On Mach’s Principle and its implementation into classical mechanics:

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abstract:
In this paper I want to address a shortcoming of classical as well as relativistic mechanics regarding its incapability to explain its a priori introduced constants G (gravitational constant) and c (speed of light). I will argue that for the first of these constants, this incapability is tight to a flawed definition of the kinetic energy. Going back on an argument given by W. Hofmann in 1904, I claim that its common definition not being relational leads to theoretical contradictions and that the correct one for each particle should involve a sum over products of all other masses in the universe and their relative velocities \[ T_i \propto \sum_j m_i m_j f_{ij}(r_{ij}) \cdot v_{ij}^2 \]; \( f \) being some function possibly depending on the relative distance. I show that an implementation of such a relational kinetic energy already leads to classical mechanics fully incorporating Mach’s principle and yielding an explicit expression for the gravitational constant G. Classical mechanics will then be based on a completely Galilean invariant Lagrangian which incorporates the induction of an isotropic inertial mass by distant matter as well as gravitoelectromagnetism. I will show, that a Lorentz-type force equation can be obtained from it. Finally, the Hofmann argument will be discussed within relativistic mechanics. I will show that the corresponding interpretation would be that of an emergent space in which the existence of space as well as that of a finite speed of light are tight to the presence of matter in the universe.

Keywords: Mach’s principle, gravitational constant, Galilean invariance, relationality, kinetic energy, origin of inertia, Gravitoelectromagnetism, isotropy of inertia, emergent space-time

1. Introduction:

Physics should always ask why the laws of nature are the way they are, and not just settle for describing them. In history it has always been this way how progress was made: An insight why a wide range of laws are the way they are, so to say what is “behind them“. Here I agree with Alexander Unzicker that especially the elimination of constants plays a central role since their appearance is always tied to opaque concepts which are postulated without explanation. The maybe best example for this is the connection between light and electromagnetism. Before Maxwell, light and electromagnetism were two different concepts existing independently from one another. On the one side, there was the concept of an electromagnetic field together with the constants \( \mu_0, \varepsilon_0 \) (from which the latter is, by the way, unexplained until today), on the other there was the concept of light together with c, the speed of light. Both of them had to be postulated since none of them could be derived from the other. This changed when Maxwell was able to show that light was an electromagnetic wave, rendering the latter concept obsolete since it was included in electromagnetism. On the other hand this fact expressed itself in the relation \( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \), effectively also rendering one constant of nature obsolete. This is a pattern in physics which has occurred many times in the past and there is no reason why it should be any different in modern physics. History has shown that major breakthroughs were always tied to explaining one of these numbers, respectively being able to derive one previously postulated concept from another. In this context, in the author’s opinion there exists an unused opportunity to achieve exactly this regarding
the gravitational constant G: Mach’s principle, which is until today implemented neither in classical physics nor in general relativity (even though features of it are actually present in general relativity). In this paper I want to show a way how to do this in classical mechanics and, in this way, obtain the gravitational constant from the equations themselves instead of putting it into them by hand. The concept which is removed shall turn out to be that of absolute space, which was postulated by Newton and is still present in classical as well as relativistic mechanics through the definition of the kinetic energy. Space will then, at least in classical mechanics, just play the role of a coordinate system in which the equations are formulated but without having any physical impact on them like in the Newtonian formulation, where it defines the notion of “rest”. In the relativistic case, it shall turn out that most likely an implementation of Mach’s principle would lead to space and time being an emergent phenomenon, which is not fundamental, but arises from whatever deeper reality: Matter creates space around itself; the entire space, which we perceive is, according to Mach’s principle, created by all particles in the universe.

2. The relationality of the kinetic energy:

We begin with a simple consideration based on an argument put forth by the physicist W. Hofmann in 1904 [1,9]: The common definitions of the kinetic energy and momentum, which read:

\[ T = \frac{1}{2} m v^2 \]  

(2.1)

and:

\[ \vec{p} = m \vec{v} \]  

(2.2)

respectively for a particle of mass m and velocity v, must be incomplete, since they depend on two absolute elements, namely \( m \) and \( \vec{v} \). As was stated by Hofmann, this can be seen by a simple thought experiment: Suppose two particles flying within an otherwise empty cosmos with a relative velocity \( \vec{v}_{12} := \vec{v}_1 - \vec{v}_2 \) towards each other. Then observer 1, co-moving with particle 1, ascribes to the system the kinetic energy and momentum:

\[ T_1 = \frac{1}{2} m_2 v_{12}^2 \]  

(2.3)

\[ \vec{p}_1 = m_2 \vec{v}_{12} \]  

(2.4)

while observer 2, co-moving with particle 2, ascribes:

\[ T_2 = \frac{1}{2} m_1 v_{12}^2 \]  

(2.5)

\[ \vec{p}_2 = m_1 \vec{v}_{12} \]  

(2.6)

Upon (inelastic) collision, both observers are able to measure the real values for both momentum and energy, which obviously disagrees with at least one of the expected ones. We explicitly note that what we mean here by energy and momentum, especially in the case of the energy, are the exact values of them and not just the ones being defined up to an arbitrary constant. This is usually done

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1) The same is also true for the relativistic kinetic energy: Only the functional dependence on the (absolute) velocity changes (in equations \( \frac{1}{2} m v^2 \rightarrow \frac{mc^2}{\sqrt{1-(v/c)^2}} \) ), but not the fact that they do depend on it. Therefore, the arguments given in this section also apply to the relativistic kinetic energy.
in Newtonian mechanics, especially in the Lagrangian formalism. These quantities can be measured upon collision (e.g. by a spring) and therefore both observers have to agree on the exact values and not just up to a constant. There are now two possible ways of resolving this problem. One is that the two masses enter symmetrically into the kinetic energy and momentum. Since both must be additive, it is also clear that the dependence on both masses must be linear, leaving the only possible choice:

\[ T_1 = T_2 = \frac{1}{2} m_1 m_2 f(r_{12}) v_{12}^2 \] (2.7)

\[ \vec{p}_1 = \vec{p}_2 = m_1 m_2 f(r_{12}) \vec{v}_{12} \] (2.8)

where \( f(r_{12}) \) accounts for a possible distance dependence. From dimensional reasons, it can be seen that:

\[ f(r_{12}) = \frac{G}{c^2 r_{12}} \] (2.9)

must hold. Now, for a cosmos with more than one particle, (2.7) and (2.8) imply that the kinetic energy as well as the momentum have to be dependent on all other particles in the universe; in equations:

\[ T_i = \frac{1}{2} \sum_{j,j \neq i} \frac{G m_i m_j}{r_{ij}} \beta_{ij}^2 \] (2.10)

\[ \vec{p}_i = \sum_{j,j \neq i} \frac{G m_i m_j}{c r_{ij}} \vec{\beta}_{ij} \] (2.11)

This is the Machian resolution of the problem since both quantities are defined with respect to all other particles in the universe. The only other possibility implies the assumption of an absolute background, with respect to which both observers are able to measure their “absolute” velocities. The kinetic energy of the particles would then read:

\[ T_i = \frac{1}{2} m_i (\vec{v}_i - \vec{v}_0)^2 \] (2.12)

\[ \vec{p}_i = m_i (\vec{v}_i - \vec{v}_0) \] (2.13)

where \( \vec{v}_0 \) is the velocity of this absolute background relative to the chosen observer\(^2\). This is the Newtonian viewpoint: An absolute space which globally defines rest. The above equations reduce to (2.1) and (2.2) if the observer is at rest relative to it (\( \vec{v}_0 = 0 \)); this is the case which is normally assumed in Newtonian mechanics when describing a system. One can simply see that this would also resolve the problem in the above thought experiment since both observers would agree on the total energy being:

\[ T = T_1 + T_2 = \frac{m_1}{2} (\vec{v}_0^{(1)})^2 + \frac{m_2}{2} (\vec{v}_{21} - \vec{v}_0^{(1)})^2 = \frac{m_1}{2} (\vec{v}_{12} - \vec{v}_0^{(2)})^2 + \frac{m_2}{2} (\vec{v}_0^{(1)})^2 \] (2.14)

