Mathematical Model of Kerr Black Hole Consciousness: via Mathematical Structure of Integrated Information Theory

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ABSTRACT

This paper uses the Model of Consciousness called Integrated Information Theory (IIT) developed by Giulio Tononi and his team and generalized Mathematical Model of IIT and Quantum IIT developed by Johannes Kleiner and Sean Tull, to develop a model of consciousness for Kerr black hole. In which I will step by step define System, State, Mechanism, Space of Proto-experience, Division & Cuts and Repertoire for Kerr Black Hole in terms of it's mass and angular momentum per unit mass using Information Geometry to be precise Ruppeiner geometry, where probability density is defined in terms of Ruppeiner metric, which is negative of hessianof entropy of Kerr black hole. In the section of black hole thermodynamics we will also discuss how entropy is the function of areaof event horizon of black hole:. Prior to that we will discuss what is event horizon and how it's radius is a function of it's mass andangular momentum per unit area. This paper can serve a pioneering work on understanding and developing a relationshipbetween cosmology and Consciousness, and as it is suggested that universe could itself be a black hole: this paper may help to develop a model for consciousness of cosmos itself.

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1 Kerr Black Hole

Depending on the parameters M and a, Kerr spacetimes are divided into three categories: slowly rotating Kerr spacetime ("slow Kerr") for $0 < a^2 < m^2$, extreme Kerr spacetime for a 2 = M2 , and rapidly rotating Kerr spacetime ("fast Kerr") for $m^2 < a^2$. In this paper, we are concerned with slow Kerr spacetimes only, since most physically interesting phenomena manifest only for this type of spacetime.

Similar to the Schwarzschild case, the points where the Kerr metric fails provide important physical insights. Notice that the Kerr metric does not fail at $r=0$, so we let r run over the whole real line.It is topologically convenient to think of the coordinates r and t as cartesian coordinates over $\mathbb{R}^2$ together with spherical coordinates θ and φ on $\mathbb{S}^2$; hence the Kerr spacetime is modeled as the product manifold $\mathbb{R}^2 \times \mathbb{S}^2$. There are three subsets of the space at which the metric fails:

1. On the *horizons*: $\Delta = 0$. For slow Kerr, this horizon equation has two distinct solutions $r_{\pm} = m \pm \sqrt{m^2 - a^2}$. Define the outer and inner horizons H_+ and H_- to be the set of points where $r = r_+$ and $r = r_-$, respectively.

2. On the *ring singularity* Σ: $\rho^2 = 0$. This terminology comes from the fact that $\rho^2 = 0$ if and only if both $r = 0$ and $\cos\theta = 0$. Thus, Σ is the cartesian product of a time axis $\mathbb{R}^1(t)$ and a circle S^1 , namely, the equatorial circle $\theta = \pi/2$ in S^2 at radius $r = 0$. Informally, the circle itself is sometimes called the ring singularity, with $\Sigma = S^1 \times \mathbb{R}^1$ its history through time.

3. On the axis A: $\sin\theta = 0$. In the sphere S^2, $\sin\theta = 0$ picks out the north pole $p(\theta = 0)$ and the south pole $-p(\theta = \pi)$. Hence in the Kerr spacetime $\mathbb{R}^2 \times \mathbb{S}^2$, the solution set to the axis equation is

$$[\mathbb{R}^2(r,t) \times \{p\}] \cup [\mathbb{R}^2(r,t) \times \{-p\}]$$

2 Black hole thermodynamics

Axiom 2.1. *S_{BH} is the entropy and A is the surface area of event horizon. Then*

$$S_{\mathrm{BH}} \propto A$$

Axiom 2.2. *The event horizon is characterized by a quantity, κ, known as the surface gravity. The surface gravity is uniformly constant over the event horizon. The black hole's surface gravity seemingly has temperature-like properties in that it has absolute zero, arbitrary scale and is defined in equilibrium. We can thus suspect that the black hole's temperature, T_{BH} is proportional to its surface*

$$T_{\mathrm{BH}} \propto \kappa$$

Now the great contribution of Hawking to black hole physics—despite all the surprises and initial incredulity—is that he convincingly and systematically derived the proportionality constant

$$S_{\mathrm{BH}} = \frac{c^3 \pi k_B A}{2\, G h}$$

$$T_{\mathrm{BH}} = \frac{h \kappa}{4\pi^2 c k_B}$$

Now setting up the parameters $c = G = k_B = 1$, $h = 2\pi = 2\,(k_B)^{-1}$

we get,

$$S_{\mathrm{BH}} = \frac{A}{4\pi} \quad \text{and} \quad T_{\mathrm{BH}} = \frac{\kappa}{2\pi}$$

2.1 Kerr black hole thermodynamics and information geometry

$A = 4\pi(r_+)^2$ and $S_{\mathrm{BH}} = \dfrac{A}{4\pi} \Rightarrow S_{\mathrm{BH}} = (r_+)^2$

$$S_{\mathrm{BH}} = m^2 \left(1 + \sqrt{1 - \left(\frac{a}{m}\right)^2}\right)^2 = \left(m + \sqrt{m^2 - a^2}\right)^2$$

2.1.1 Thermodynamic geometry

Definition 2.1. *The* Ruppeiner metric *is defined as the negative of the Hessian of the entropy function with respect to the thermodynamic system's mechanically conserved quantities:*

$$g^R_{ij} = -\partial_i \partial_j S(x)$$

Where $i,j\in\{M,a\}$ in the case of Kerr blackholes

let Ω be the number of (equiprobable) microstates consistent with a given macroscopic state. Boltzmann argued that the macroscopic entropy is given by

$$S=k_B\ln\Omega$$

Einstein rewrote this equation as

$$P\propto e^{S/k_B}$$

where p is the probability decity that the given macrostate will be realized. We can Taylor expand the entropy around an equilibrium state, taking into account that the entropy has a maximum there, and introduce the Hessian matrix.

$$g_{ij}^R(x)=-\partial_i\partial_jS(x)$$

Here x stands for the n extensive variables shifted so that they take the value zero at equilibrium. The matrix is positive definite if the entropy is concave. If we normalize the resulting probability distribution (and set $k_B=1$) we obtain

$$P(x)=\frac{\sqrt{\det(g^R)}}{(2\pi)^{\frac{n}{2}}}e^{-\frac{1}{2}\sum_{ij}g_{ij}^Rx^ix^j}$$

In the case kerr blace holes $n=2$, and $i,j\in\{M,a\}$

2.1.2 Flatness theorem

It is observable that there are seemingly geometrical patterns of thermodynamic geometries for black hole families. It is then natural to investigate why some thermodynamic geometries are flat, whilst the others are not. In information geometry we can define a metric in some preferred affine coordinate system by

$$g_{\mathrm{ij}}^I=\partial_i\partial_j\psi$$

where $\psi=-S$, entropy.

