Distribution of Integrals of Wiener Paths

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Abstract

With a new proof approach, we show that the normal distribution with mean zero and variance $1/3$ is the distribution of the integrals $\int_{[0,1]} W_t \, dt$ of the sample paths of Wiener process $W$ in $C([0,1], \mathbb{R})$.

Keywords: Brownian motion; classical Wiener space; integrals of Wiener paths; Wiener process

MSC 2020: 60G17; 60G15; 60F05; 26A42

1 Introduction

In contrast with the notion of “martingale (stochastic) integration” associated with Wiener measure, attention is less directed to the integrals of the sample paths of Wiener process $W$ in $C([0,1], \mathbb{R})$. Since every realization of $W$ is a continuous function on a compact interval, it always makes sense to speak of the integral of a Wiener path; investigating the integrals of Wiener paths, in particular the distribution of such integrals (which is evidently possible and is justified in what follows), is then a natural move.

In the present short communication, we supply a new proof of

Theorem *. If $W$ is Wiener process in $C([0,1], \mathbb{R})$, then

$$\int_{[0,1]} W_t \, dt \sim N(0, 1/3).$$

2 Proof

Throughout, let $C_w$ be the metric space $C([0,1], \mathbb{R})$ equipped with the uniform metric; and let $W$ be Wiener process in $C_w$.

We now give

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Proof (of Theorem *). For all \( f, g \in C_w \), we have
\[
\left| \int (f - g) \right| \leq \sup_t |f(t) - g(t)|;
\]
so the integration operator \( \int \) is (uniformly) continuous on \( C_w \).

If \( X_1, X_2, \ldots \) are independent identically distributed standard normal random variables, let \( \hat{W}^n \) be for each \( n \in \mathbb{N} \) the “Donsker process” obtained by linear interpolation between the \( \frac{1}{\sqrt{n}} \)-scaled cumulative sums of \( X_1, \ldots, X_n \) such that the resulting process fixes the origin, so that the sequence \( (\hat{W}^n)_{n \in \mathbb{N}} \) satisfies the assumptions of Donsker’s theorem (Theorem 8.2 in Billingsley [1], for concreteness). The continuous mapping theorem and Donsker’s theorem then jointly imply the weak convergence
\[
\int \hat{W}_t^n \, dt \rightharpoonup \int W_t \, dt.
\]

Let \( S_0 := 0; \) and let \( S_j := \sum_{i=1}^j X_i \) for all \( 1 \leq j \leq n \) and all \( n \in \mathbb{N} \). If \( n \in \mathbb{N} \), then we have
\[
\int \hat{W}_t^n \, dt = \sum_{j=1}^n \int_{(j-1)/n}^{j/n} \hat{W}_t^n \, dt,
\]
and we have \( \hat{W}_{j/n}^n = S_j / \sqrt{n} \) for each \( 0 \leq j \leq n \). Given any \( 1 \leq j \leq n \), we have
\[
\int_{(j-1)/n}^{j/n} \hat{W}_t^n \, dt = \frac{1}{\sqrt{n}} \int_{(j-1)/n}^{j/n} \tau S_j + (1 - \tau)S_{j-1} \, d\tau
\]
\[
= \frac{1}{\sqrt{n}} \left( S_j \frac{\tau^2}{2} \bigg|_{(j-1)/n}^{j/n} + \frac{1}{n} S_{j-1} - S_{j-1} \frac{\tau^2}{2} \bigg|_{(j-1)/n}^{j/n} \right).
\]
Summing the last term above over each \( 1 \leq j \leq n \) gives
\[
\int \hat{W}_t^n \, dt = \frac{1}{n^{3/2}} \left( nX_1 + (n-1)X_2 + \cdots + X_n \right) - \frac{1}{2n^{5/2}} S_n.
\]

The last term in (2) vanishes in probability by the continuous mapping theorem and the usual weak law of large numbers.

If \( n \in \mathbb{N} \), the sum of the independent normal random variables \( (n - j + 1)X_j \) with \( 1 \leq j \leq n \) in (2) is the normal random variable with mean zero and variance \( 1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6 \). If \( \kappa := 2^{3/2}\Gamma(2)/\sqrt{\pi} \), then
\[
\sum_{j=1}^n \mathbb{E}((n - j + 1)X_j)^3 = \kappa \sum_{j=1}^n j^3 = \kappa \frac{n^2(n+1)^2}{4},
\]
which grows more slowly than \( n(n+1)(2n+1)/6)^{3/2} \) as \( n \to \infty \). The classical Lyapunov central limit theorem (e.g. p. 332, Shiryaev [2], for concreteness) and the continuous
mapping theorem together imply that
\[
\frac{1}{n^{3/2}} \left( nX_1 + (n-1)X_2 + \cdots + X_n \right)
\]
\[
= \sqrt{\frac{n(n+1)(2n+1)}{6}} \left( \sqrt{\frac{n(n+1)(2n+1)}{6}} \right)^{-1} \left( nX_1 + (n-1)X_2 + \cdots + X_n \right)
\]
\[
\sim N(0, 1/3).
\]

Upon applying the continuous mapping theorem once more, we have
\[
\int \hat{W}_t^\alpha \, dt \sim N(0, 1/3)
\]
from (2). But then from [1] and the uniqueness of weak limit it follows that
\[
\int W_t \, dt \sim N(0, 1/3)
\]
as desired.

References