\(^2\) It is worth mentioning, that both approaches include the case of \( \vec{v}_0 = -\frac{1}{M} \sum_{i \in U} m_i \vec{v}_i \) being the velocity of the center of mass of the universe. This approach was made by Lynden-Bell&Katz [11] and does also resolve the Hofmann problem. It yields for the kinetic energy (2.10) \( T = \frac{1}{4 M} \sum_{j \neq i} m_i m_j \vec{v}_{ij}^2 \) and therefore corresponds to the choice

\[ f(r_{12}) = \frac{1}{2 M} = \text{const} \]. It does, however, not enable one to calculate the gravitational constant \( G \) since this constant will still appear in the total Lagrangian in the potential energy.
where \( \vec{v}_0^{(1)} \) and \( \vec{v}_0^{(2)} \) are the velocities relative to the absolute background as seen by observer 1 and 2 respectively. However, there are three objections against this resolution. The first two are of empirical nature. The first one is, that for both particles being able to agree on (2.14), it is necessary for them to being able to directly interact via a measurement with the absolute (background) space. But until today, no such direct measurement of space has been successfully performed. The second one is that if such an absolute background globally defines what “rest” means, it will make itself noticeable according to eq. (2.13): An inertial reference frame could be only a such which is un-accelerated relative to the background. Now, since most of the local systems (e.g. galaxies) throughout the universe are accelerated relative to one another, at most one of these can agree with the background system. In all others, there would occur non-inertial contributions to, for example, the movement of the stars which is clearly not observed. We will see in section 4, that, in order to correctly obtain the Newtonian formulation of mechanics, described by a Lagrangian of the form:

\[
L_k = \frac{m_k}{2} v_k^2 - V_k
\]  

the kinetic energy needs to have the form (2.10)\(^3\). The last objection is of theoretical nature: If we compare the formulation according to Mach (2.10), (2.11) and the one according to Newton (2.12), (2.13) one can see that the latter needs to postulate the absolute background system which the former one doesn’t. There, this system is induced by something which is present anyway: the particles themselves. Consequently, according to Occam’s razor, the Machian alternative is much more likely to be realized in nature as the Newtonian one.

3. A Lagrangian of the universe:

From this we can already construct a Lagrange function of the universe by adding the potential energy:

\[
V = -\frac{1}{2} \sum_{j \neq i} G m_i m_j / r_{ij}
\]

Since now in \( L \), both terms \( T \) and \( V \) depend on \( G \) and the Lagrange function remains invariant under rescaling, we can leave it out and write:

\[
L = T - V = \frac{1}{2} \sum_{j \neq i} \frac{m_i m_j}{r_{ij}} (1 + \vec{\beta}_{ij}^2)
\]

This is (apart from the factor \( G \)) the same as used in the „Inertia free mechanics“ by Hans-Jürgen Treder [2,4]. As we see, this expression does not involve a gravitational constant. We will show in the following section, that it comes out of the equations in a natural way.

4. Gravitoelectromagnetism:

We now want to show that expression (3.2) includes the phenomena of gravitoelectromagnetism as well as an inertia induction according to Mach’s principle. The inertial mass will turn out to be isotropic, despite the Lagrangian being dependent on the relative velocities of the particles. The latter fact will just give rise to gravitomagnetic phenomena.

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\(^3\) We shall see that the obtained expressions will also hold in arbitrary accelerated frames of reference. The potential \( V_k \) will, of course, be different in all of these systems.
If we define the gravitomagnetic and electric potentials in the classical way (but, again, without the constant G) via:

\[
\tilde{A}_k := \sum_{j \neq k} \frac{m_j}{r_{kj}} \beta_j \\
\Phi_k := \sum_{j \neq k} \frac{m_j}{r_{kj}}
\]  (4.1)

then indeed by using the second binomial identity \(\tilde{\beta}^2 = \tilde{\beta}_k^2 + \tilde{\beta}_j^2 - 2 \langle \tilde{\beta}_k, \tilde{\beta}_j \rangle\) and gathering together all terms involving the k th particle, we get for its Lagrangian:

\[
L_k = \frac{1}{2} m_k^* \dot{v}_k^2 + m_k \Phi_k - 2 m_k \langle \tilde{A}_k, \tilde{\beta}_k \rangle + \sum_{j \neq k} m_k \frac{m_j}{r_{kj}} \beta_j^2
\]  (4.3)

The first three terms in this expression differ from the common Lagrangian for a particle in an electromagnetic field only by an additional factor of 2 at the magnetic term (the correct relativistic value is 4) and the fact that an „inertial mass“ is given by

\[
m_k^* = m_k \frac{2 \Phi_k}{c^2}
\]  (4.4)

Expression (4.4) is isotropic, showing that the relational Lagrangian does not lead to an anisotropic inertial mass\(^5\). Mach’s principle is satisfied since the inertial mass is determined by distant masses. It can be seen that this, as well as gravitomagnetic phenomena (the 3rd term in (4.3)), are a direct consequence of the kinetic energy having the form (2.10) instead of just being \(m_k \frac{1}{2} v_k^2\). Therefore, an inertia-induction in the sense of Mach’s principle does indeed arise from demanding the Galilean invariance of kinetic energy.

We now determine the equations of motion by using (4.3) and apply the Euler-Lagrange equations:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{v}_k} - \frac{\partial L}{\partial r_k} = 0
\]  (4.5)

Neglecting the last term in (4.3), which is quadratic in \(\beta_j\), we obtain:

\[
\frac{2 \Phi_k}{c^2} \frac{\partial \dot{v}_k}{\partial t} = \tilde{E}_k - 2 \tilde{\beta}_k \times \tilde{B}_k
\]  (4.6)

where the gravolectric and magnetic fields are given by:

\[
\tilde{E}_k := \tilde{\nabla}_k \Phi_k + \frac{2}{c} \frac{\partial}{\partial t} \tilde{A}_k
\]  (4.7)

\[
\tilde{B}_k := \tilde{\nabla}_k \times \tilde{A}_k + \tilde{\beta}_k \times \tilde{\nabla}_k \Phi_k
\]  (4.8)

If we divide through the factor on the left hand side in eq. (4.6):

\[
\frac{\partial \dot{v}_k}{\partial t} = \frac{c^2}{2 \Phi_k} (\tilde{E}_k - 2 \tilde{\beta}_k \times \tilde{B}_k)
\]  (4.9)

and compare the expressions on the right hand side with the common definitions of the gravolectric and magnetic fields, we find:

4) This quantity does not have the dimension of a mass since we cancelled out G in the Lagrangian (3.2). But is does play exactly this role, since we could have let G remain in the equations and then see later that it cancels out when calculating the gravitational acceleration. Just like the normal mass does due to the equivalence principle.