Now, the main question is when is an *information metric flat*?

Theorem 2.1. *Flatness theorem : The information metric defined through*

$$g_{ij}^I(x,y)=\partial_i\partial_j\psi(x,y)\qquad\text{and}\qquad\psi(x,y)=-x^a\_f(x^b\_y)$$

with $b=-1$ and $a\neq1$, x and y are coordinates on the state space and f is some smooth function, is flat.

Proof.

We change coordinates on state space $(x,y)\to(\psi,\sigma)$

where $\psi=x^a\_f(x^b\_y)$, as metioned above, and $\sigma=x^b\_y$ and taking $b=-1$

we get we get a flat manifold. □

By flatness theorem themodynamic geometry of Kerr black hole is flat.

3 The Mathematical Structure Of IIT 3.0

Integrated Information Theory (IIT), developed by Giulio Tononi and collaborators, has emerged as one of the leading scientific theories of consciousness. The more generel mathematical model of IIT (3.0 the latest version) was developed by Johannes Kleiner (Munich Center for Mathematical Philosophy) and Sean Tull (University of Cambridge), in the rest of the paper we will review it and apply it to kerr black hole, to know the degree of cousiouness in them which may also give us an idea if universe is consiouss or not.

3.1 Systems

Definition 3.1. *A system class (a collection of sets, or more generaly mathematical objects)* Sys *is a class each of whose elements S, called systems, come with the following data:*

1. *a set* $\mathrm{St}(S)$ *of states;*
2. *for every* $s \in \mathrm{St}(S)$*, a set* $\mathrm{Sub}_s(S) \subset \mathrm{Sys}$ *of subsystems and for each* $M \in \mathrm{Sub}_s(S)$ *an induced state* $s|_M \in \mathrm{St}(M)$
3. *a set* $\mathbb{D}_S$ *of decompositions, with a given trivial decomposition* $1 \in \mathbb{D}_S$*;*
4. *for each* $z \in \mathbb{D}_S$ *a corresponding cut system* $S^z \in \mathrm{Sys}$ *and for each state* $s \in \mathrm{St}(S)$ *a corresponding cut state* $s^z \in \mathrm{St}(S^z)$.
5. Sys *contains a distinguished empty system, denoted I,and that* $I \in \mathrm{Sub}(S)$ *for all S.*

Axiom 3.1. $\mathrm{Sub}_s(S) \simeq \mathrm{Sub}_{s'}(S)$ *for all* $s, s' \in \mathrm{St}(S)$ *and* $\mathrm{Sub}_s(S) \simeq \mathrm{Sub}_{s^z}(S^z)$ *for all* $z \in \mathbb{D}_S$*. Here* $\simeq$ *indicates bijections.*

3.2 Experience

Definition 3.2. *An experience space is a set E with:*

1. *an intensity function* $||.||: E \to \mathbb{R}^+$
2. *a distance function* $d: E \times E \to \mathbb{R}^+$
3. *a scalar multiplication* $\mathbb{R}^+ \times E \to E$*, denoted* $(r, e) \to r\bullet e$*, satisfying*

   I. $||r\bullet e|| = r\bullet ||e||$

   II. $r\bullet(s\bullet e) = (rs)\bullet e$

   III. $1\bullet e = e$

   for all $e \in E$ *and* $r, s \in \mathbb{R}^+$

Note 3.1. the distance function does not necessarily have to satisfy the axioms of a metric. While this and further natural axioms such as $d(r\bullet e, r\bullet f) = r\bullet d(e, f)$ might hold, they are not necessary for the IIT algorithm

Example 3.1. Any metric space (X, d) may be extended to an experience space $\bar{X} := X \times \mathbb{R}^+$ in various ways. E.g., one can define $||(x, r)|| = r$, $r\bullet(x, s) = (x, rs)$ and define the distance as

$$\bar{d}((x,r),(y,s)) = (r-s)\, d(x,y)$$

This is the definition used in classical IIT

Definition 3.3.

1. *For experience spaces E and F, we define the product to be the space* $E \times F$ *with*

   - *distance*

     $$d((e,f),(e',f')) = d(e,e') + d(f,f')$$

   - *intensity,*

     $$||(e,f)|| = \max\{||e||, ||f||\}$$

   - *and scalar multiplication*

     $$r\centerdot(e,f) = (r\centerdot e, r\centerdot f)$$

2. *Generalization of the product of expirence space is* $\prod_i E_i$ *where i belongs to a ring or field or subset of them e.g. set of integers with:*

   - *distance*

     $$d((e_i)_i, (f_i)_i) = \sum_i d(e_i, f_i)$$

   - *intensity,*

     $$||(e_i)_i|| = \max\{||e_i||\}_i$$

   - *scalar product*

     $$r(e_i)_i = (r e_i)_i$$

3.3 Repertoires

Definition 3.4. *Let D denote any set with a distinguished element* 1*, for example the set* D_S *of decompositions of a system S, where the distinguished element is the trivial decomposition* $1 \in \mathbb{D}_S$*. Let e be an element of an experience space E. Then a decomposition of e over D is a mapping* $\bar{e}: D \to E$ *with* $\bar{e}(1) = e$*.*

For subsystems $M, P \in \mathrm{Sub}_s(S)$*, define* $\mathbb{D}_{M,P} := \mathbb{D}_M \times \mathbb{D}_P$ *. This set describes the decomposition of both subsystems simultaneously. It has a distinguished element* $1_{MP} = (1_M, 1_P)$*.*

Definition 3.5. *A* cause-effect repertoire *at S is given by a choice of experience space* $\mathbb{PE}(S)$*, called the space of proto-experiences, and for each* $s \in \mathrm{St}(S)$ *and* $M, P \in \mathrm{Sub}_s(S)$*, a pair of elements*

$$\mathrm{caus}_s(M,P), \mathrm{eff}_s(M,P) \in \mathbb{PE}(S)$$

and for each of them a decomposition over $\mathbb{D}_{M,P}$

So, to futher elobrate cause-effect repertoire at S is experience space $\mathbb{PE}(S)$ with

