod } 10^{n+1})\), then \(\exists a_{1}(n) \equiv \left(2^{4 \cdot 5^{n+1} + 1} - 1\right)(\text{mod } 10^{n})\). Thus, if the \((n + 1)\)-th rightmost digit of \(\alpha'_{1}\) (see Equation (2)) is nonzero, then the unique base \(a_{1}(n) \leq \left(2^{4 \cdot 5^{n+1} + 1} - 1\right)(\text{mod } 10^{n})\) corresponds to the desired \(\tilde{a}_{1}(n)\).
In general, \( \forall n \geq 2 \), we have that \( a_1(n) \equiv 51 \pmod{10^2} \Rightarrow V(a_1) = \{ n : (2^n \mid (a_1 + 1) \wedge 5^n \mid (a_1 - 1)) \wedge (2^{n+1} \mid (a_1 + 1) \vee 5^{n+1} \mid (a_1 - 1)) \} \), while \( a_2(n) \equiv 1 \pmod{10^2} \Rightarrow V(a_2) = \{ n : (10^n \mid (a_1 - 1) \wedge 10^{n+1} \mid (a_1 - 1)) \} \), and these are the only cases for \( s_2 = 1 \).

It follows that, \( \forall n \geq 2, 10^n + 1 \geq a_2(n) > 5^n + 1 \) (since \( \forall n \in \mathbb{N} : 5^n + 1 \equiv 1 \pmod{10} \)).

Similarly to Equation (6), if \( s_2 = 9 \), we have

\[
a_9(n) = \begin{cases} (2 \cdot 5^{2n} - 1)(\pmod{10^n}) + j_n \cdot 10^n, & \forall j_n \neq (2 \cdot 5^{2n+1} - 1)(\pmod{10^n}) - (2 \cdot 5^{2n-1})(\pmod{10^n}) \\
10^n - 1 + k \cdot 10^n, & \forall k \equiv \{0, 1, 2, 3, 4, 5, 6, 7, 8\}(\pmod{10^n}) \end{cases}
\]  

(7)

As previously shown, if \( n : (2 \cdot 5^{2n} - 1)(\pmod{10^n}) \neq (2 \cdot 5^{2n+1} - 1)(\pmod{10^n}) \), then \( \exists! a_9(n) \leq (2 \cdot 5^{2n} - 1)(\pmod{10^n}) \). Thus, \( \forall n \geq 2 \), we have that \( a_9(n) \equiv 49 \pmod{10^2} \Rightarrow V(a_9) = \{ n : (2^n \mid (a_9 - 1) \wedge 5^n \mid (a_9 + 1)) \wedge (2^{n+1} \mid (a_9 - 1) \vee 5^{n+1} \mid (a_9 + 1)) \} \), while \( a_9(n) \equiv 99 \pmod{10^2} \Rightarrow V(a_9) = \{ n : (10^n \mid (a_9 + 1) \wedge 10^n + 1 \mid (a_9 + 1)) \} \).

Consequently, \( \forall n \geq 2, 10^n > a_9(n) > 5^n - 1 \).

We point out that, as shown in Proposition 6 (see the case \( s_{(n+1)} = 0 \)),

\[
n : \frac{(2^{4 \cdot 5^{n+1} + 1} - 1)(\pmod{10^n}) - (2^{4 \cdot 5^{n+1}} - 1)(\pmod{10^n})}{10^n} \equiv 0 \pmod{10^n}
\]

\[
\Rightarrow (2^{4 \cdot 5^{n+1} + 1} - 1) \equiv (2^{4 \cdot 5^{n+1} + 1} - 1)(\pmod{10^n}) \Rightarrow V((2^{4 \cdot 5^{n+1} + 1} - 1)(\pmod{10^n})) > n,
\]

and similarly

\[
n : \frac{(2 \cdot 5^{2n+1} - 1)(\pmod{10^n}) - (2 \cdot 5^{2n} - 1)(\pmod{10^n})}{10^n}
\]

\[
\Rightarrow (2 \cdot 5^{2n} - 1) \equiv (2 \cdot 5^{2n+1} - 1)(\pmod{10^n}) \Rightarrow V((2 \cdot 5^{2n} - 1)(\pmod{10^n})) > n
\]

(e.g., \( V(163574218751) = V((2^{4 \cdot 5^{12} + 1} - 1)(\pmod{10^{12}})) = 13) \).

