# A Solution for Finding Composite Numbers in an Unending Sequence Starting with Prime Numbers

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**Abstract:** A non-terminating sequence like 31, 331, 3331, 33331, … starts with first seven terms as prime numbers, while the $8^{th}$ term, which is 333333331, can be expressed as 17 x 19607843. Using Fermat's Little Theorem, it can be easily proved that there are many more terms in this sequence that are not prime numbers. This paper puts forward a solution to find factors of composite numbers in all such sequences without using Fermat's Little theorem or divisibility tests. The solution uses a prime number only once to scan all the terms of the unending sequence together, to check if any term is divisible by that prime number instead of checking every term separately, hence reduces the computational complexity. The solution finds the smallest number of the sequence which is divisible by a particular prime number and also proves that it cannot be assumed that all the terms of such sequences will be prime numbers.

**Keywords:** Prime Numbers, Integer Factorization, Divisibility, Factors, Composite Numbers, Sequences and Series, Number Theory.

## I. Introduction

Checking if a prime number divides a certain number is an easy task. However, checking if a prime number divides every term of an increasing unending sequence, in order to find a composite term, becomes a tedious task. This paper, not only simplifies this task, but also proves that for an unending sequence that starts with the first few terms as prime numbers, it cannot be assumed that all the terms in the sequence will be prime. Just like the Sieve of Eratosthenes, the method starts with the smallest prime number and checks if it is a factor of any number in the sequence. The only difference being that it checks every possible term of the sequence in a single iteration, instead of checking each term separately for a particular prime number. By doing this, one of two possible outcomes for every prime number is obtained. The first outcome is that the prime number is not a factor of any number in the sequence and the second outcome is that we find the smallest number in the sequence that is divisible by that prime number. As soon as we find the smallest composite term, we can say that this unending sequence doesn't consist of all prime numbers.

## II. Preliminaries

### 2.1 Theorems

The following two well known theorems are used:

**Theorem 1.1:** If a positive integer N is divisible by another positive integer M, and we add any multiple of M with N, then the resultant integer P = N+ k*M is still divisible by M, where k is positive integer.

**Theorem 1.2:** If a positive integer N is not divisible by another positive integer M, and we add any multiple of M with N, then the resultant integer P = N+ k*M is still not divisible by M, where k is positive integer.

## III. Method

We consider the infinite sequence S ≡ 31, 331, 3331, …. as example to demonstrate the solution.

1) Start with the smallest prime number denoted by $p_i$. 2 is the smallest number, but as the unit's place will never be even, we go to the next prime number for this example i.e. $p_i = 3$.
2) Assume that we are adding multiples of $p_i$ to the first term of the sequence which is divisible by $p_i$, which is unknown. We just consider the last few digits of this unknown number such that it is the smallest possible value greater than $10 \times p_i$. In the example, $10 \times 3 = 30$, and the smallest term greater than 30 is 31, so we start with 31. The required term of the sequence can have any number of 3's and the units place as 1, but initially we consider 31. Make note of the place at which this term lies in sequence. For 31, it is the first term. Hence k=1.
3) If the units digit is non zero, then add the smallest multiple of $p_i$ to the selected last few digits of the term of the sequence such that the resultant sum has the face value of units place as 0. For the sum to have unit's place =0, $3 \times 3 = 9$ and $31 + 9 = 40$.
4) If the units digit is zero, then discard it, until the number has a non-zero units' digit.
5) If the last digit of every term of the sequence is even or 5, then all of them are composite numbers, hence such series are not under consideration. The smallest multiple of $p_i$ will lie within its first 10 multiples as all numbers ending in 1, 3, 7, 9 have their first 10 multiples with unique unit's digit, and they repeat this unique cycle.
6) Now increase the length of the selected digits by including additional digit with maximum place value and then discard the units digit to get back the same length. In the example, 40 was obtained which becomes 340, and after discarding the units digit, we get 34. Every time we increase the length of the selected digit by 1, increment the value of k by 1. In case multiple 0's are discarded, then make sure the condition (2) of the method is satisfied before adding a multiple of $p_i$ by increasing the length of the term.
7) Now repeat step 3,4 and 5 until either the same sum appears more than once or $p_i$ is obtained.
8) If the same sum appears, move to the next value of $p_i$ as no value in the sequence will be divisible by this prime number.
9) If $p_i$ or any of its first 10 multiples is obtained, then the $k^{th}$ term of the sequence is divisible by $p_i$.