* an intensity function $||.||: \mathbb{PE}(S) \to \mathbb{R}^+$
* a distance function $d: \mathbb{PE}(S) \times \mathbb{PE}(S) \to \mathbb{R}^+$
* a scalar multiplication $\mathbb{R}^+ \times \mathbb{PE}(S) \to \mathbb{PE}(S)$, denoted $(r, \mathrm{caus}_s(M,P)) \to r\,\text{.}\mathrm{caus}_s(M,P)$, satisfying,

   I. $||r\,\text{.}\,\mathrm{caus}_s(M,P)|| = r\,\text{.}\,||\,\mathrm{caus}_s(M,P)\,||$

   II. $r\,\text{.}\,(s\,\text{.}\,\mathrm{caus}_s(M,P)) = (rs)\,\text{.}\,\mathrm{caus}_s(M,P)$

   III. $1\,\text{.}\,\mathrm{caus}_s(M,P) = \mathrm{caus}_s(M,P)$

   for all $\mathrm{caus}_s(M,P) \in \mathbb{PE}(S)$ and $r, s \in \mathbb{R}^+$ similarly for $\mathrm{eff}_s(M,P)$.

Decoposition of $\mathrm{caus}_s(M,P)$ over $\mathbb{D}_{M,P}$ is the mapping $\overline{\mathrm{caus}_s}(M,P): \mathbb{D}_{M,P} \to \mathbb{PE}(S)$, such that $\overline{\mathrm{caus}_s}(M,P)(1_{MP}) = \mathrm{caus}_s(M,P)$, similarly for $\mathrm{eff}_s(M,P)$.

Definition 3.6. *7. A* cause-effect structure *is a specification of a cause-effect repertoire for every* $S \in \mathrm{Sys}$ *such that.*

$$\mathbb{PE}(S) = \mathbb{PE}(S^z) \quad \text{for all} \quad z \in \mathbb{D}_s$$

The names ‘cause’ and ‘effect’ highlight that the definitions of $\mathrm{caus}_s(M,P)$ and $\mathrm{eff}_s(M,P)$ in classical and quantum IIT describe the causal dynamics of the system. More precisely, they are intended to capture the manner in which the ‘current’ state s of the system, when restricted to M, constrains the ‘previous’ or ‘next’ state of P, respectively

3.4 Integration

I have now introduced all of the data required to define an IIT; namely, a system class along with a cause-effect structure. From this, we will give an algorithm aiming to specify the conscious experience of a system. Before proceeding to do so, we introduce a conceptual short-cut which allows the algorithm to be stated in a concise form. This captures the core ingredient of an IIT, namely the computation of how integrated an entity is.

Definition 3.7. *Let E be an experience space and e an element with a decomposition over some set D. The* integration *level of e relative to this decomposition is*

$$\phi(e) := \min_{1 \neq z \in D} d(e, \bar{e}(z))$$

Definition 3.8. *The* integration scaling *of e is then the element of E defined by*

$$\iota(e) := \phi(e)\,\text{.}\,\hat{e}$$

where ˆe denotes the normalization of e, defined as

$$\hat{e} := \begin{cases} \frac{1}{\|e\|} \cdot e & \text{if } \|e\| \neq 0 \\ e & \text{if } \|e\| = 0 \end{cases} \square$$

Finally, the integration scaling of a pair e_1, e_2 of such elements is the pair.

$$\iota(e_1, e_2) := (\phi(e_1) \cdot \hat{e_1}, \phi(e_2) \cdot \hat{e_2})$$

where $\phi := \min(\phi(e_1), \phi(e_2))$ is the minimum of their integration levels.

Definition 3.9. *Let S be a system in a state $s \in \mathrm{St}(S)$ and assume that for every $M \in \mathrm{Sub}_s(S)$ an element e_M of some experience space E_M with a decomposition over some D_M is given. We call $(e_M)_{M \in \mathrm{Subs}(S)}$ a collection of decomposable elements, and denote it as $(e_M)_M$.*

Definition 3.10. *The core of the collection $(e_M)_M$ is the subsystem $C \in \mathrm{Sub}(S)$ for which $\phi(e_C)$ is maximal.*$\square$ *The core integration scaling of the collection is $\iota(e_C)$.*

i.e. $\phi(e_C) = \max\left\{\phi(e_M) := \min_{1 \neq z \in D} d(e_M, \bar{e}_M(z)) : M \in \mathrm{Sub}(S)\right\}$ *and*

Definition 3.11. *The core integration scaling of a pair of collections $(e_M, f_M)_M$ is $\iota(e_C, f_D)$, where C, D are the cores of $(e_M)_M$ and $(f_M)_M$, respectively.*

3.5 Constructions - Mechanism Level

Let $S \in \mathrm{Sys}$ be a physical system whose experience in a state $s \in \mathrm{St}(S)$ is to be determined. The first level of the algorithm involves fixing some subsystem $M \in \mathrm{Subs}(S)$, referred to as a '*mechanism*', and associating to it an object called its '*concept*' which belongs to the *concept space.*

$$\mathbb{C}(S) := \mathbb{PE}(S) \times \mathbb{PE}(S)$$

For every choice of $P \in \mathrm{Sub}_s(S)$, called a '*purview*', the repertoire values $\mathrm{caus}_s(M, P)$ and $\mathrm{effs}(M, P)$ are elements of $\mathbb{PE}(S)$ with given decompositions over $\mathbb{D}_{M,P}$. Fixing M, they form collection of decomposable elements

$$\mathrm{caus}_s(M) := (\mathrm{caus}_s(M, P))_{P \in \mathrm{Sub}(S)}$$

$$\mathrm{eff}_s(M) := (\mathrm{eff}_s(M, P))_{P \in \mathrm{Sub}(S)}$$

Definition 3.12. *The concept of M, $\mathbb{C}_{S,s}(M)$ is then defined as the core integration scaling of this pair of collections,*

$$\mathbb{C}_{S,s}(M) = \iota(\mathrm{caus}_s(M, P^c), \mathrm{eff}_s(M, P^e)) \square$$

where Q (and R) are the core cause (and effect) perviews defined as follows.

$$P^c \in \{P : \max\{\phi(\mathrm{caus}_s(M, P)), P \in \mathrm{Sub}(S)\}\}$$

similarly for P^e, where

$$\phi(\mathrm{caus}_s(M, P)) := \min_{1 \neq z \in \mathbb{D}_{M,P}} d(\mathrm{caus}_s(M, P), \overline{\mathrm{caus}_s}(M, P)(z))$$