Let us consider the case \( s_1 = 5 \). From [14], we know that, \( \forall n \in \mathbb{N} \setminus \{0, 1\} \),

\[
a_5(n) = \begin{cases} 2^n \cdot \left(5 + 2 \cdot \sin \left(\frac{\pi}{2} \cdot n\right) + 4 \cdot \cos \left(\frac{\pi}{2} \cdot n\right)\right) + 1 + k \cdot 5 \cdot 2^{n+1}, & \forall k \in \mathbb{N}_0 \\
2^n \cdot \left(5 - 2 \cdot \sin \left(\frac{\pi}{2} \cdot n\right) - 4 \cdot \cos \left(\frac{\pi}{2} \cdot n\right)\right) - 1 + k \cdot 5 \cdot 2^{n+1}, & \forall k \in \mathbb{N}_0
\end{cases}
\]

(8)

Equation (8) implies that

\[
\tilde{a}_5(n) \leq 9 \cdot 2^n + 1,
\]

(9)

and the last inequality (trivially) holds because, \( \forall n \in \mathbb{N} \),

\[
\max \left(\pm x \cdot \cos \left(\frac{\pi}{2} \cdot n\right) \pm y \cdot \sin \left(\frac{\pi}{2} \cdot n\right)\right) = \max(|x|, |y|).
\]

If \( s_1 = 4 \) or \( s_1 = 6 \), for the reasons already discussed in the previous subsection, we have, respectively,

\[
a_4(n) = 5^n - 1 + k \cdot 2 \cdot 5^n, \forall k \equiv \{0, 1, 3, 4\}(\pmod{5});
\]

(10)

\[
a_6(n) = 5^n + 1 + k \cdot 2 \cdot 5^n, \forall k \equiv \{0, 1, 3, 4\}(\pmod{5}).
\]

(11)

Equations (10)&(11) imply that, \( \forall n, a_4(n) = a_6(n) - 2 \).

Thus, \( \min(\tilde{a}_4(n), \tilde{a}_6(n)) = \tilde{a}_4(n) = 5^n - 1 \).
Now, we study the cases $s_1 = 2$ and $s_1 = 8$. We have,
\[
V(a_{(2,8)}) = \{ n : (5^n | (a_{(2,8)}^2 + 1) \land 5^{n+1} \nmid (a_{(2,8)}^2 + 1) \}
\]
\[
\Rightarrow a_{(2,8)}(n) = \sqrt{5^n \cdot c_{a_{(2,8)}}(n) - 1}. \quad (12)
\]

Since $c_{a_{(2,8)}}(n) \in \mathbb{N} - \{0\}$ for any $n$, Equation (12) states that $\min(\tilde{a}_2(n), \tilde{a}_8(n)) \geq \sqrt{5^n - 1}$.

More specifically, picking any value of $n$, the constraint that $c_{a_{(2,8)}} = \frac{a_{(2,8)}^2 + 1}{5^n}$ have to return a positive integer (as $a$) let us calculate the solutions (taking the natural logarithm) from
\[
n = \frac{\ln(\frac{a_{(2,8)}^2 + 1}{5^n})}{\ln(5)}. \quad (13)
\]

Equation (13) provides also a valid upper bound for the constant congruence speed of $a_{(3,7)}$, since, for every $\bar{n}$, $a_{(2,3,7,8)}^2 + 1 = \prod_{p \neq 5} p_j^{a_j} \cdot 5^\bar{n} \geq \prod_{p \neq 5} p_j^{a_j} \cdot 5^{V(a_{(2,3,7,8)})}$ (where $p_j$ represents the prime divisors of $a_{(2,3,7,8)}^2 + 1$ which are not equal to 5, while every $q$ indicates how many times the corresponding $p$ appears in the factorization of $a_{(2,3,7,8)}^2 + 1$ [16]).