5) This, together with eq. (4.4), was already obtained by K. Retzlaff [4].
We have derived the "gravitational constant" from the equations. Unlike in common Newtonian physics, it comes out of the equations naturally, it does not have to be put in by hand a priori. This allows us to shed light on the physical reason for the existence of the constant G in the usual Newtonian law of gravity (as well as in General relativity). This, together with the physical reason for the expression (4.10) for G will be discussed in the next section.

The presence of gravitomagnetic phenomena can be interpreted in a simple way: Since the Lagrangian (3.2) depends on the relative velocities, what "rest" locally means is defined not only by the gravity fields of the surrounding bodies, but also by their accelerations and velocities. This gives rise to gravitomagnetic forces as well as the induction term in (4.7). If the particle is far away from other masses, then the right side of (4.9) is zero, and we re-obtain Newton’s first law:

\[ \frac{\partial \vec{v}_k}{\partial t} = 0 \]

Also, eq. (4.9) states that a particle is always free falling towards the rest of the universe. This can be seen by expressing eq. (4.9) in the rest-system of the particle. There, \( \vec{v}_k = \vec{v}^* = 0 \) holds and we obtain:

\[ \vec{E}_k = 0 \]  

(4.11)

This, together with the gravitoelectromagnetic equations\(^6\) (4.6-4.8) being valid in an arbitrary accelerated frame, was already used by Sciama to explain inertia and derive expression (4.10) for the gravitational constant [5]. However, both were postulated in his work; here, it is just the result of the Euler-Lagrange equations when applied to the Lagrangian (3.2).

We now want to show the similarity between the Lagrangian (4.3) and the Einstein-Infeld-Hoffmann (EIH) Lagrangian of the general theory of relativity; especially with respect to the terms which have the form \( 1 \pm \frac{2G \delta \Phi}{c^2} \) and appear again and again in GR. Consider therefore some mass distribution in the foreground, causing a potential \( \delta \Phi \), and the background including all other masses of the universe, causing a potential \( \Phi_0 \). The latter will be assumed to be much farther away than the former and therefore can be considered approximately constant. We write:

\[ \Phi = \Phi_0 + \delta \Phi \]

If we plug it into the Lagrangian (4.3) and keep in mind that \( \Phi_0 \) is constant, we obtain:

\[ L_k = \frac{1}{2} m_k (1 + \frac{\delta \Phi_k}{\Phi_0}) \vec{v}_k^2 + \frac{1}{2} m_k c^2 (1 + \frac{\delta \Phi_k}{\Phi_0}) - m_k \frac{c^2}{\Phi_0} \frac{\delta \Phi_k}{\Phi_0} \langle \vec{A}_k, \vec{B}_k \rangle + m_k \frac{c^2}{2 \Phi_0} \sum_{j \neq k} \frac{m_j}{r_{kj}} \beta_j^2 \]  

(4.12)

Now, in this expression, the classical gravitational potential is just the term \( \frac{1}{2} m_k c^2 \frac{\delta \Phi_k}{\Phi_0} \). If we compare it with the Newtonian potential:

\[ \frac{1}{2} m_k c^2 \frac{\delta \Phi_k}{\Phi_0} = G m_k \delta \Phi_k \]

we find\(^7\):

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\(^6\) Apart from a factor of 2 standing by the vector potential; the relativistic value of this number is 4.

\(^7\) An expression like this was obtained by various authors [5,6].
As said, the "gravitational constant" now comes out of the equations naturally. The Lagrangian (4.5) can be written in the following form:

\[ L_k = \frac{1}{2} m_k v_k^2 + \frac{1}{2} m_k c^2 - 2 m_k G \langle \vec{A}_k, \vec{B}_k \rangle + \sum_{j \neq k} G m_k m_j \frac{r_{kj}}{r_{ij}} \beta_j^2 \]  

\[ m_k^* = m_k \left( 1 + \frac{2 G \Phi_k}{c^2} \right) \]  

All the terms of (4.14) are also present, although with slightly different numerical factors, in the second order \((1/c^2)\) expansion of general relativity; the Einstein-Infeld-Hoffmann (EIH) equations:

\[ L_{EIH} = \frac{m}{2} \left( 1 + \frac{3 G \Phi}{c^2} \right) v^2 + G \Phi - \frac{7}{2} m G \langle \vec{B}, \vec{A} \rangle + \frac{3}{2} \sum_{j \neq k} G m_k m_j \frac{r_{kj}}{r_{ij}} \beta_j^2 
\]

\[ + m \frac{v^2}{8} \frac{1}{c^2} \frac{G^2 \Phi}{c^2} \frac{7}{2} \sum_{j \neq k} G m_k m_j \langle \vec{B}_k, \vec{n}_{kj} \rangle \langle \vec{B}_j, \vec{n}_{kj} \rangle \]  

The terms in the first row altogether appear in (4.14), the terms in the second row are higher relativistic corrections (the first two) and corrections due to retardation (the last one). These effects are both not considered in the classical approach and the corresponding terms obviously don’t appear. The terms of the first row\(^8\), however, do appear in our classical approach and arise due to the kinetic energy having the form (2.10) instead of the usual Newtonian one. Therefore these terms have a Machian origin in our theory and it is suggestive that they have in General relativity, too. We saw in the derivation of (4.14) that in our theory the „1“ in (4.15) is the part of the mass which arises from the background of the universe, the second part arises due to a perturbing mass in the foreground. As said, the corresponding terms also exist in general relativity. There, the „1“ comes from the Minkowski-metric representing the flat space-time background, the small term again comes from a perturbing foreground mass. Assuming now that these terms indeed have a Machian origin in GR too, it would imply that the flat background space-time (the Minkowski-metric) is induced by the background mass distribution of the universe. This would also explain, why GR is only Machian if appropriate boundary conditions are applied: The “appropriate boundary conditions” (e.g. flat Minkowski space at infinity) themselves are a result of a more complete, Machian theory of gravity. This theory, when evaluated locally with a fore- and background massdistribution, just as was done in this section with our classic theory, would then yield the equations of GR for the foreground masses and a background space induced by the background masses. Then constants (\(G\), and maybe also \(c\)) would be determined by expressions similar to the one derived for \(G\) in this work (4.10). We will discuss this further in section 7.

The Lagrangian (3.2) could be extended to include electromagnetic phenomena as well by simply adding the electromagnetic potential divided by \(G\) (since we rescaled the whole Lagrangian (3.2) by this factor):

\[ V_{el} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4 \pi \varepsilon_0 G r_{ij}} \]  

\[ G = \frac{c^2}{2 \Phi_0} \]  

\[ m_0 = m_0 c^2 \] has a gravitational origin due to Mach’s principle.

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9) The effects which they represent are gravitoelectromagnetism, the induction of inertial mass due to the presence of matter and a “rest energy” of \( E_0 = \frac{1}{2} m_0 c^2 \). The factor \( \frac{1}{2} \) suggests, that at least a fraction of the relativistic rest energy \( E_0 = m_0 c^2 \) has a gravitational origin due to Mach’s principle.
This yields:

\[ L = \frac{1}{2} \sum_{j \neq i} m_j m_i \left( 1 + \beta_{ij}^2 \right) - \eta \left( \frac{me^2}{e} \right)^2 \frac{q_i q_j}{r_{ij}} \tag{4.18} \]

where \( \eta \) is the relative strength of electric and gravitational force between two protons; the first dimensionless number of Dirac:

\[ \eta' = \frac{1}{4 \pi \varepsilon_0 G} \left( \frac{e}{m_e} \right)^2 \approx 10^{42} \tag{4.19} \]

Despite being able to derive the gravitational constant (4.10), this one still enters the equations without explanation. Since it already pops up in a classical Lagrangian, one expects it to have an explanation which has no need of quantum mechanics, or relativity. This awaits further discussion.

