MY COLLECTION OF PAPERS WRITTEN TO TRY TO RESOLVE THE $abc$ CONJECTURE

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Abstract. — In this book, I present my collection of 23 papers written, with different approaches to try to resolve the $abc$ conjecture and others conjectures related to it like $c < rad^2(abc)$.

This monograph can give an idea about the advancement of the comprehension of the conjectures related to the problem cited above.

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Résumé (Collection d’Articles Essayant de Résoudre la Conjecture $abc$ (Novembre 2018 - Novembre 2020))

Dans ce monograph, nous présentons 23 différents articles écrits pour essayer de résoudre la conjecture $abc$ et d’autres en relation comme celle de $c < rad^2(abc)$.

Ce recueil peut donner une idée sur l’avancement de la compréhension des conjectures objet du problème.
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1.1. Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i - 1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a.\text{rad}(a)$$

The $ABC$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $ABC$ conjecture is given below:

**Conjecture 1.** — ($ABC$ Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending only of $\epsilon$.

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students. Our proof will use the Fermat’s Last Theorem approved by Andrew John Wiles in 1993 [3].

We recall the Fermat’s Last Theorem:
CHAPTER 1. FERMAT’S LAST THEOREM ➞ PROOF OF THE ABC CONJECTURE

**Theorem 1.** — *The equation :

\[ x^n + y^n = z^n \]

has no solutions with \( x, y, z \) all nonzero, relatively prime integers with \( n > 2 \) a positive integer.

The negation of the last theorem is:

*It exists \( A, B, C \) relatively prime integers and \( n > 2 \) a positive integer so that :

\[ A^n + B^n = C^n \]

1.2. Methodology of the proof

We denote :

(6) \hspace{1cm} A: Fermat’s Last Theorem

(7) \hspace{1cm} B: ABC Conjecture

and we use the following property (the contra-positive law, [4]) :

\[ A(\text{False}) \implies B(\text{False}) \iff B(\text{True}) \implies A(\text{True}) \]

From the right equivalent expression in the box above, as \( A \) (FLT) is true, then \( B \) (ABC Conjecture) is true.

1.3. Proof of the conjecture

(33) We suppose that FLT is false, then it exists \( A, B, C \) positive coprime integers and \( m \) a positive integer > 2 such:

\[ A^m + B^m = C^m \]

the integers \( A, B, C, m \) are supposed large integers. We consider in the following that \( A > B \). Now, we use the ABC conjecture for equation (9). We choose the value of \( \epsilon \approx 0.001 \), then it exists the constant \( K(\epsilon) > 0 \), we want to find if :

\[ C^m \leq K(\epsilon) \text{rad}(A^m.B^m.C^m)^{1+\epsilon} \]

(10)

\[ C^m \leq K(\epsilon) (\text{rad}(A).\text{rad}(B).\text{rad}(C))^{1+\epsilon} \]

But \( \text{rad}(A) \leq A < C, \text{rad}(B) \leq B < C \) and \( \text{rad}(C) \leq C \), then we write (10) as :

(11) \hspace{1cm} \[ C^m \leq K(\epsilon) (\text{rad}(A).\text{rad}(B).\text{rad}(C))^{1+\epsilon} \implies C^m \leq K(\epsilon)C^{\epsilon(1+\epsilon)} \]
1.3.1. Case $K(\epsilon) \leq 1$

In this case, we obtain:

\[(12)\quad C^m < C^{3(1+\epsilon)}\]

As $\epsilon \ll 1 \implies 3(1 + \epsilon) \ll m$, then $C^m > K(\epsilon)\text{rad}(A^m.B^m.C^m)^{1+\epsilon}$ and the $ABC$ conjecture is false. Using the right member of the property (8), we obtain:

\[(13)\quad \boxed{ABC \text{ Conjecture True} \implies \text{FLT True}}\]

But as FLT holds, hence $ABC$ Conjecture is true.

1.3.2. Case $K(\epsilon) > 1$ and $C^m > K(\epsilon)$

In this case, let $\epsilon \approx 0.001$ and we suppose that $K(\epsilon) > 1$. As FLT is supposed false, we consider that it exits a solution of (9) such that $C^m > K(\epsilon)$ with $C^m \gg C K(\epsilon)$ that means $\exists \lambda$ a positive constant depending of $C$ such $C^m = \lambda.K(\epsilon)$ and $\lambda \approx C_h$ with $(m - h) \leq \frac{m}{2}$. Then :

\[(14)\quad C^m < K(\epsilon)C^{3(1+\epsilon)}\]

The last equation can be written as :

\[(15)\quad \lambda < C^{3(1+\epsilon)}\]

The equation (15) indicates that we can write $\lambda \approx C^3 \implies m < 6$, then the contradiction with $6 \ll m$. Hence :

\[C^m > K(\epsilon)\text{rad}(A^m.B^m.C^m)^{1+\epsilon}\]

and the $ABC$ conjecture is false. Using the right member of the property (8), we obtain:

\[(16)\quad \boxed{ABC \text{ Conjecture True} \implies \text{FLT True}}\]

But as FLT holds, hence $ABC$ Conjecture is true.