## IV. Results

1) Step by step solution for $p_i = 13$ and S ≡ 31, 331, 3331, ….
   i. Initially choose the smallest term in the sequence greater than $10 \times 13$, i.e. 130.
   ii. The second term, i.e. 331 satisfies this condition, so k = 2. Now we add the smallest multiple of 13 to 331 such that the sum has units place = 0.
   iii. 39 is the multiple which satisfies the above requirement and 331 + 39 = 370.
   iv. We discard the 0 in the unit's place and get 37. Now we increment k by 1 and borrow an additional digit at the left-hand side of 37 to get 337.
   v. Now adding 13 to 337 as the resultant will have unit's place 0 and get 350.
   vi. Discarding 0 from the unit's place and borrowing a digit at the left, we get 335. K is incremented by 1.
   vii. Now adding 65 to 335 we obtain 400.
   viii. Discarding the unit's place and borrowing another digit at the left-hand side, we get 340. K is incremented by 1.
   ix. The units place is already 0, so we discard it and obtain 34.
   x. Borrowing a digit, 334 is obtained. Increment k by 1.
   xi. Adding 26, 360 is obtained. 0 is discarded to get 36.
   xii. A digit is borrowed, and 336 is obtained. Increment k by 1.
   xiii. Adding 104, 440 is obtained. Discard the 0 to get 44.

xiv. Borrow a digit and increment k to obtain 344.
xv. Add 26. We get 370, which was already obtained in step iii. As this cycle will continue, so we stop. Hence, there is no number is this sequence divisible by 13.

2) Step by step solution for $p_i = 17$ and $S \equiv 31, 331, 3331, \ldots.$

i. Initially choose the smallest term in the sequence greater than $10 \times 17$, i.e. 170.
ii. The second term, i.e. 331 satisfies this condition, so k = 2. Now we add the smallest multiple of 17 to 331 such that the sum has units place = 0.
iii. 119 is the multiple which satisfies the above requirement and 331 + 119 = 450.
iv. We discard the 0 in the unit's place and get 45. Now we increment k by 1 and borrow an additional digit at the left-hand side of 45 to get 345.
v. Now adding 85 to 345 as the resultant sum will have unit's place 0, 430 is obtained.
vi. Discarding 0 from the unit's place and borrowing a digit at the left, we get 343. K is incremented by 1.
vii. Now adding 17 to 343 we obtain 360.
viii. Discarding the unit's place and borrowing another digit at the left-hand side, we get 336. K is incremented by 1.
ix. Adding 34 and discarding 0 from the resultant, 37 is obtained.
x. Borrowing a digit, 337 is obtained. Increment k by 1.
xi. Adding 153, 490 is obtained. 0 is discarded to get 49.
xii. A digit is borrowed, and 349 is obtained. Increment k by 1.
xiii. Adding 51, 400 is obtained. Discard the 0 to get 40.
xiv. Borrow a digit and increment k to obtain 340
xv. The unit's place is already 0, so after discarding it, 34 is obtained.
xvi. As 34 is $2 \times 17$, the $k^{th}$ term of the sequence is divisible by 17.
xvii. Initially k was 2, and it got incremented 6 times in the process, so the $8^{th}$ term of the sequence is a multiple of 17.