5. Explanation of G:

The derived equations allow us to give an explanation of why G shows up in the gravitational law and how the expression (4.10):

\[ G = \frac{c^2}{2 \Phi_k} \]

comes about. Therefore, we consider some particle k in its rest frame. There, the equation of motion reads, according to (4.11):

\[ \vec{E}_k = 0 \tag{5.1} \]

\[ \vec{E}_k = \nabla_k \Phi_k + \frac{2}{c} \frac{\partial \vec{A}_k}{\partial t} \tag{5.2} \]

From eq. (5.1), we can see that a particle always moves in way relative to the universe that it does not experience any force. From eq. (5.2) we can see that this means, that it does always accelerate in way that the induction caused by the entire universe, as seen by this particle, cancels the gravitational field, as seen by this particle. We now take a look at the vectorpotential, which is given by (4.1):

\[ \vec{A}_k := \sum_{j \neq k} \frac{m_j}{r_{kj}} \vec{\beta}_j \tag{5.3} \]

In this expression, the velocities \( \beta_j \) are the velocities of the other particles in the universe as seen from the restframe of the particle k. We can express each of these as a sum of two velocities: One being that of the particle relative to the rest frame defined by all masses\(^{10}\) (we call it \( \beta_{j,0} \)) and the other being that of k relative to this frame (we call it \( \beta_{k,0} \)):

\[ \vec{\beta}_j = \vec{\beta}_{j,0} + \vec{\beta}_{j,0} \tag{5.4} \]

Plugging this into the vectorpotential yields:

\(^{10}\) This “rest frame defined by all particles” is the generalisation of the centre of mass system in Newtonian mechanics to the mechanics presented in this paper, which are built on the kinetic energy (2.10). It has the similar property that \( \sum_{i \in U} m_i \vec{v}_i \approx 0 \) since all velocities are roughly isotropically distributed in this frame, what is just what is implied by the system being “at rest relative to the universe”.

\[ \vec{A}_k = \Phi_k \cdot \vec{\beta}_{k,0} + \sum_{j \neq k} \frac{m_j}{r_{kj}} \vec{\beta}_{j,0} \]  

(5.5)

Here, the second part is just the vectorpotential as seen from the rest frame, it approximately vanishes due to the isotropic distribution of the velocities of the particles in this frame. The first term arises due to the acceleration of the particle relative to the restframe and is the one which is interesting here. Plugging (5.5) into (5.2) and using that \( \Phi_k \approx \text{const} \) (all other masses are assumed to be far away), we obtain:

\[ 0 = \vec{\nabla}_k \Phi_k + \frac{2\Phi_k}{c} \frac{\partial \vec{\beta}_{k,0}}{\partial t} \]

(5.6)

Solving for the acceleration yields Newtons law of gravitation:

\[ \frac{\partial \vec{\beta}_{k,0}}{\partial t} = -\frac{c^2}{2\Phi_k} \vec{\nabla}_k \Phi_k = -G \vec{\nabla} \Phi_k \]

(5.7)

As can be seen from eq. (5.6) and (5.7) the existence of the gravitational constant arises from the resistance of the universe against the particles acceleration, caused by the induction effect of the universe. Its strength is proportional to \( \frac{2\Phi_k}{c^2} \), which, in turn, yields the expression (4.10) for G.

In fact, its not the gravitational force which has G built into it, but all accelerations have built in the factor 1/G which expresses the aforementioned resistance of the entire universe against them. This “resistance against acceleration” is what the term “inertia” actually means. We can therefore say that mechanics built on the kinetic energy (2.10) realises Mach’s principle in the way that “inertia here is caused by matter over there (throughout the entire universe)”. This also explains why gravitational accelerations are so small: Since there is such an enormous amount of matter in the universe, already very tiny accelerations lead to a large (opposed) induction force which is sufficient to cancel the gravitational force. If there was considerably less matter, gravity would be predicted to be much stronger. E.g. if the universe consisted only of the milky way, then gravity would be roughly \( 10^7 \) times stronger than it is in our universe, at least if c would keep its known value in such a situation. This was already pointed out by Sciama [5].

We can also approximately calculate the value of the (local) gravitational constant from:

\[ G = \frac{c^2}{2\Phi} \]

(5.8)

by neglecting a possible foreground contribution. For the contribution of the background we have for an approximately homogeneous universe:

\[ \Phi_0 = \int_{K, \in R^3} \frac{\rho_i}{r_{ij}^{3/2}} d^3 \vec{r}_j \approx 4\pi \rho_0 \int_0^{R_u} r \, dr = \frac{3}{2} \frac{M_u}{R_u} \]

(5.9)

where \( M_u, R_u \) are (visible) mass and radius of the observable universe. We obtain\(^{11}\):

\[ G_0 \approx \frac{1}{3} \frac{R_u c^2}{M_u} \]

(5.10)

Indeed, inserting \( M_u \approx 10^{53} \text{ kg} \), \( R_u \approx 4 \cdot 10^{26} \text{ m} \) and \( c \approx 3 \cdot 10^8 \frac{\text{ m}}{\text{ s}} \) we obtain:

\[ G \approx 6 \cdot 10^{-4} \frac{m^3}{\text{ kg s}^2} \]

\(^{11}\) Here, one can see that if the universe indeed consisted only of the milky way we had \( M_u \approx 10^{41} \text{ kg} \) and \( R_u \approx 10^{21} \text{ m} \), and therefore roughly (ignoring it’s completely different, non-isotropic structure when seen from the earth) \( G \approx 6 \cdot 10^{-4} \frac{m^3}{\text{ kg s}^2} \), \( 10^7 \) times the known value.
which is an agreement to a very good accuracy since the mass and the radius of the universe are only known by orders of magnitude, and the approximation of a homogeneous universe, if at all, only holds on the largest scales. Also, no relativistic effects have been considered. The mere fact that such a relation holds to such an accuracy is another reason in favour of every theory providing a relation like (5.8) since it is most unlikely that its numerical validity is just a mere coincidence, rather than the result of a deeper rooted mechanism (for example Mach’s principle).

6. The relativistic kinetic energy:

We now make an attempt to discuss the relativistic case of the Hofmann argument. This turns out to be somewhat complicated since one needs to find expressions which are relational and Lorentz invariant at the same time. Astonishingly, the commonly known expression for the relative velocities:

\[
\beta_{ij}^{\mu} = \frac{\sqrt{\left[ (u_i)^\mu (u_j)^\mu \right]^2 - 1}}{(u_i)^\mu (u_j)^\mu} = \frac{\sqrt{\beta_i^2 - (\vec{\beta}_i \times \vec{\beta}_j)^2}}{1 - \langle \vec{\beta}_i, \vec{\beta}_j \rangle}
\]  

(6.1)

satisfies this properties; it is relational and at the same time Lorentz invariant. On the other hand, for spatial quantities such an expression is impossible to construct. This already hints into a direction that rapidity should be a more fundamental quantity than space and time. We later want to argue (section 7) that space and time are emergent phenomena in the sense of Mach’s principle that „without matter there is no space“ and maybe also no time.

