Assuming $c < \text{rad}^2(abc)$ Implies $abc$ Conjecture True

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In this paper about the $abc$ conjecture, assuming the condition $c < \text{rad}^2(abc)$ holds, and the constant $K(\epsilon)$ is a smooth decreasing function and having a derivative for $\epsilon \in ]0, 1[$, then we give the proof of the $abc$ conjecture.

To the memory of my Father who taught me arithmetic

To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$; $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i - 1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Rightarrow a = \mu_a \text{rad}(a)$$

The $abc$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $abc$ conjecture is given below:

Conjecture 1.1. ($abc$ Conjecture): Let $a, b, c$ positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that:

$$c < K(\epsilon).\text{rad}^{1+\epsilon}(abc)$$

$K(\epsilon)$ depending only of $\epsilon$.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in September 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the $abc$ conjecture is due to the incomprehensibility how the prime factors are organized in $c$ giving $a, b$ with $c = a + b$.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. It is the key to resolve the $abc$ conjecture. In my paper, we assume that the last conjecture holds, and the constant $K(\epsilon)$ for $\epsilon \in ]0, 1[$ is a smooth function. The paper is organized as follows: in the second section, we begin by presenting some proprieties of the constant $K(\epsilon)$, then we give the proof of the $abc$ conjecture.
2 The Proof of the abc Conjecture

Let a, b, c positive integers relatively prime with c = a + b, a > b, b ≥ 2. We denote $R = \text{rad}(abc), I = ]0, 1].$ For $c < R$, it is trivial that the abc conjecture holds. In the following, we consider the triples $(a, b, c)$ with $a, b, c$ relatively coprime and $c > R$. As we assume that $c < R^2$, it follows that $\forall \epsilon \geq 1$, it suffices to take $K(\epsilon) = 1$ and $c$ satisfies $c < K(\epsilon)R^{1+\epsilon}$ and the abc conjecture is true.

2.1 Properties of the constant $K(\epsilon)$

- From the definition of the abc conjecture, above, the constant $K(\epsilon)$ is a positive real number, and for every $\epsilon > 0$, it exists a number $K(\epsilon)$ dependent only of $\epsilon$.

- In the following, we consider that $\epsilon \in I$. We can say that $K$ is a function $K : \epsilon \in I \rightarrow K(\epsilon) \in ]0, +\infty[$, so that $c < K(\epsilon)R^{1+\epsilon}$ holds, if the abc conjecture is true. Assuming that $c < R^2$ is satisfying, we can adopt that $K(\epsilon) = 1$, because $c < K(\epsilon)R^{1+\epsilon}$.

- We observe that $K(\epsilon) > 1$ if $\epsilon \in I$. If not, we consider the example $9 = 8 + 1$, we take $\epsilon = 0.2$, then $c < K(0.2)R^{1.02} < 1.01^2$. But $c = 9 > 6^{1.2} \approx 8.58$, then the contradiction.

- We take one value $\epsilon \in I$, let one triplet $(a, b, c)$ so that $c > R$ and $c < K(\epsilon)R^{1+\epsilon}$. When $\epsilon < \epsilon' < R^{1+\epsilon}$, the last formula is satisfied if $K(\epsilon') < \epsilon$. If not, in the case where $K(\epsilon)$ continues in $\epsilon', \epsilon$ exists $c' > c$ with $\text{rad}(abc) = \text{rad}(a'b'c') = R$ that verifies $c' > K(\epsilon')R^{1+\epsilon'}$ for some $0 < \epsilon' < \epsilon$. It follows the contradiction. Let the example:

$$1 + 2^3 \times 7 = 3^2 \times 5^2 \iff 1 + 224 = 225 = c, R = \text{rad}(abc) = 2 \times 3 \times 5 \times 7 = 210 < c$$

$$2^5 + 3^3 = 3 \times 5^3 \iff 32 + 343 = 375 = c' > c, \text{rad}(a'b'c') = R, c' - c = 150 \approx 66\%c.$$

Fix one $\epsilon_0 \in I$, and if the abc conjecture holds, we have $225 < K(\epsilon_0)210^{1+\epsilon_0}$, if $K(\epsilon) < \epsilon$ for $\epsilon < \epsilon_0, \exists \epsilon' < \epsilon_0$, so that $375 < K(\epsilon')210^{1+\epsilon'}$ is not satisfied. Then the contradiction.