Also,

$$\iota\Big(\mathrm{caus}_s(M, P^c), \mathrm{eff}_s(M, P^e)\Big) = \Big(\phi(\mathrm{caus}_s(M, P^c))\, \widehat{\mathrm{caus}}_s(M, P^c), \phi(\mathrm{eff}_s(M, P^e))\, \widehat{\mathrm{eff}}_s(M, P^e)\Big)$$

and where $\widehat{\mathrm{caus}}_s(M, P^c)$ *is normalisation of* $\mathrm{caus}_s(M, Q)$ *and similarly for* $\widehat{\mathrm{eff}}_s(M, P^e)$

It is an element of $\mathbb{C}(S)$. Unravelling our definitions, the concept thus consists of the values of the cause and effect repertoires at their respective 'core' purviews P^c , P^e , i.e. those which make them 'most integrated'. These values $\mathrm{caus}(M, P^c)$ and $\mathrm{eff}(M, P^e)$ are then each rescaled to have intensity given by the minima of their two integration levels.

3.6 Constructions - System Level

Definition 3.13. *all concepts of a system are collected to form its* $\mathbb{Q}$*-shape, defined as*

$$\mathbb{Q}_s(S) := (\mathbb{C}_{S,s}(M))_{M \in \mathrm{Subs}(S)}$$

This is an element of the space

$$\mathbb{E}(S) = \mathbb{C}(S)^{\,n(S)}$$

where $n(S) := |\mathrm{Sub}_s(S)|$*, the cardinality of* $\mathrm{Sub}_s(S)$*,which is finite and independent of the state s according to our assumptions..*

Note 3.2. We can also define a $\mathbb{Q}$-shape for any cut of S. Let $z \in \mathbb{D}_S$ be a decomposition, S^z the corresponding cut system and s z be the corresponding cut state. We define

$$\mathbb{Q}_s(S^{\,z}) := (\mathbb{C}_{S^z, s^z}(M))_{M \in \mathrm{Sub}_{s^z}(S^z)}$$

Because

$$\mathbb{PE}(S) = \mathbb{PE}(S^{\,z}) \quad \text{for all} \quad z \in \mathbb{D}_s,$$

and since the number of subsystems remains the same when cutting, $\mathbb{Q}$s(S z) is also an element of $\mathbb{E}$(S). This gives a map

$$\bar{\mathbb{Q}}_{S,s} : \mathbb{D}_S \to \mathbb{E}(S)$$

$$z \to \mathbb{Q}_s(S^z)$$

which is a decomposition of $\mathbb{Q}_s(S)$ over $\mathbb{D}_S$.

Definition 3.14. *Considering this map for every subsystem of S gives a collection of decompositions defined as:*

$$\mathbb{Q}(S, s) := (\bar{\mathbb{Q}}_{M, s|_M})_{M \in \mathrm{Sub}_s(S)}$$

This is the system level-object of relevance and is what specifies the experience of a system according to IIT.

Definition 3.15. *The actual experience of the system S in the state* $s \in \mathrm{St}(S)$ *is*

$$\mathbb{E}(S, s) := \text{Core integration scaling of}\, \mathbb{Q}(S, s)$$

i.e.

$$\mathbb{E}(S,s)=\iota(\bar{\mathbb{Q}}_{C,s|_C})=\Phi(\bar{\mathbb{Q}}_{C,s|_C})\hat{\bar{\mathbb{Q}}}_{C,s|_C}$$

where C is,

$$C\in\left\{K\colon\Phi(\bar{\mathbb{Q}}_{K,s|_K})=\max\left\{\Phi(\bar{\mathbb{Q}}_{M,s|_M}):=\min_{1\neq z\in\mathbb{D}_M} d(\bar{\mathbb{Q}}_{M,s|_M},\bar{\mathbb{Q}}_{M^z,s^z|_{M^z}})\colon M\in\mathrm{Sub}_s(S)\right\}\right\}$$

and $\hat{\bar{\mathbb{Q}}}_{C,s|_C}$ is normalised $\bar{\mathbb{Q}}_{C,s|_C}$.

The definition implies that $\mathbb{E}(S,s)\in\mathbb{E}(M)$, where $M\in\mathrm{Sub}_s(S)$ is the core of the collection $\mathbb{Q}(S,s)$, called the major complex. It describes which part of the system S is actually conscious. In most cases there will be a natural embedding $\mathbb{E}(M)\to\mathbb{E}(S)$ for a subsystem M of S, allowing us to view $\mathbb{E}(S,s)$ as an element of $\mathbb{E}(S)$ itself. Assuming this embedding to exist allows us to define an Integrated Information Theory concisely in the next section.

3.7 Integrated Information Theories

Definition 3.16. *An Integrated Information Theory is determined as follows. The data of the theory is a system class* Sys *along with a cause-effect structure. The theory then gives a mapping*

$$\mathrm{Sys}\xrightarrow{\mathbb{E}}\mathrm{Exp}$$

into the class Exp *of all experience spaces, sending each system S to its space of experiences $\mathbb{E}(S)$ defined as,*

$$\mathbb{E}(S)=\mathbb{C}(S)^{\,n(S)}$$

and a mapping

$$\begin{aligned}\mathrm{St}(S)&\to\mathbb{E}(S)\\ s&\to\mathbb{E}(S,s)\end{aligned}$$

which determines the experience of the system when in a state s, defined above. The quantity of the system's experience is given by

$$\Phi(S,s)=\|\mathbb{E}(S,s)\|$$

and the quality of the system's experience is given by the normalized experience $\mathbb{E}(S,s)$. The experience is located in the core of the collection $\mathbb{Q}(S,s)$, called major complex, which is a subsystem of S.

4 Quantum IIT and Kerr Black Hole

In this section we will try use the framework of Quantum IIT and it apply on black hole to develop a model for consiouness of black holes.

4.1 Systems for Kerr Black Hole

1. We first describe the system class underlying classical IIT. Physical systems S are considered to be built up of several components $\mathcal{H}_1,\ldots,\mathcal{H}_n$, called elements

   In case of Kerr black hole elements are $\mathcal{H}_m,\mathcal{H}_a\subset\mathbb{R}$ where m reperests mass and a represents angular momentum per unit mass.