If we now, analogously to what is done in the canonic transition to relativistic mechanics, replace:

\[
m_i \frac{m_j}{2} v_{ij}^2 \rightarrow -m_i m_j c^2 \sqrt{1 - \beta_{ij}^2}
\]  

(6.2)

we have indeed:

\[
m_i \frac{m_j}{r_{ij}} \sqrt{1 - \beta_{ij}^2} \approx m_i \frac{m_j}{r_{ij}} \frac{1}{2} \frac{m_i m_j}{c^2 r_{ij}} v_{ij}^2
\]

which is just the non-relativistic expression (2.10). Now, also \( \frac{1}{r_{ij}} dt \) has to be replaced by some relational quantity which is an invariant, so that the whole action remains invariant. Here exists a problem that it is not possible to find such a quantity in space and time coordinates. We can only find a quantity which is Lorentz-invariant and contains, in lowest order, the relational quantity \( \frac{1}{r_{ij}} \).

It can simply be found by seeing that:

\[
inv = \phi_\mu \, dx^\mu = c \, dt \phi_i - \langle \vec{d}x, \vec{A}_i \rangle = c \, dt \left( \phi_i - \langle \vec{\beta}_i, \vec{A}_i \rangle \right)
\]  

(6.3)

holds. Therefore, in eq. (2.10) we have to replace:

\[
\frac{1}{r_{ij}} \rightarrow \frac{1}{r_{ij}} \left( 1 - \langle \vec{\beta}_i, \vec{\beta}_j \rangle \right)
\]

to get a Lorentz invariant action. Last but not least, to take into account retardation effects, all arguments must be taken as functions of the retarded time (for each particle j):

\[
X \rightarrow \left[ X \right] := X(t - \frac{\left[ r_{ij} \right]}{c})
\]
This yields for the action:

\[ S = \sum_{j \neq i} \int \frac{\sqrt{[(u_i)_\mu (u_j)_\mu]^2 - 1}}{(u_i)_\mu (u_j)_\mu} (\phi_{ij})_\mu \, dx_i^\mu \]  

(6.4)

or, when written in 3 dimensional form and after using the identity:

\[ \sqrt{1 - \beta_{ij}^2} = \frac{\sqrt{1 - \beta_i^2} \sqrt{1 - \beta_j^2}}{1 - \langle \vec{\beta}_i, \vec{\beta}_j \rangle} \]

we obtain for the lagrangian:

\[ L = T - V \]

\[ T = - \sum_{i \neq j} \frac{m_i m_j}{r_{ij}} \sqrt{1 - \beta_i^2} \sqrt{1 - \beta_j^2} \]  

(6.5)

\[ V = - \nu \cdot \sum_{i \neq j} \frac{m_i m_j}{r_{ij}} (1 - \langle \vec{\beta}_i, \vec{\beta}_j \rangle) \]  

(6.6)

If we now use the Euler-Lagrange equations to calculate the generalised momentum, we obtain:

\[ \vec{P}_k = \frac{\partial L}{\partial \vec{v}_k} = \frac{2 [\Phi_k^{(rel)}]}{c^2} \cdot \frac{m_k \vec{v}_k}{\sqrt{1 - \beta_k^2}} - 2 \nu \frac{m_k}{c} [\vec{A}_k] \]  

(6.7)

with:

\[ \Phi_k^{(rel)} = \sum_{j \neq i} \frac{m_j}{r_{kj}} \sqrt{1 - \beta_j^2} \]  

(6.8)

Eq. (6.7) is the familiar expression for the relativistic momentum, apart from the fact that the inertial mass is again, like in the non-relativistic case, induced by distant matter according to Mach’s principle:

\[ m_i^* = m_k \frac{2 [\Phi_k^{(rel)}]}{c^2} \]  

(6.9)

It can be seen that there occurs an additional factor \( \sqrt{1 - \beta_j^2} \) as well as retardation compared to (4.4). If one neglects retardation, the inertial mass is still isotropic. Taking retardation into account, the inertial mass becomes anisotropic, in disagreement with observation. We will discuss this issue in section 8.

The Lagrangian obtained from (6.5) and (6.6) can be made to agree with the second order EIH-Lagrangian by setting \( \nu = \frac{4}{3} \). Expanding up to second order and doing the same calculation with back- and foreground mass distribution, as was done in section 4 leads to:

\[ L_k = \frac{1}{2} m_k (1 + \frac{3 G \delta \Phi_k}{c^2}) v_k^2 + \frac{1}{3} m_k c^2 + G \delta \Phi_k - \frac{7}{2} m_k G \langle \vec{A}_k, \vec{\beta}_k \rangle \]

\[ + \frac{1}{2} \sum_{j \neq k} G \frac{m_j}{r_{kj}} (3 \beta_j^2 - \langle \vec{\beta}_j, \vec{n}_j \rangle \cdot \langle \vec{\beta}_j, \vec{n}_j \rangle) \]  

(6.10)

which agrees with the EIH-Lagrangian (4.16) up to terms of second order v/c (apart from a constant). The absolute terms (which are proportional to a single absolute mass \( m_k \)) are induced...
here, as already in the non-relativistic case, by the background of the universe. In the following section, we want to discuss what this means in the framework of GR.

7. Mach’s principle, emergent space-time and the finite speed of light:

In the previous sections we saw how the implementation of Mach’s principle gave rise to a background induced by distant matter. We now want to discuss what interpretation this Machian framework would give to general relativity. Therefore we first take a look at the Schwartzschild-solution given by:

\[
\frac{ds^2}{c^2} = \frac{1}{1 - \frac{2GM}{rc^2}} - \frac{1}{1 - \frac{2GM}{rc^2}} dx^2 \tag{7.1}
\]

It can be seen that the presence of a single (foreground) mass always decreases \( g_{00} \) and increases \( g_{11} \), where the former is a scale factor for the observed product \( c \cdot dt \) and the latter for the observed unit length in radial direction. Consequently, the latter is a scale factor for the amount of observed space. Now, since a single mass has this effect, all other masses in the universe must have too. Since our observer is within the combined gravitational field of all these other masses, the local values of \( g_{00} \) and \( g_{11} \), which hold in the absence of our single foreground mass, must be generated by all these other masses. But these are just the components of the Minkowski metric \( \eta_{00} \) and \( \eta_{11} \). More precisely: According to the prior given arguments, the value of \( g_{00} \) must have been lowered by all other masses to its local value \( g_{00} \) from a higher value \( (g_{00})_0 \), which would hold in the absence of these masses. Since no finite value would be in any way preferred over another, it is most suggestive that \((g_{00})_0 = \infty\). The same arguments apply to \( \eta_{11} \) (and consequently, also to \( \eta_{22}, \eta_{33} \)), just with it being increased from a lower value \( (\eta_{11})_0 \). Again, with no finite value being preferred over another, it is suggestive that \((\eta_{11})_0 = 0\) holds.

I now want to show, how the aforementioned behaviour can be obtained qualitatively when interpreting (7.1) in the sense of Mach’s principle. In the non-relativistic section we’ve already encountered terms which have the form (compare eqs. (4.12) and (4.15)):

\[
1 + \frac{2G\delta\Phi}{c^2} = 1 + \frac{\delta\Phi}{\Phi_0} \tag{7.2}
\]

In these expressions, the 2nd term was caused by a foreground mass (distribution) \( \delta\Phi \), whereas the 1st one was caused by the background mass distribution \( \Phi_0 \). To get (7.1) into this form, we make a coordinate transformation:

\[
r \rightarrow \xi : = r - r_g \tag{7.3}
\]

with \( r_g = \frac{2GM}{c^2} \) the gravitational radius of the mass M. This yields:

---

12) Since in general relativity always only the product \( c \cdot dt \) enters, it is completely equivalent, if one speaks of the unit time or the speed of light changing. However, I want to focus on the speed of light here, since I’m especially interested in the question why there exists a finite speed of light, and therefore the constant c.