As shown in Section 2.1, we can easily improve the aforementioned upper bound referring to the commutative ring of the 10-adic integers, giving an explicit formula for $V(a_{(3,7)}) = n$ in the same way as we have already done for $V(a_{(1,9)})$. For this purpose, let $V(a_{(3,7)}) = n \leq \bar{n}$.

Since $a_7 = h - r = \alpha_3'$ and $a_7'' = h + r = -\alpha_3''$ (where $h(n) \approx 5^2$ and $r(n) \approx 2^{5^n}$), if $s_1 = 3$, then
\[
a_3(n) = \begin{cases}
(5^n - 25^n)(\bmod 10^n) + j_n \cdot 10^n, & \forall j_n \neq \frac{(5^{2n+1} - 25^{n+1})(\bmod 10^{n+1}) - (5^{2n} - 25^n)(\bmod 10^n)}{10^n} \\
-(5^n + 25^n)(\bmod 10^n) + j_n \cdot 10^n, & \forall j_n \neq \frac{(5^{2n+1} + 25^{n+1})(\bmod 10^{n+1}) - (5^{2n} + 25^n)(\bmod 10^n)}{10^n}
\end{cases},
\]
while the case $s_1 = 7$ is covered by Equation (15)
\[
a_7(n) = \begin{cases}
(5^n - 25^n)(\bmod 10^n) + j_n \cdot 10^n, & \forall j_n \neq \frac{(5^{2n+1} - 25^{n+1})(\bmod 10^{n+1}) - (5^{2n} - 25^n)(\bmod 10^n)}{10^n} \\
(5^n + 25^n)(\bmod 10^n) + j_n \cdot 10^n, & \forall j_n \neq \frac{(5^{2n+1} + 25^{n+1})(\bmod 10^{n+1}) - (5^{2n} + 25^n)(\bmod 10^n)}{10^n}
\end{cases}. \quad (15)
\]

Furthermore (as a consequence of Proposition 6), $a_{(3,7)}(\bar{n}) \neq (a_{(3,7)}(\bmod 10^\bar{n}) + 5 \cdot 10^{\bar{n}-1})(\bmod 10^\bar{n}) \Rightarrow V(a_{(3,7)}) = \{ \bar{n} : (2^\bar{n} | (a_{(3,7)}^2 - 1) \land 5^\bar{n} | (a_{(3,7)}^2 + 1)) \land (2^{\bar{n}+1} \nmid (a_{(3,7)}^2 - 1) \lor 5^{\bar{n}+1} \nmid (a_{(3,7)}^2 + 1)) \};$ in particular, $a_{(3,7)}(\bar{n}) = (a_{(3,7)}(\bmod 10^\bar{n}) + 5 \cdot 10^{\bar{n}-1})(\bmod 10^\bar{n}) \Rightarrow \bar{n} - 1 \leq V(a_{(3,7)}(b)) \leq \bar{n}$ for any sufficiently large $b(a)$ (e.g., if Conjecture 1 holds, then $b \geq \text{len}(a) + 2$ is a sufficient condition for $V(a_{(3,7)}(b)) = V(a_{(3,7)})$).

It follows that, $\forall n \geq 2$, $\min(\tilde{a}_3(n), \tilde{a}_7(n)) > \sqrt{5^n - 1}$ (since $5^n - 1$ is even).

In order to complete the (constant) congruence speed map, we only need a formula for $a_2(n)$ and $a_8(n)$, as shown in Equations (6), (7), (8), (10), (11), (14), and (15).