2. Each element $\mathcal{H}_i$ comes with a state space S, i.e. $\mathrm{St}(S)$ defined as.

$$\mathrm{St}(S)=\mathcal{S}(\mathcal{H}_s)$$

where:

I. $\mathcal{H}_s=\bigoplus_{i=1}^{n}\bigotimes_{i=1}^{n}\mathcal{H}_i$

In case of Kerr black hole:

$$\mathcal{H}_S=\bigoplus_{i\in\{m,a\}}\bigotimes_{i\in\{m,a\}}\mathcal{H}_i,$$

where $\mathcal{H}_m,\mathcal{H}_a\subset\mathbb{R}$ such that $\mathcal{H}_m\bigcup\mathcal{H}_a=\mathcal{H}:=\mathbb{R}$ and $\mathcal{H}_m\bigcap\mathcal{H}_a=\varnothing$

II. $\mathcal{S}(\mathcal{H}_s)\subset L(\mathcal{H}_s)$ describes the positive semidefinite Hermitian operators of unit trace on $\mathcal{H}_s$, aka density matrices

In case of Kerr black hole elements :

$$\mathcal{S}(\mathcal{H}_S)=\{\tilde{P}:\mathcal{H}_S\to\mathbb{R}_0^+\,|\,\mathrm{tr}(P)=1\}$$

also

$$P(m,a)=\frac{\sqrt{\det(g_{ij}^R)_{i,j\in\{m,a\}}}}{2\pi}e^{-\sum_{i,j\in\{m,a\}}g_{ij}^R x_i x_j}\quad\text{for all } m\otimes a\in\bigotimes_{i\in\{m,a\}}\mathcal{H}_i$$

and let $\varphi:\mathcal{H}_m\times\mathcal{H}_a\to\mathcal{H}_S$ using universal property of algebra $P=\tilde{P}\circ\varphi$ for the sake of simlicity we will be using P instead of $\tilde{P}$, it wouldn't make much difference.. Hence we would beconsidering $\mathcal{S}(\mathcal{H}_S)=\{P\}$

where $x_m:=m$, $x_a:=a$ and $(g_{ij}^R)_{i,j\in\{m,a\}}$ is *Ruppeiner metric* mesitioned is section 5.1.1 Thermodynamic geometry.

III. we define trace as, this has been done taking into considration that $P,Q\in\mathcal{S}(\mathcal{H}_S)$

before defining trace we will make some co-ordenate tranformations, in the following manner:

Let $f:(0,m)\to\mathbb{R}$, be a bijection such that $\tilde{f}(0)=-\infty$ and $\tilde{f}(m)=\infty$, $\tilde{f}$ is dmooth fuction and $\tilde{f}'(a)>0$, for all a so that f would be strictly incrising function and $\max\{\tilde{f}(a)\}=\tilde{f}(m)$ and $\min\{\tilde{f}(a)\}=\tilde{f}(0)$. where f is restriction of $\tilde{f}$ over $(0,m)$.

example of such a f is

$$f(a)=\ln(a(m-a)^{-1})$$

and let $h$be f^{-1} in our example

$$a=h(x)=\frac{m}{e^{-x}+1}\text{ and }h'(x)=\frac{me^{-x}}{(e^{-x}+1)^2}\text{ for all } y\in\mathbb{R}$$

Let $u:(a,\infty)\to\mathbb{R}$, be a bijection such that $\tilde{u}(a)=-\infty$ and $\tilde{u}(\infty)=\infty$, $\tilde{u}$ is smooth fuction and $\tilde{u}'(a)>0$, for all a so that $\tilde{u}$ would be strictly incrising function and $\max\{\tilde{u}(a)\}=\tilde{u}(m)$ and $\min\{\tilde{u}(a)\}=\tilde{u}(0)$. where u is restriction of $\tilde{u}$ over $(0,m)$.

example of such a u is

$$u(a)=\ln(m(m-a))$$

and let v be u^{-1} in our example.

$$m = v(x) = \frac{a + \sqrt{a^2 + 4\_e^y}}{2} \text{ and } v'(x) = e^y (a^2 + 4\_e^y)^{-\frac{1}{2}} \text{ for all } y \in \mathbb{R}$$

taking $a = \alpha(z, y), m = \mu(z, y)$. And in perticular example of ours using above four equations and taking $x + y = z$ would lead to

$$a = \alpha(z, y) := \frac{e^z}{\sqrt{e^y + e^z}} \text{ for all } z \in \mathbb{R},\ y \in \mathbb{R}$$

$$m = \mu(z, y) := \sqrt{e^y + e^z} \text{ for all } z \in \mathbb{R},\ y \in \mathbb{R}$$

a) $\operatorname{tr}_a P\,(m) = \int_0^m P(m, a)\, da,$

which leads to:

$$\operatorname{tr}_a P = \int_{\mathbb{R}} P(m, h(x)) h'(x)\, dx$$

b) $\operatorname{tr}_m P\,(a) = \int_a^\infty P(m, a)\, dm$

which leads to:

$$\operatorname{tr}_a P\,(a) = \int_{\mathbb{R}} P(v(y), a) v'(y)\, dy$$

c) $\operatorname{tr}_a (P \otimes Q)\,(a) = \int_0^m P(m, a) Q\,(m, a) da$, similarly with m

d) $\operatorname{tr}_\emptyset P(a, m) = P(a, m)$

e) $\operatorname{tr}_S P = \int_{\mathcal{H}_m \times \mathcal{H}_a} P(a, m) da\, dm = \int_{\mathbb{R}} \int_{\mathbb{R}} P(\alpha, \mu)\_\det(J)\_dx\, dy = 1$ where J is jacobean

3. The time evolution of the system is again given by a time evolution operator, which here is assumed to be a trace preserving completely positive map

$$\mathcal{T}: L(\mathcal{H}_S) \rightarrow L(\mathcal{H}_S)$$

i.e.

$$\mathcal{T} \in \{\mathcal{F}: L(\mathcal{H}_S) \rightarrow L(\mathcal{H}_S) | \operatorname{tr}(\mathcal{F}(P)) = \operatorname{tr}(P) = 1\}$$

and if P is positive definate the $\mathcal{T}(P)$ is also positive definate.

In case of Kerr black hole:

$$\mathcal{T}(P)(a, m) \quad \text{is} \quad e^{-\tau \frac{d}{dt}}(P(a, m)) = P(a_\tau, m_\tau)$$

where τ is time lapsed and $\frac{d}{dt}$ is derivative with respect to time given mass and aungular moment per unit time are fuction of time.

m_τ is mass at time τ, similarly a_τ is the anuglar momentum per unit mass at time τ.