13) This also applies to the components \( g_{22} \) and \( g_{33} \) which are just not changed by our foreground mass, since it causes a spherical symmetric field. This, however, does not apply to the background masses and consequently, these masses will, according to the same arguments, also generate these components.
which allows us to interpret the expressions occurring in the metric in terms of Mach’s principle. The coordinate transformation (7.3) corresponds to measuring the radial distance from the gravitational radius \( r_g \). If we now again write \( \phi_0 = \sum_{j \neq i, k} \frac{m_j}{\xi} \) and \( \delta \phi = \frac{m_k}{\xi} \) and use
\[
\frac{2G}{c_0^2} = \frac{1}{\phi_0}
\]
we obtain:

\[
g_{00} = \left( \frac{c}{c_0} \right)^2 - \frac{1}{1 + \frac{2GM}{\xi c_0^2}} \frac{\phi_0}{\delta \phi}
\]

(7.5)

From eq. (7.5) follows:

\[
c^2 \cdot \phi = c_0^2 \cdot \phi_0 = \text{const} =: \Lambda
\]

(7.6)

The constant \( \Lambda \) can be expressed through locally observed constants \( G \) and \( c_0 \) via to:

\[
\Lambda = \frac{c_0^4}{2G}
\]

From (7.6) one can easily obtain a differential equation for the observed speed of light:

\[
\frac{\Delta}{c^2} = \frac{4 \pi}{\Lambda} \cdot \rho = \frac{8 \pi G}{c_0^4} \cdot \rho
\]

(7.7)

Equations (7.6) as well as (7.7) connect the finiteness of the speed of light to the presence of matter in the universe. If there was no matter in the universe, the speed of light would be infinite. Indeed, it holds:

\[
\lim_{M \to 0} g_{00} = \text{const} \cdot \lim_{M \to 0} \frac{1}{\Phi} = \infty
\]

as was already suggested at the beginning of this section, in agreement with Mach’s principle: The existence of a locally observed finite speed of light has its origin in all the remainder matter in the universe. Analogously, for the observed space, we can write down the relation\(^{15}\):

\[
g_{11} = \frac{\eta_{11}}{\phi_0} = \text{const}
\]

(7.8)

where \( g_{11} \) is the known 11-component of the metric tensor with:

\[
ds^2 = g_{11} \cdot dx^1 \cdot dx^1 = g_{11} \cdot d\xi^2
\]

This leads to:

\[
g_{11} = \text{const} \cdot \Phi
\]

(7.9)

\(^{14}\) We’ve chosen the symbol \( \phi \) for the “potential” here, since it does, contrary to the \( \Phi \) in previous sections, depend on \( r_{ij} \) but on the shifted \( \xi \), according to (7.3).

\(^{15}\) We restrict our discussion on the radial coordinate of \( dx^2 \), for the angular coordinates all arguments are qualitatively the same, just quantitatively with \( \Phi, \Phi_0 \to \Phi^2, \Phi_0^2 \)
with \( \text{const} = \frac{2G}{c_0^2} \) when expressed through locally observed constants. In differential form we have:

\[
\Delta g_{11} = \frac{8\pi G}{c_0^2} \rho \tag{7.10}
\]

Equations (7.9) and (7.10) (together with the similar equations for the angular components, which we didn’t write down) connect the existence of space with the presence of matter in the universe. If there was no matter in the universe, no space would exist. Indeed, it holds:

\[
\lim_{M \to 0} g_{11} = \text{const} \quad \lim_{M \to 0} \Phi = 0 \tag{7.11}
\]

This again realises Mach’s principle in the way that the existence of local space is tight to the presence of matter in the universe. If we calculate the momentum according to Euler-Lagrange equations as

\[
\frac{\partial L}{\partial \dot{x}} = \dot{p}
\]

using the Lagrange-function corresponding to (7.4) which reads:

\[
L = -mc_0^2 \sqrt{g_{00} - g_{11} \left( \frac{v_x}{c} \right)^2 - g_{22} \xi^2 \left( \frac{v_\Omega}{c} \right)^2} \tag{7.12}
\]

we obtain:

\[
p_{\xi} = \frac{m \cdot g_{11} \cdot v_{\xi}}{\sqrt{g_{00} - g_{11} \left( \frac{v_x}{c} \right)^2 - g_{22} \xi^2 \left( \frac{v_\Omega}{c} \right)^2}} \approx m \cdot g_{11} \cdot v_{\xi} \tag{7.13}
\]

In the 2\textsuperscript{nd} step, for the sake of clarity, we approximated \( \sqrt{g_{00} - g_{11} \left( \frac{v_x}{c} \right)^2 - g_{22} \xi^2 \left( \frac{v_\Omega}{c} \right)^2} \approx \sqrt{g_{00}} \approx 1 \) which is basically taking the non-relativistic limit; however, it changes nothing with respect to what we want to show. If we now insert the expression (7.9) which we found for \( g_{11} \), we see that:

\[
p_{\xi} = m \cdot \text{const} \cdot \Phi \cdot v_{\xi} \tag{7.14}
\]

Like in expression (4.4) which we obtained for the classical case, we see that now the inertial mass is proportional to \( \Phi^2 \) containing the mass-distribution of the entire universe. The same can be shown for the other two components \( p_\theta, p_\phi \) which turn out to be proportional to \( \Phi^2 \). Of course, the derived equations are only approximations. It is surely not exactly true, that:

\[
\frac{2G}{c^2} = \frac{1}{\Phi_0}
\]

This can be seen if we chose more than one mass being in the foreground F, we then had:

\[
\delta \phi = \sum_{j \in F} \frac{m_j}{p_{\delta j}} = \sum_{j \in F} \frac{m_j}{r_{ij} \left( \frac{r_g}{r_{ij}} \right)}
\]

\[
\phi_0 = \sum_{j \in U \setminus F} \frac{m_j}{p_{\delta j}} = \sum_{j \in U \setminus F} \frac{m_j}{r_{ij} \left( \frac{r_g}{r_{ij}} \right)}
\]
(U being all masses in the universe). By plugging this into (7.5) and expanding up to 2nd order one can easily see that it does not agree in the 2nd order with the $g_{00}$ obtained by general relativity, which reads\(^{16}\):

$$g_{00}^{(2)} = \left( \frac{c}{\c_0} \right)^2 = 1 - \frac{2G \delta \Phi}{c_0^2} + \frac{1}{2} \left( \frac{2G \delta \Phi}{c_0^2} \right)^2 + \frac{3G}{c_0^2} \sum_{j \in F} \frac{m_j \delta \Phi_j}{r_{ij}} + \frac{m_j \beta_j^2}{r_{ij}} \quad (7.12)$$

This, however, changes nothing in the qualitative argumentation: It can also be seen from the 2nd order expansion (7.12) that the presence of masses does lower the value of $g_{00}$. For the components $g_{\alpha \alpha}$ it is also still true that the presence of multiple masses increases their value. From there one can reason in the same way as was done for one mass.

It is to be expected, since the gravity field of the universe is a very strong field, that the contribution of non-linear effects to the local space-time background will dominate. But exactly these non-linear effects are the ones which cannot be correctly described by general relativity since there occur two singularities in the Schwarzschild metric, a coordinate singularity at $r = r_g$ and a real singularity at $r = 0$. The latter is already present in Newtonian mechanics and shows that there is something fundamentally wrong in both theories which becomes significant when approaching the mentioned limit. The fact, that coordinate singularity at $r = r_g$ is an expression of the impossibility of defining a rigid coordinate system for $r \leq r_g$, supports the idea that what is wrong has something to do with space itself. That is just an emergent phenomenon which acts as we know it on most scales throughout the universe, but breaks down in such extreme situations as a black hole.