Let $\gamma_2(n) := (2^n(\bmod 10^n))(\bmod 5^{n+1})$. Let $\gamma_8(n) := (-2^n(\bmod 10^n))(\bmod 5^{n+1})$.

Consequently, $5^n | (\gamma_2(n) + \gamma_8(n))$. Let $u_2(n)$ be equal to
\[
\begin{cases}
2 \cdot m & \text{if } \gamma_2(n) \equiv 2(\bmod 10), & \text{where } m \in \mathbb{N}_0 : \gamma_2(n) > 2 \cdot m \cdot 5^n \land \gamma_2(n) < 2 \cdot (m + 1) \cdot 5^n \\
2 \cdot m - 1 & \text{if } \gamma_2(n) \equiv 7(\bmod 10), & \text{where } m \in \mathbb{N}_0 : \gamma_2(n) > (2 \cdot m - 1) \cdot 5^n \land \gamma_2(n) < (2 \cdot m + 1) \cdot 5^n.
\end{cases}
\]
Let \( u_8(n) \) be equal to
\[
\begin{align*}
2 \cdot m & \iff \gamma_8(n) \equiv 8 \pmod{10}, & \text{where } m \in \mathbb{N}_0 : \gamma_8(n) > 2 \cdot m \cdot 5^n \land \gamma_8(n) < 2 \cdot (m + 1) \cdot 5^n \\
2 \cdot m - 1 & \iff \gamma_8(n) \equiv 3 \pmod{10}, & \text{where } m \in \mathbb{N}_0 : \gamma_8(n) > (2 \cdot m - 1) \cdot 5^n \land \gamma_8(n) < (2 \cdot m + 1) \cdot 5^n .
\end{align*}
\]

Then, \( \forall n, \)
\[
\begin{align*}
a_2(n) &= \gamma_2(n) - u_2(n) \cdot 5^n + k \cdot 2 \cdot 5^n, & \forall k \not\equiv \frac{a_2(n+1) - a_2(n)}{2 \cdot 5^n} \pmod{5}; \\
a_8(n) &= \gamma_8(n) - u_8(n) \cdot 5^n + k \cdot 2 \cdot 5^n, & \forall k \not\equiv \frac{a_8(n+1) - a_8(n)}{2 \cdot 5^n} \pmod{5},
\end{align*}
\]
where \( a_2(n) := \gamma_2(n) - u_2(n) \cdot 5^n \), and \( a_8(n) := \gamma_8(n) - u_8(n) \cdot 5^n .
\]
Moreover, \( \forall n \in \mathbb{N} - \{0\} \), we have \( a_2(n) = 2 \cdot 5^n \cdot (\beta(n) - a_8(n)) \), where \( \beta(n) \in \{1, 2\} \).

In conclusion, if \( V(a) = 1 \), then
\[
a(1) \equiv \{2, 3, 4, 6, 8, 9, 11, 12, 13, 14, 16, 17, 19, 21, 22, 23\} \pmod{25}.
\]

Therefore, we have mapped all the bases \( a \) such that \( V(a, b) = V(a) = n \).

The constant congruence speed formula that we have shown in the present section confirms also Hypothesis 1 and Hypothesis 2, stated in Reference [14], as \( V(a) \geq 2 \Rightarrow \mathcal{P}(V(a)) = 10^{V(a)+1} \) (see [14], Equation (10)). So, we are finally ready to prove that \( n \geq 2 \Rightarrow a_{\{1,2,3,4,5,6,7,8,9\}}(n) = a_5(n) = \left(2^n \cdot ((-1)^{n-1} + 2) - i^{n(n-1)}\right) \), and this will be the goal of Section 3.