4. We also dfine $\mathbb{S} := \{\emptyset, m, a, S\}$ and $\mathrm{Sub}(S) \subset \mathbb{S}$

4.2 Subsystems for Kerr Black Hole

1. Subsystems are again defined to consist of subsets M of the elements of the system, with corresponding Hilbert space

$$\mathcal{H}_M = \bigoplus_{i \in M} \bigotimes_{i \in M} \mathcal{H}_i$$

In case of kerr black hole Subsystem:

$$\mathcal{H}_M \in \{\emptyset, \mathcal{H}_m, \mathcal{H}_a, \mathcal{H}_S\}$$

2. The time-evolution $\mathcal{T}_M : L(\mathcal{H}_M) \to L(\mathcal{H}_M)$ is defined as

$$\mathcal{T}_M(P) = \mathrm{tr}_{M^\perp}(\mathcal{T}(\mathrm{tr}_{M^\perp}(P) \otimes P))$$

we define,

$$\mathcal{T}_M(P)(e_M, e_{M^\perp}) := \mathrm{tr}_{M^\perp}(\mathcal{T}(\mathrm{tr}_{M^\perp}(P)(e_M)\, P(\cdots, e_{M^\perp})))$$

this implies,

$$(\mathcal{T}_M(P)(e_M, e_{M^\perp}))(e_M) := \mathrm{tr}_{M^\perp}(\mathcal{T}(\mathrm{tr}_{M^\perp}(P)(e_M)\, P(e_M, e_{M^\perp})))$$

where $M, M^\perp \in \mathbb{S} := \{\emptyset, a, m, S\}$ and $e_M \in \{M\}$ for all $M \in \mathbb{S}$ also $M^\perp = S \backslash M$

In case of kerr black hole

Case I $M = m$ i.e. $M^\perp = a$

$$(\mathcal{T}_m(P)(m, a))(m) = \mathrm{tr}_a\left(e^{-\tau \frac{d}{dt}}(\mathrm{tr}_a(P)\,(m) P(m, a))\right)$$

let, m_τ is mass at time τ, similarly a_τ is the anuglar momentum per unit mass at time τ. cosidering $\mathrm{tr}_a(P)\,(m)P(m,a))$ as a single funtion of a and m. Also, using the equation mentioned in section 7.1.3, we get:

$$(\mathcal{T}_m(P)(m, a))(m) = \mathrm{tr}_a((\mathrm{tr}_a(P)\,(m_\tau) P(m_\tau, a_\tau)))$$

Since, m_τ are independet of a

$$(\mathcal{T}_m(P)(m, a))(m) = ((\mathrm{tr}_a(P)\,(m_\tau))\mathrm{tr}_a(P(m_\tau, a_\tau)))$$

or

$$(\mathcal{T}_m(P)(m, a))(m) = ((\mathrm{tr}_a(P)\,(m_\tau))\mathrm{tr}_{a_{-\tau}}(P(m_\tau, a)))$$

Similarly,

$$(\mathcal{T}_a(P)(a, m))(a) = ((\mathrm{tr}_m(P)\,(a_\tau))\mathrm{tr}_m(P(a_\tau, m_\tau)))$$

In general we can say that:

$$(\mathcal{T}_M(P)(e_M, e_{M^\perp}))(f) := \mathrm{tr}_{M^\perp}(\mathrm{tr}_{M^\perp}(P)(e_{M_\tau})\, P(f_\tau, e_{M_\tau^\perp}))$$

where e_{M_τ} is the evolution of e_M at time τ similary with $e_{M_\tau^\perp}$ and f_τ where $e_M \in M$, $e_{M^\perp} \in M^\perp$ and $f_\tau \in S$.

4.3 Decompositions and Cuts

Decompositions are also defined via partitions $z=(D, D^\perp)\in \mathbb{D}_S$ of the set of elements N into two disjoint subsets D and $D^\perp$ whose union is N. For any such decomposition, the cut system $S^{(D,D^\perp)}$ is defined to have the same set of states but time evolution

$$\mathcal{T}^{(D,D^\perp)}(P)=\mathcal{T}\left(\mathrm{tr}_{D^\perp}(P)\otimes\frac{1}{\dim(\mathcal{H}_{D^\perp})}\mathbf{1}_{\mathcal{H}_{D^\perp}}\right)$$

Which implies,

$$\mathcal{T}^{(D,D^\perp)}(P)\,(e_D, e_{D^\perp})=\frac{1}{\dim(\mathcal{H}_{D^\perp})}\left(e^{-\tau\frac{d}{dt}}(\mathrm{tr}_{D^\perp}(P)(e_D)\ \mathbf{1}_{\mathcal{H}_{D^\perp}}(e_{D^\perp}))\right)$$

Which implies,

$$\mathcal{T}^{(D,D^\perp)}(P)\,(e_D, e_{D^\perp})=\frac{1}{\dim(\mathcal{H}_{D^\perp})}(\mathrm{tr}_{D^\perp}(P)(e_{D_\tau}))e_{D_\tau^\perp}$$

So, in the case of kerr black hole, $D, D^\perp \in \{\emptyset, m, a, S\}$ and $e_D =$

example $D=m$ i.e. $D^\perp = a$

$$\mathcal{T}^{(m,a)}(P)\,(m,a)=\frac{1}{\dim(\mathcal{H}_a)}(\mathrm{tr}_a(P)(m_\tau))a_\tau$$

4.4 Proto-Experiences for Kerr Black Hole

Definition 4.1. *For any $P, Q\in S(\mathcal{H}_S)$, the trace distance defined a*

$$d(P,Q)=\frac{1}{2}\mathrm{tr}_S\sqrt{(P-Q)^2}=\frac{1}{2}\mathrm{tr}_S|P-Q|$$

$$\mathbb{PE}(S):=\overline{S(\mathcal{H}_S)}$$

where $\overline{\mathcal{S}(\mathcal{H}_S)}:=\mathcal{S}(\mathcal{H}_S)\times\mathbb{R}^+$

* *an intensity function* $||.||:\overline{\mathcal{S}(\mathcal{H}_S)}\to\mathbb{R}^+$