8. The isotropy of inertia:

We have seen that, according to (4.4), the inertial mass is given by:

$$m^* = m \frac{2\Phi_k}{c^2} \quad (8.1)$$

Since this expression is a scalar, it follows that in the classical theory, inertial mass is isotropic. This fact is required by the high precision measurements in the Hughes-Drever experiments which state (to present date) an upper bound for possible mass anisotropies of $\frac{\Delta m}{m} \leq 10^{-34}$ \[^9\]. Nevertheless, expression (8.1) can generate a splitting of states in a quantum system, like was expected in those experiments. Consider therefore some quantum system on the earth and have a mass $M$ in some distance $R$ of the earth (e.g. the sun) in front of the background masses of the universe. In this situation we then have, according to (4.8)\(^{17}\):

$$m^* = m \left( 1 + \frac{2GM}{c^2|\vec{r} + \vec{R}|} \right) \quad (8.2)$$

Here $\vec{r}$ is the position vector pointing from the origin of the quantum system to the particle, $\vec{R}$ the vector pointing from the disturbing mass to the origin of the quantum system. We can Taylor-expand the potential to 2nd order (since in every case $|\vec{r}| \ll |\vec{R}|$ holds):

$$\frac{M}{|\vec{r} + \vec{R}|} = \frac{M}{R} - \frac{M}{R^3} \langle \vec{r}, \vec{R} \rangle + \frac{M}{2R^5} \left( 3 \langle \vec{r}, \vec{R} \rangle^2 - r^2 R^2 \right) + O \left( \frac{r^3}{R} \right)$$

\(^{16}\)Compare, for example, Landau & Lifshitz, Volume II: “The classical theory of fields”, § 106. The equations of motion in second order, eq. (106.13)

\(^{17}\)For the sake of clarity, we suppress the index $k$ in this paragraph.
and plug this into (8.2) to obtain:

$$m^* = m \left( 1 + \frac{2GM}{c^2 R} - \frac{2GM}{c^2 R^2} \langle \vec{r}, \vec{R} \rangle + \frac{GM}{c^2 R^3} (3 \langle \vec{r}, \vec{R} \rangle^2 - r^2 R^2) \right) \quad (8.3)$$

This, together with the one-particle kinetic energy in (4.7) $\frac{m^*}{2} v^2$, leads to a perturbed Hamiltonian for the quantum system:

$$\hat{H} = \hat{H}_0 + \delta \hat{H} \quad (8.4)$$

where:

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + V \quad (8.5)$$

is the commonly known Hamiltonian with a constant mass and:

$$\delta \hat{H} = \left( -\frac{2GM}{c^2 R^3} \langle \vec{R}, \langle \vec{r}, \frac{\hat{p}^2}{2m_{0,0}} \rangle \rangle + \frac{GM}{c^2 R^2} (3 \langle \vec{r}, \vec{R} \rangle^2 - r^2 R^2) \right) \frac{\hat{p}^2}{2m} \quad (8.6)$$

is the perturbation through the mass being dependent on the distance to other masses (in this approximation only the disturbing mass $M$). This leads to an energy shift of:

$$\Delta E = -2 \frac{GM}{c^2 R^3} \langle \vec{R}, \langle \vec{r}, \frac{\hat{p}^2}{2m_{0,0}} \rangle \rangle + \frac{GM}{c^2 R^2} \langle (3 \langle \vec{r}, \vec{R} \rangle^2 - r^2 R^2) \cdot \frac{\hat{p}^2}{2m} \rangle \quad (8.7)$$

The first term only “survives”, if the system is anisotropic itself. More precisely, if the square of the wave function of the system, when expressed in spherical coordinates and expanded in spherical harmonics, provides either terms of uneven order in cos($\theta$) or depends on the polar angle $\phi$. Especially, this implies that if angular momentum is conserved, then the first term in (8.7) vanishes, since the wave functions are proportional to a single spherical harmonic. Since it is $\langle \frac{\hat{p}^2}{2m} \rangle \sim E$ by orders of magnitude we then have for the relative magnitude of the shift:

$$\frac{\Delta E}{E} \sim \frac{GM}{R^3 c^2} d^2 \quad (8.8)$$

where $d$ is a characteristic scale of the quantum system considered. For the disturbing mass being the earth, we have $M \approx 6 \cdot 10^{24}$ kg $R \approx 6 \cdot 10^6$ m and for a nuclear quantum system, which is typically used for the type of experiments considered, we have $d \approx 10^{-15}$ m. This yields:

$$\frac{\Delta E}{E} \sim \frac{GM}{R^3 c^2} d^2 \approx 10^{-53} \quad (8.9)$$

which is far beyond the present experimental precision. However, if the quantum system has the above mentioned anisotropies, we have:

$$\frac{\Delta E}{E} \sim \frac{GM}{R^3 c^2} d \cdot \Gamma \approx 10^{-31} \cdot \Gamma \quad (8.10)$$

Here $\Gamma$ is the relative magnitude of the contribution of terms of uneven order in cos($\theta$), or such dependent on the polar angle $\phi$, to the energy of the system. Considering the experimental limits stated at the beginning of this section, this effect can in principle be observed in a quantum system which satisfies the condition, that these anisotropic terms contribute by a sufficient amount to its total energy. It must be mentioned, that the same splitting is also expected by general relativity, since it contains the same term which was investigated above; more precisely it is (compare eq. (4.16)):
in the situation considered. Obviously, this expression does lead to the same predictions (apart from a factor of 3/2).

The situation of the inertial mass being anisotropic is even worse in the relativistic case, when retardation effects are taken into account. In the absence of them, there does not change much: In this case, the inertial mass is still given by the expression (8.1), just with:

\[ \Phi_k \rightarrow \Phi_k^{(rd)} = \sum_{j \neq i} \frac{m_j}{r_{kj}} \sqrt{1 - \beta_j^2} \]

to which the above discussion applies since the only new terms which occur are powers in the scalar quantity \( \beta_j^2 \). However, the situation does change when retardation effects are taken into account. If we expand this expression around \( t_{\text{ret}} = t \) to second order and use that the Lagrangian stays invariant under adding a total time derivative, we obtain:

\[ \Phi_k^{(\text{rel})} = \sum_{j \neq i} \frac{m_j}{r_{kj}} \langle \vec{\beta}_k , \vec{\beta}_j \rangle \]

stating that inertia is dependent on the angle between the particles’ velocity and the velocities of the masses in the universe. This leads, assuming one mass in the vicinity of the particle to be dominating, to a relative anisotropy of magnitude:

\[ \frac{\delta E}{E} \sim \frac{G M}{2 R c^2} |\vec{\beta}| |\vec{\beta}| \]

where \( M \) is, again, the disturbing body’s mass, \( \beta \) its velocity relative to the laboratory and \( R \) its distance to it. E.g. for an electron in a hydrogen atom, the anisotropy caused by the galactic centre would be:

\[ \frac{\delta E}{E} \sim 10^{-16} \]

which is clearly above the upper limit. Indeed, this is problematic anyway when trying to implement Mach’s principle and retardation at the same time: Since gravitational action travels at a finite speed \( c \), which state of the universe exactly does determine a particles’ inertia? The current one, or the retarded? Again, the same term as (8.12) is also present in general relativity [7]. Therefore it is also affected by the problem of the anisotropic mass[18]). This problem is widely ignored. Considering how the anisotropy arose in the relativistic discussion in section 5, it is suggestive that in general relativity too it arises due to retardation effects. In the author’s opinion, this is an argument in favour of gravity being, at least in some way, an action-at-a-distance: If, to whatever reason, retardation effects would effectively cancel out, it would both solve the problem what state would determine a particles inertia as well as remove the anisotropy terms. This awaits further discussion and will necessarily be part of a relativistic implementation of Mach’s principle.