\section{Smallest \( a(n) \) for any \( V(a_{\{1,2,3,4,5,6,7,8,9\}}) = n \)}

In this section, we prove the last conjecture stated in [14].

\begin{thm}
Let \( a(n) := \min\{a_{\{1,2,3,4,5,6,7,8,9\}} : V(a_{\{1,2,3,4,5,6,7,8,9\}}) = n\} \). \( \forall n \in \mathbb{N} - \{0,1\} ,
\[
\tilde{a}(n) = \begin{cases} 
2^n \cdot \left(5 + 2 \cdot \sin \left(\frac{\pi \cdot n}{2}\right) + 4 \cdot \cos \left(\frac{\pi \cdot n}{2}\right)\right) + 1 & \text{iff } n \equiv \{2, 3\} \pmod{4} \\
2^n \cdot \left(5 - 2 \cdot \sin \left(\frac{\pi \cdot n}{2}\right) - 4 \cdot \cos \left(\frac{\pi \cdot n}{2}\right)\right) - 1 & \text{iff } n \equiv \{0, 1\} \pmod{4}
\end{cases}
\]
If \( n = 0 \), then \( a(n) = \tilde{a}(n) = 1 \); while \( \tilde{a}(1) = 2 \).
\end{thm}

\begin{proof}
From Section 2.2 (see Equations (6) to (17)), we know that, \( \forall n \geq 2 ,
\[
\begin{align*}
\tilde{a}_1(n) &> 5^n + 1; \\
\tilde{a}_9(n) &> 5^n - 1; \\
\tilde{a}_{\{4,6\}}(n) &\geq \min\{\tilde{a}_4(n), \tilde{a}_6(n)\} = \tilde{a}_4(n) = 5^n - 1; \\
\tilde{a}_{\{3,7\}}(n) &\geq \min\{\tilde{a}_3(n), \tilde{a}_7(n)\} > \sqrt{5^n - 1}; \\
\tilde{a}_{\{2,8\}}(n) &\geq \min\{\tilde{a}_2(n), \tilde{a}_8(n)\} > \sqrt{5^n - 1}.
\end{align*}
\]
Hence,
\[
\tilde{a}_1(n) > \tilde{a}_2(n) > \tilde{a}_3(n) > \tilde{a}_4(n) > \tilde{a}_5(n) > \tilde{a}_6(n) > \tilde{a}_7(n) > \tilde{a}_8(n) > \tilde{a}_9(n) > \tilde{a}_5(n) \geq \sqrt{5^n - 1}.
\]
\end{proof}
On the other hand, Equation (9) implies that \( \forall n \in \mathbb{N} - \{0, 1\} : \tilde{a}_5(n) > 9 \cdot 2^n + 1, \) since
\[
\tilde{a}_5(n) = \begin{cases} 
2^n \cdot \left(2 \cdot \cos \left(\frac{\pi \cdot (n-1)}{2}\right) - 4 \cdot \sin \left(\frac{\pi \cdot (n-1)}{2}\right) + 5\right) + 1 & \text{iff } n \equiv \{2, 3\}(\text{mod } 4) \\
2^n \cdot \left(4 \cdot \sin \left(\frac{\pi \cdot (n-1)}{2}\right) - 2 \cdot \cos \left(\frac{\pi \cdot (n-1)}{2}\right) + 5\right) - 1 & \text{iff } n \equiv \{0, 1\}(\text{mod } 4). 
\end{cases}
\]

Thus, in order to prove the main statement of Theorem 1, it is sufficient to check the inequality \( \sqrt{5^n - 1} > 9 \cdot 2^n + 1, \) observing that it is certainly true for every \( n \geq 20 \) (since \( \sqrt{5^n - 1} = 9 \cdot 2^5 + 1 \Rightarrow 19.693374 < x < 19.693375 \)). So we only need to verify that, \( \forall n \in \{2, 19\} \), \( \tilde{a}_5(n) < \bar{a}_{\{1,2,3,4,6,7,8,9\}}(n) \), and the values are listed in Table 1 (see Equations (6) to (17)).