  $||(P,r)||=r$ *for all* $(P,r)\in\overline{\mathcal{S}(\mathcal{H}_S)}$

* *a distance function* $\bar{d}:\overline{\mathcal{S}(\mathcal{H}_S)}\times\overline{\mathcal{S}(\mathcal{H}_S)}\to\mathbb{R}^+$

  $d((P,r),(Q,s))=|r-s|\,d(p,q)$ *for all* $(p,r)\in\overline{\mathcal{S}(\mathcal{H}_S)}$

* *a scalar multiplication* $\overline{\mathcal{S}(\mathcal{H}_S)}\to\overline{\mathcal{S}(\mathcal{H}_S)}$, *denoted* $(r,\bar{P})\to r\centerdot\bar{P}$, *satisfying* $r\centerdot(P,s)=(P,rs)$

  I. $||r\centerdot\bar{P}||=r\centerdot||\bar{P}||$

  II. $r\centerdot(s\centerdot\bar{P})=(rs)\centerdot\bar{P}$

  III. $1\centerdot\bar{P}=\bar{P}$

  for all $\bar{P}=(P,s)\in\overline{\mathcal{S}(\mathcal{H}_S)}$ *and* $r,s\in\mathbb{R}^+$

***Note* 4.1.** *in the case of Kerr black holes for each $(P,r)\in\overline{\mathcal{S}(\mathcal{H}_S)}$, following condition is imposed on r as to make evey eqally probale event have same intesity, also this will ensure that the distance between two equally probable states is zero.*

$$r\in\{n[\![P(m,a)]\!]\,\kappa\colon [\![P(m,a)]\!]=\{P(\mu,\alpha)\colon P(\mu,\alpha)=P(m,a)\}\}$$

where $n[\![P(m,a)]\!]$ is cardinality or dimeaion of $[\![P(m,a)]\!]$ and κ some constant.

4.5 Repertoires for Kerr black Hole

Let a system S in state $P\in\mathrm{St}(S)$ be given. we utilize maps caus′ s and eff′ s which here map subsystems M and N to St(N). They are defined

$$\mathrm{eff}_P'(M,N)=\mathrm{tr}_{N^\perp}\mathcal{T}\left(\mathrm{tr}_{M^\perp}(P)\otimes\frac{1}{\dim(\mathcal{H}_{M^\perp})}\mathbf{1}_{\mathcal{H}_{M^\perp}}\right)$$

$$\mathrm{caus}_P'(M,N)=\mathrm{tr}_{N^\perp}\mathcal{T}^*\left(\mathrm{tr}_{M^\perp}(P)\otimes\frac{1}{\dim(\mathcal{H}_{M^\perp})}\mathbf{1}_{\mathcal{H}_{M^\perp}}\right)$$

$\mathcal{T}^*$ is hermitian adjoint of $\mathcal{T}$.

In case of kerr blackhole $\mathcal{T}^*\equiv e^{-\tau\frac{d}{dt}}$, so that $\mathcal{T}^*\mathcal{T}=\text{identity}$

Also, In case of kerr blackhole $M,N\in\{\emptyset,m,a,S\}$ and using the fact that

$$\mathcal{T}\left(\mathrm{tr}_{M^\perp}(P)\otimes\frac{1}{\dim(\mathcal{H}_{M^\perp})}\mathbf{1}_{\mathcal{H}_{M^\perp}}\right)=\mathcal{T}^{(M,M^\perp)}(P)$$

we get

$$\mathrm{eff}_P'(M,N)(e_M,e_{M^\perp})=\frac{1}{\dim(\mathcal{H}_{M^\perp})}\mathrm{tr}_{N^\perp}((\mathrm{tr}_{M^\perp}(P)(e_{M_\tau}))e_{M_\tau^\perp})$$

similary

$$\mathrm{caus}_P'(M,N)(e_M,e_{M^\perp})=\frac{1}{\dim(\mathcal{H}_{M^\perp})}\mathrm{tr}_{N^\perp}((\mathrm{tr}_{M^\perp}(P)(e_{M_{-\tau}}))e_{M_{-\tau}^\perp})$$

$$\mathrm{eff}_P'(\emptyset,N)(e_S)=\frac{1}{\dim(\mathcal{H}_S)}\mathrm{tr}_{N^\perp}((\mathrm{tr}_S(P))(e_{S_\tau}))=\frac{1}{\dim(\mathcal{H}_S)}\mathrm{tr}_{N^\perp}(e_{S_\tau})$$

since, $\mathrm{tr}_S(P)=1$

Similarly,

$$\mathrm{caus}_P'(\emptyset,N)(e_S)=\frac{1}{\dim(\mathcal{H}_S)}\mathrm{tr}_{N^\perp}(e_{S_{-\tau}})$$

$$\mathrm{eff}_P(M,N)(e_M,e_M^\perp)(e_{S_\tau})=\frac{1}{\dim(\mathcal{H}_{M^\perp})\dim(\mathcal{H}_S)}\mathrm{tr}_{N^\perp}((\mathrm{tr}_{M^\perp}(P)(e_{M_\tau}))e_{M_\tau^\perp})\mathrm{tr}_{N^\perp}(e_{S_\tau})$$

similarly,

$$\mathrm{caus}_P(M,N)(e_M,e_M^\perp)(e_{S_\tau})=\frac{1}{\dim(\mathcal{H}_{M^\perp})\dim(\mathcal{H}_S)}\mathrm{tr}_{N^\perp}((\mathrm{tr}_{M^\perp}(P)(e_{M_{-\tau}^\perp}))e_{M_{-\tau}^\perp})\mathrm{tr}_{N^\perp}(e_{S_{-\tau}})$$

Definition 4.2.