9. Local variations of the strength of gravity:

According to what was said in the previous sections, mechanics built on Mach’s principle would predict that different measurements of the gravitational constant cannot yield exactly the same value unless the distances to the surrounding masses stay constant during the measurements. Indeed, the Lagrangian (4.3) predicts that the strength of gravity in the close vicinity of masses is weaker and in

\[ L_{\text{EH}} \propto \frac{m}{2} \left( 1 + 3 \cdot \frac{G \delta \Phi}{c^2} \right) v^2 = \frac{m}{2} \left( 1 + \frac{3G M}{c^2 |r + R|} \right) v^2 \]

\[ \Phi_k \rightarrow \Phi_k^{(rd)} = \sum_{j \neq i} \frac{m_j}{r_{kj}} \sqrt{1 - \beta_j^2} \]

\[ \Phi_k^{(\text{rel})} \sim \sum_{j \neq i} \frac{m_j}{r_{kj}} \langle \vec{\beta}_k , \vec{\beta}_j \rangle \]

\[ \frac{\delta E}{E} \sim 10^{-16} \]

\[ 18) \text{This was already mentioned by Treder [3] and Retzlaff [4].} \]
the greater distance of them stronger as compared to Newtonian theory. To see this, we stay in the situation given in section 4. We then have, according to (5.1), for the acceleration of some test particle in lowest order (the higher order terms have no impact on the statement):

\[ \frac{\partial \vec{v}_k}{\partial t} = G \cdot \vec{\nabla} \delta \Phi_k \]  

(9.1)

\[ G = \frac{1}{2 G_0 \delta \Phi} - \frac{G_0}{1 + \frac{2 G_0 \delta \Phi}{c^2}} \]  

(9.2)

Since this is exactly the same expression which one obtains from the EIH-Lagrangian (up to a factor of 3/2 and the fact that the gravitational constant is defined as \( G_0 \) in above equation), all the following statements also apply to general relativity. One can simply derive now that an observer at some point \( \vec{r}_0 \) (e.g. the Earth) within a foreground mass distribution who performs a measurement of gravity’s strength \( G_0 \) and interpolates according to Newton’s law gets the acceleration wrong by:

\[ \delta \vec{a} = (G(\vec{r}) - G(\vec{r}_0)) \vec{\nabla} \Phi \]  

(9.3)

or by Taylor expanding \( G \) up to first order in:

\[ \delta \vec{a} = G_0 \frac{\delta \Phi(\vec{r}_0) - \delta \Phi(\vec{r})}{\Phi_0} \vec{\nabla} \Phi = \frac{\delta \Phi(\vec{r}_0) - \delta \Phi(\vec{r})}{\Phi_0} \vec{a}_N \]  

(9.4)

where \( \vec{a}_N \) is the gravitational acceleration expected by Newton’s law. From this equation can be seen the following: The observer who extrapolates Newton’s law of gravitation would, by doing so, overestimate gravity’s strength in the closer vicinity of masses than he found himself in, and underestimate it at a greater distance from them.

Let us now look what eq. (9.4) would imply for the strength of the gravitational surface acceleration of the Earth. Since the Earth orbits the Sun, we have \( \delta \Phi = \Phi_s + \Phi_E \) (the contributions of other planets are negligible) and for the surface acceleration of the Earth, \( \vec{a}_N \approx - \frac{G_0 M_E}{R_E^2} = \vec{g} \). The potential of the sun is given by \( \Phi_s = \frac{M_s}{r} \), where \( r \) is the distance between the Earth and the Sun. Plugging this into (9.4) together with:

\[ r = a \frac{(1 - e^2)}{1 + e \cdot \cos(\theta)} \]

where \( a = 1 AU \) is the great half axis of the Earth-Sun system and \( e \) the eccentricity of the Earth’s orbit, we obtain:

\[ \delta \vec{g} = \frac{2 G_0 M_S}{a c^2} \frac{e}{1 - e^2} (\cos(\theta_0) - \cos(\theta)) \vec{g}_0 \]  

(9.5)

Here we have again inserted (4.6). \( \theta_0 \) is the angle corresponding to the Earth’s position where the initial measurement was made. According to (8.5), the strength of the Earth’s surface acceleration will vary periodically over a year, having its minimum value when the earth is in its perihelion. The relative magnitude of this effect is:

\[ \frac{2 G_0 M_S}{a c^2} e \approx 10^{-10} \]
The same effect does occur in every system whose distance changes over time relative to a massive object nearby. Since this effect occurs due to inertial mass being determined by gravitation of other masses according to eq. (4.15), it’s a direct expression of Mach’s principle. A confirmation of this effect would therefore also be a confirmation of this principle. Finally, we want to note that also the perihelion shift of Mercury is caused by the mass induction term in general relativity, which is, as already mentioned, of the same form as (4.15), just with a factor of 3/2 standing by the second addend. Since these terms are clearly of Machian nature, also the precession of Mercury can already be regarded as a strong evidence for Mach’s principle. However, it is not mandatory since other terms (e.g. the first relativistic correction to the kinetic energy) also yield a contribution to the precession, and therefore this observation does not necessarily imply the presence of terms like (4.15).

10. Conclusions:

I have shown a way how to implement Mach’s principle into classical mechanics, and argued that kinetic energy should be redefined as was proposed by Hofmann. It would cover a variety of phenomena already in classical mechanics, which are currently solely attributed to general relativity, namely those which are of Machian nature. Above all, it explains the nature of one arbitrary number, the gravitational constant. I therefore think that classical mechanics should be formulated in this way. Consequently, also with respect to relativistic theories, it would be worthwhile to stretch beyond general relativity: It does neither explain the gravitational constant, which is, to my understanding, a result of not correctly incorporating Mach’s principle, nor does it explain the speed of light c, which is, to my understanding a consequence of not understanding the origin of gravity and special relativity. Apart from not correctly accounting for the isotropy of inertia, it also causes some well known problems, especially in cosmology: those disagreements with observations, which are covered up by terms like dark matter, dark energy, inflation and so on. Last but not least, it’s solutions have singularities, which must not appear in any complete theory; they are a mere consequence of general relativity still being tight to the concept of space and being unable to deal with situations, in which this concept breaks down. Nature is regular everywhere and so should be its laws.

If we are to understand phenomena like dark matter or dark energy, the only way we can hope to one day succeed is to get an understanding of why the laws of physics are the way they are. As said at the beginning, i think the constants of nature play a key role in understanding; therefore they have to be questioned in particular. By finding more expressions like the one linking c and the electromagnetic constants, or the one obtained here for G, it should be possible to actually explain and calculate all dimensional constants in the laws of nature, instead of setting them to unity, which is a faulty practice anyway since it simply sweeps away something which is unexplained. The dimensionless quantities which then remain may ultimately provide a link between physics and pure mathematics.

References:


[4] Klaus Retzlaff, Projekt Machsches Prinzip (Trägheitsfreie Mechanik) der Astronomischen Gesellschaft Magdeburg e.V.