<table>
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<tr>
<th>( n = V(\alpha) )</th>
<th>( \bar{a}_5(n) )</th>
<th>( \bar{a}_{{1,2,3,4,6,7,8,9}}(n) )</th>
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</tr>
<tr>
<td>19</td>
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</tr>
</tbody>
</table>

Table 1: Comparison between the smallest \( \alpha(n) \) congruent modulo 10 to 5, whose constant congruence speed is equal to \( n \leq 19 \), and the minimum \( \alpha(n) \equiv \{1, 2, 3, 4, 6, 7, 8, 9\}(\text{mod } 10) \).

As it follows from Equations (9) and (18), \( \forall n \in \mathbb{N} - \{0, 1\} \), \( \bar{a}(n) := \bar{a}_{\{1,2,3,4,5,6,7,8,9\}}(n) = \tilde{a}_5(n) \).

Therefore, in order to complete the proof, it is sufficient to observe that \( V(2) = 1 \) and \( V(1) = 0 \) (see [14]). \( \square \)

**Corollary 1.** Let \( \bar{a}(n) := \min(\alpha_{\{1,2,3,4,5,6,7,8,9\}}) : V(\alpha_{\{1,2,3,4,5,6,7,8,9\}}) = n \), and let \( i^2 = -1 \).

\[
\bar{a}(n) = 2^n \cdot ((-1)^{n-1} + 2)^{i^n \cdot (n-1)}.
\]

(19)

**Proof.** The statement of Corollary 1 easily follows from Theorem 1.
Since, in September 2020, Bruno Berselli noted that Sequence A337392 of the OEIS is given by \( a(n) = (2 - (-1)^n) \cdot 2^n + i^{(n+1) \cdot (n+2)} \) (see Formula in Reference [12]), it trivially follows that Equation (5) can be further simplified if we prove the claim;

\[
\bar{a}(n) = \begin{cases} 
2^n \cdot \left( 5 + 2 \cdot \sin \left( \frac{\pi \cdot n}{2} \right) + 4 \cdot \cos \left( \frac{\pi \cdot n}{2} \right) \right) + 1 & \text{iff } n \equiv \{2,3\} \pmod{4} \\
2^n \cdot \left( 5 - 2 \cdot \sin \left( \frac{\pi \cdot n}{2} \right) - 4 \cdot \cos \left( \frac{\pi \cdot n}{2} \right) \right) - 1 & \text{iff } n \equiv \{0,1\} \pmod{4} 
\end{cases}
\]

\[
= 2^{n+1} + \left( \sin \left( \frac{\pi \cdot (n+1) \cdot (n+2)}{2} \right) \right) - 2^n \cdot \sin(\pi \cdot n) \cdot i - 2^n \cdot \cos(\pi \cdot n) + \cos \left( \frac{\pi \cdot (n+1) \cdot (n+2)}{2} \right). \quad (20)
\]

Hence \( 2^n \cdot \cos(\pi \cdot n) - i \cdot 2^n \cdot \sin(\pi \cdot n) = -2^n \cdot e^{i\pi n} \) implies that

\[
\bar{a}(n) = 2^{n+1} - 2^n \cdot e^{i\pi n} + e^{i\frac{\pi}{2} \cdot (\pi \cdot (n+3) + 2)}. \quad (21)
\]

Since \( e^{i\pi} + 1 = 0 \Rightarrow e^{i\frac{\pi}{2}} = i \) and \( e^{i\pi n} = (-1)^n \), it follows that

\[
\bar{a}(n) = 2^{n+1} - 2^n \cdot (-1)^n + i^{(n+3) + 2}. \quad (22)
\]

Thus, Berselli’s formula is correct and we have

\[
\bar{a}(n) = 2^{n+1} + 2^n \cdot (-1)^{n-1} - i^n \cdot (n+3). \quad (23)
\]

Therefore, in order to confirm Equation (19) and conclude the proof, it is sufficient to observe that \( i^{n \cdot (n+3)} = i^{n \cdot (n-1)}. \)

\[ \Box \]