3) Solved Example for $p_i = 3$ and $S \equiv 37, 337, 3337, \ldots.$

| Iteration Number | K | $p_i$ | Initial Term for the Iteration | Multiple of $p_i$ to be added | Sum | Result after discarding |
|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 37 | 3 | 40 | 4 |
| 2 | 2 | 3 | 34 | 6 | 40 | STOP |

In the above example, as the sum 40 is obtained more than once, the cycle will repeat without giving a multiple of $p_i$. Hence, no term of the sequence $S \equiv 37, 337, 3337, \ldots.$ is divisible by 3.

4) Solved Example for $p_i = 7$ and $S \equiv 37, 337, 3337, \ldots.$

| Iteration Number | K | $p_i$ | Initial Term for the Iteration | Multiple of $p_i$ to be added | Sum | Result after discarding |
|---|---|---|---|---|---|---|
| 1 | 2 | 7 | 337 | 63 | 400 | 40 |
| 2 | 3 | 7 | 340 | - | 340 | 34 |
| 3 | 4 | 7 | 334 | 56 | 390 | 39 |
| 4 | 5 | 7 | 339 | 21 | 360 | 36 |
| 5 | 6 | 7 | 336 | 14 | 350 | 35 |

In the above example, the result obtained is 35 which is the $5^{th}$ multiple of 7, so we stop. As the value of k is 6, hence the $6^{th}$ term of the sequence $S \equiv 37, 337, 3337, \ldots.$ is divisible by 7.

5) Solved Example for $p_i$ = 23 and S ≡ 31, 331, 3331, ….

| Iteration Number | K | $p_i$ | Initial Term for the Iteration | Multiple of $p_i$ to be added | Sum | Result after discarding |
|---|---|---|---|---|---|---|
| 1 | 2 | 23 | 331 | 69 | 400 | 40 |
| 2 | 3 | 23 | 340 | - | 340 | 34 |
| 3 | 4 | 23 | 334 | 46 | 380 | 38 |
| 4 | 5 | 23 | 338 | 92 | 430 | 43 |
| 5 | 6 | 23 | 343 | 207 | 550 | 55 |
| 6 | 7 | 23 | 355 | 115 | 470 | 47 |
| 7 | 8 | 23 | 347 | 23 | 370 | 37 |
| 8 | 9 | 23 | 337 | 23 | 360 | 36 |
| 9 | 10 | 23 | 336 | 184 | 520 | 52 |
| 10 | 11 | 23 | 352 | 138 | 490 | 49 |
| 11 | 12 | 23 | 349 | 161 | 510 | 51 |
| 12 | 13 | 23 | 351 | 69 | 420 | 42 |
| 13 | 14 | 23 | 342 | 138 | 480 | 48 |
| 14 | 15 | 23 | 348 | 92 | 440 | 44 |
| 15 | 16 | 23 | 344 | 46 | 390 | 39 |
| 16 | 17 | 23 | 339 | 161 | 500 | 50 |
| 17 | 18 | 23 | 350 | - | 350 | 35 |
| 18 | 19 | 23 | 335 | 115 | 450 | 45 |
| 19 | 20 | 23 | 345 | 115 | 460 | 46 |

In the above example, as 46 is twice of 23, we stop. This means that $k^{th}$ term i.e. $20^{th}$ term of the sequence S ≡ 31, 331, 3331, …. is divisible by 23.

## V. Conclusion

Factorizing large numbers has a lot of applications in fields like cryptography where very large prime numbers are used to form a composite number with two prime factors for encoding messages. To break this code, an attacker should find the factors of this large composite number, which is an arduous task. New approaches and faster computational speeds of computers is making it easier to break these codes. However, prime factorization will always play an important role in this area. With the help of this solution, it becomes easier to find a term which is divisible by a smaller prime number in unending sequences following a pattern of repeating digit with one constant digit in the unit's place. As the first few terms of the sequence are prime, the paper also shows that it cannot be assumed that all the terms will be prime.

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