- In 1996, A. Nitaj had confirmed that the constant $K(\epsilon)$ verifies [4]:

$$\lim_{\epsilon \rightarrow 0} K(\epsilon) = +\infty$$

It follows that the function $K(\epsilon)$ is a decreasing function.

2.2 The proof of the abc conjecture

Proof. Let us suppose that $K(\epsilon)$ is a smooth decreasing function having a derivative in every point $\in ]0, 1]$. We denote : 

$$Y_\epsilon(\epsilon) = \text{Log}K(\epsilon) + (1 + \epsilon)\text{Log}R - \text{Log}c$$

We obtain $\lim_{\epsilon \rightarrow 1} Y_\epsilon(\epsilon) = 2\text{Log}R - \text{Log}c = y_1 > 0$, assuming $c < R^2$, and $\lim_{\epsilon \rightarrow 0} Y_\epsilon(\epsilon) = +\infty$. The derivative of $Y_\epsilon(\epsilon)$ gives:

$$Y'_\epsilon(\epsilon) = \frac{K'(\epsilon)}{K(\epsilon)} + \text{Log}R$$

We have the following cases:

i) - If $Y'_\epsilon(\epsilon) > 0$ for all $\epsilon \in ]0, 1[$, then $Y$ is an increasing function of $\epsilon$. It follows the contradiction because $\lim_{\epsilon \rightarrow 0} Y_\epsilon(\epsilon) = +\infty$.

ii) - If $Y'_\epsilon(\epsilon) < 0$ for all $\epsilon \in ]0, 1[$, then $Y$ is a decreasing function of $\epsilon$. It follows $\forall \epsilon, Y_\epsilon(\epsilon) > 0 \implies c < K(\epsilon)R^{1+\epsilon}$ is satisfied. As $c$ is an arbitrary integer with the condition $c > R$, we deduce that the abc conjecture is true.

iii) - We suppose that $Y'_\epsilon(\epsilon) = 0$ for some $\epsilon_0 \in ]0, 1[$. This is possible because as $K(\epsilon)$ is a decreasing function, $K'(\epsilon) < 0$, so that we obtain the equation:

$$- \frac{K'(\epsilon_0)}{K(\epsilon_0)} = \text{Log}R$$

* If $Y_\epsilon(\epsilon_0)$ is positive, then $Y_\epsilon(\epsilon) > 0$. As above, we deduce that the abc conjecture holds.
** If \( Y_c(\epsilon_0) \) is negative, then it exists two values \( \epsilon_1, \epsilon_2 \) with \( 0 < \epsilon_1 < \epsilon_0 < \epsilon_2 < 1 \), so that \( Y_c(\epsilon_1) = Y_c(\epsilon_2) = 0 \). It follows that \( c = K(\epsilon_1)R^{\epsilon_1}.\text{rad}(abc) \). Suppose that \( K(\epsilon_1)R^{\epsilon_1} \) is an integer, we obtain that \( a, b, c \) are not coprime. Then the contradiction.

Then, the \( abc \) conjecture holds for \( \forall \epsilon \in I \).

\[ \square \]

Q.E.D

End of the mystery!

3 Conclusion

Finally, assuming \( c < R^2 \), and choosing the constant \( K(\epsilon) \) as a smooth function, having a derivative, we have given an elementary proof that the \( abc \) conjecture is true. We can announce the important theorem:

**Theorem 3.1.** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), assuming \( c < \text{rad}(abc) \), then for each \( \epsilon > 0 \), there exists \( K(\epsilon) \) such that:

\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \]  

(7)

where \( K(\epsilon) \) is a constant depending only of \( \epsilon \) and varying smoothly.

\[ \square \]

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References