**Remark 2.** Corollary 1 provides also a short proof of Theorem 1, since

\[
\bar{a}(n) = 2^n \cdot ((-1)^{n-1} + 2) - i^n \cdot (n-1) \leq 2^n \cdot (1 + 2) + 1. \quad (24)
\]

Thus, \( \sqrt{5^n - 1} > 3 \cdot 2^n + 1 \) holds for any \( n \geq 10. \)

**Corollary 2.** \( \forall n \in \mathbb{N} - \{0,1\} \) and \( \forall k \in \mathbb{N}_0, \)

\[
a_5(n) = \left( (2^n \cdot ((-1)^{n-1} + 2) - i^n \cdot (n-1)) \lor (2^n \cdot ((-1)^n + 8) + i^n \cdot (n-1)) \right) + k \cdot 10 \cdot 2^n. \quad (25)
\]

**Proof.** Equation 5 and Corollary 1 (Berselli’s formula) imply that

\[
a_5(n) = 2^n \cdot ((-1)^{n-1} + 2) - i^n \cdot (n-1) + k \cdot 10 \cdot 2^n \cup \left\{ \begin{aligned}
2^n \cdot \left( 5 + 2 \cdot \sin \left( \frac{\pi \cdot n}{2} \right) + 4 \cdot \cos \left( \frac{\pi \cdot n}{2} \right) \right) + 1 + k \cdot 10 \cdot 2^n & \text{ iff } n \equiv \{0,1\} \pmod{4} \\
2^n \cdot \left( 5 - 2 \cdot \sin \left( \frac{\pi \cdot n}{2} \right) - 4 \cdot \cos \left( \frac{\pi \cdot n}{2} \right) \right) - 1 + k \cdot 10 \cdot 2^n & \text{ iff } n \equiv \{2,3\} \pmod{4}
\end{aligned} \right. 
\]

Since, \( \forall n \geq 2, \) it easy to verify (as shown in the proof of the aforementioned Corollary 1) that

\[
= 2^n \cdot \left( 2^3 + \cos(\pi \cdot n) + i \cdot \sin(\pi \cdot n) \right) + \cos \left( \frac{\pi \cdot n \cdot (n - 1)}{2} \right) + i \cdot \sin \left( \frac{\pi \cdot n \cdot (n - 1)}{2} \right) 
\]

\[
= 2^n \cdot ((-1)^n + 8) + i^n \cdot (n-1),
\]

the statement of Corollary 2 follows. \( \Box \)
4 Conclusion

The congruence speed of the integer tetration \( b^a \) certainly does not depend on \( b, \forall a \in \mathbb{N} - \{0\} \) which is not a multiple of 10, if \( b \) is larger than \( a \) (i.e., the criterion \( b > a \) always holds), and we conjecture that a stricter sufficient condition for \( V(a, b) = V(a) \) is \( b \geq \text{len}(a) + 2 \). Thus, let us take any \( b = b(a) \) that assures that \( V(a, b) = V(a) \); then Equations (6), (7), (10), (11), (14), (15), (16), (17), and (25) return the set of all the bases \( a \) whose (constant) congruence speed is any given \( V(a) \in \mathbb{N} - \{0\} \), and we know from [14] that \( V(a) = 0 \leftrightarrow a = 1 \). Under any additional constraint on specific congruence classes modulo 10 for \( a \), we can easily find \( \bar{a}(V(a)) \), the smallest \( a \equiv \{1, 2, 3, 4, 5, 6, 7, 8, 9\}(\text{mod } 10) \) whose constant congruence speed is equal to any given positive integer. Since \( \bar{a}(0) = 1, \bar{a}(1) = 2 \), and \( \bar{a}(V(a) \geq 2) = 2^n \cdot (\pm 1)^{n-1} + 2 - i^n(n-1) \) [12], we can finally conclude that the conjecture stated in Reference [14] is true.

In the present paper we only considered radix-10, but our results can be clearly extended to different numeral systems, as shown by [1] which was inspired by [16]; this observation suggests a topic for the next research article.

References