1. $\overline{\mathrm{eff}}_P(M,N)(z_M,z_N):=\mathrm{eff}_P'(M_z,N_z)\otimes\mathrm{eff}_P'(M_{z^\perp},N_{z^\perp})\otimes\mathrm{eff}_p'(\emptyset,N^\perp)$

   where $M_z\bigcup M_{z^\perp}=M$ and $M_z\bigcap M_{z^\perp}=\emptyset$ similarly for N, hence $M_{z^\perp}=M_z^\perp$

Hence,

I. $\mathrm{eff}'_P(M_z,N_z)(e_{M_z},e_{M_z^\perp})=\frac{1}{\dim(\mathcal{H}_{M_z^\perp})}\mathrm{tr}_{N_z^\perp}(\,(\mathrm{tr}_{M_z^\perp}(P)(e_{M_{z_\tau}})\,)e_{M_{z_\tau}^\perp})$

II. $\mathrm{eff}'_P(M_{z^\perp},N_{z^\perp})(e_{M_z^\perp},e_{M_z})=\frac{1}{\dim(\mathcal{H}_{M_z})}\mathrm{tr}_{N_z}\Big(\Big(\mathrm{tr}_{M_z}(P)\big(e_{M_{z_\tau}^\perp}\big)\Big)e_{M_{z_\tau}}\Big)$

III. $\mathrm{eff}'_P(\emptyset,N^\perp)(e_S)=\frac{1}{\dim(\mathcal{H}_S)}\mathrm{tr}_N(\,e_{S_\tau})$

since each of the above three quatities are numbers we get

$$\frac{1}{\dim(\mathcal{H}_{M_z^\perp}\otimes\mathcal{H}_{M_z}\otimes\mathcal{H}_S)}\mathrm{tr}_{N_z^\perp}(\,(\mathrm{tr}_{M_z^\perp}(P)(e_{M_{z_\tau}})\,)e_{M_{z_\tau}^\perp})\mathrm{tr}_{N_z}\Big(\Big(\mathrm{tr}_{M_z}(P)\big(e_{M_{z_\tau}^\perp}\big)\Big)e_{M_{z_\tau}}\Big)\mathrm{tr}_N(\,e_{S_\tau})$$

2. similarly, $\overline{\mathrm{caus}}_P(M,N)(z_M,z_N):=\mathrm{caus}'_P(M_z,N_z)\otimes\mathrm{caus}'_P(M_{z^\perp},N_{z^\perp})\otimes\mathrm{caus}'_p(\emptyset,N^\perp)$

$$\frac{1}{\dim(\mathcal{H}_{M_z^\perp}\otimes\mathcal{H}_{M_z}\otimes\mathcal{H}_S)}\mathrm{tr}_{N_z^\perp}\Big(\,(\mathrm{tr}_{M_z^\perp}(P)(e_{M_{z_{-\tau}}})\,)e_{M_{z_{-\tau}}^\perp}\Big)\mathrm{tr}_{N_z}\Big(\Big(\mathrm{tr}_{M_z}(P)\big(e_{M_{z_{-\tau}}^\perp}\big)\Big)e_{M_{z_{-\tau}}}\Big)\mathrm{tr}_N(\,e_{S_{-\tau}})$$

4.6 Constructions - Mechanism Level

We know that concept space, $\mathbb{C}(S):=\mathbb{PE}(S)\times\mathbb{PE}(S)$, and we also know

$$\mathrm{caus}_P(M):=(\mathrm{caus}_P(M,N))_{N\in\mathrm{Sub}(S)}$$

$$\mathrm{eff}_P(M):=(\mathrm{eff}_P(M,N))_{N\in\mathrm{Sub}(S)}$$

Definition 4.3. *The concept of M, $\mathbb{C}_{S,P}(M)$ is then defined as the core integration of pair of collection:*

$$\mathbb{C}_{S,P}(M)=\iota(\mathrm{caus}_P(M,N^c),\mathrm{eff}_P(M,N^e))$$

where N^c(and N^e) are the core cause (and effect) perviews defined as follows:

$$N^c\in\{N\colon\max\{\phi(\mathrm{eff}_P(M,N)),N\in\mathrm{Sub}(S)\}\}$$

similarily for N^e, where

$$\phi(\mathrm{eff}_P(M,N)):=\min_{1\neq z\in\mathbb{D}_{M,N}}d(\mathrm{eff}_P(M,N),\overline{\mathrm{eff}}_P(M,N)(z))$$

And we know that,

$$d(\mathrm{caus}_P(M,N),\overline{\mathrm{caus}}_P(M,N)(z))=\frac{1}{2}\mathrm{tr}_S|\mathrm{caus}_P(M,N)-\overline{\mathrm{caus}}_P(M,N)(z)|$$

Similarly we can cunctruct system level model which would be more completed and from the we can eveluate $\boldsymbol{\Phi}$ for the black hole and get degree of consiounesss.

5 Appendix: Geometry Of The Kerr Space Time

Definition 5.1. *Boyer-Lindquist coordinates*

$$x^0=t,\quad x^1=\sqrt{r^2+a^2}\sin\theta\cos\phi,\quad x^2=\sqrt{r^2+a^2}\sin\theta\sin\phi,\quad x^3=r\cos\theta$$

***Note* 5.1.** *let "m" is mass and "a" is angular monentum per unit mass of black holw.*

Definition 5.2.

1. $\rho^2 = r^2 + a^2 \cos^2\theta$

2. $\Delta = r^2 + a^2 - 2mr,$

Definition 5.3. *metric tensor of Kerr Space Time*

1. $g_{tt} = -1 + 2mr/\rho^2 = -1 + 2\dfrac{mr}{r^2 + a^2\cos^2\theta}$

2. $g_{rr} = \rho^2/\Delta = \dfrac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - 2mr}$

3. $g_{\theta\theta} = \rho^2 = r^2 + a^2 \cos^2\theta$

4. $g_{\phi\phi} = \left(r^2 + a^2 + \left(\dfrac{2mra^2\sin^2\theta}{\rho^2}\right)\right)\sin^2\theta = \left(r^2 + a^2 + \left(\dfrac{2mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\right)\sin^2\theta$

5. $g_{\phi t} = g_{t\phi} = \dfrac{-2mra\sin^2\theta}{\rho^2} = \dfrac{-2\,\mathrm{mr}\,a\sin^2\theta}{r^2 + a^2\cos^2\theta}$

6. $g_{ij} = 0$ *for all* $i \neq j$ *and* $i, j \notin \{t, \phi\}$

Note 5.2.

1. Kerr space time is Minkoski space-time if $m = 0 = a$

2. Kerr space time is Schwarzschild space-time if $a = 0$

Biblography

1. THE MATHEMATICAL STRUCTURE OF INTEGRATED INFORMATION THEORY: JOHANNES KLEINER AND SEAN TULL

2. GEOMETRY OF THE KERR BLACK HOLES: ANH HOANG TRONG NAM

3. Information geometries in black hole physics Narit Pidokrajt (PhD thesis)

4. From the Phenomenology to the Mechanisms of Consciousness: Integrated Information Theory 3.0 : Masafumi Oizumi, Larissa Albantakis, Giulio Tononi

5. Mathematical Models of Consciousness: Johannes Kleiner