1.3.3. Case $K(\epsilon) > 1$ and $C^m < K(\epsilon)$

We consider $\epsilon = 0.001$ and we suppose that $K(\epsilon) > 1$. As FLT is supposed false, we consider that it exits a unique solution of (9) such that $C^m < K(\epsilon)$:

\[(17)\quad C^m = A^m + B^m\]

We obtain that:

\[(18)\quad C^m < K(\epsilon)\text{rad}(A^m.B^m.C^m)^{1+\epsilon}\]

and the $ABC$ conjecture is true for $C^m = A^m + B^m$, but there is a contradiction because the hypothesis of the beginning used for the proof is false, then this case is
to reject.

The proof of the $ABC$ conjecture is achieved.

1.4. Conclusion

In the mathematical literature, the $ABC$ conjecture, assumed true, is used to approve the Fermat’s Last Theorem, in our paper, we have given a proof that the $ABC$ conjecture is true using the Fermat’s Last Theorem. We can announce the important theorem:

**Theorem 2.** — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending only of $\epsilon$. 


CHAPTER 2

THE ABC CONJECTURE: A PROOF OF
\[ C < \text{rad}^2(ABC) \]

To the memory of my Father who taught me arithmetic.

2.1. Introduction and notations

Let \( a \) a positive integer, \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
(20) \quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i-1}
\]

We note:

\[
(21) \quad \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a)
\]

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given above:

**Conjecture 2.** — (ABC Conjecture): For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(22) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

where \( K \) is a constant depending only of \( \epsilon \). We know that numerically,

\[
\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751 ([2]).
\]

Here we will give a proof that:

\[
(23) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.
2.2. A Proof of the condition (23)

Let $a, b, c$ positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$. If $c \leq \text{rad}(ab)$ then we obtain:

\[(24) \quad c \leq \text{rad}(ab) < \text{rad}^2(abc)\]

and the condition (23) is verified.

In the following, we suppose that $c > \text{rad}(ab)$.

2.2.1. Case $c = a + 1$

\[(25) \quad c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac)\]

2.2.1.1. $\mu_a = 1$

In this case, $a = \text{rad}(a)$, it is immediately truth that :

\[(26) \quad c = a + 1 < 2a < \text{rad}(a) \text{rad}(c) < \text{rad}^2(ac)\]

Then (25) is verified.

2.2.1.2. $\mu_a \neq 1, \mu_a < \text{rad}(a)$

we obtain :

\[(27) \quad c = a + 1 < 2\mu_a \text{rad}(a) \Rightarrow c < 2\text{rad}^2(a) \Rightarrow c < \text{rad}^2(ac)\]

Then (25) is verified.

2.2.1.3. $\mu_a \geq \text{rad}(a)$

We have $c = a + 1 = \mu_a \text{rad}(a) + 1 \leq \mu_a^2 + 1 < \text{rad}^2(ac)$. We suppose that $\mu_a^2 + 1 \geq \text{rad}^2(ac) \Rightarrow \mu_a^2 > \text{rad}^2(a) \text{rad}(c) \geq 3\text{rad}^2(a) \Rightarrow \mu_a > \sqrt{3}\text{rad}(a) \geq 2\text{rad}(a)$ then $\mu_a > 2\text{rad}(a)$, that is the contradiction with $\mu_a \geq \text{rad}(a)$. We deduce that $c \leq \mu_a^2 + 1 < \text{rad}^2(ac) \Rightarrow c < \text{rad}^2(ac)$ and the condition (25) is verified.

2.2.2. $c = a + b$

We can write that $c$ verifies:

\[
c = a + b = \text{rad}(a).\mu_a + \text{rad}(b).\mu_b = \text{rad}(a).\text{rad}(b) \left( \frac{\mu_a}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} \right) \Rightarrow \\
(28) \quad c = \text{rad}(a).\text{rad}(b).\text{rad}(c) \left( \frac{\mu_a}{\text{rad}(b).\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a).\text{rad}(c)} \right) 
\]
We can write also:

\[ c = \text{rad}(abc) \left( \frac{\mu_a}{\text{rad}(b) \cdot \text{rad}(c) k} + \frac{\mu_b}{\text{rad}(a) \cdot \text{rad}(c)} \right) \]

To obtain a proof of (25), one method is to prove that:

\[ \frac{\mu_a}{\text{rad}(b) \cdot \text{rad}(c)} + \frac{\mu_b}{\text{rad}(a) \cdot \text{rad}(c)} < \text{rad}(abc) \]

2.2.2.1. \( \mu_a = \mu_b = 1 \)

In this case, it is immediately truth that:

\[ \frac{1}{\text{rad}(a_i)} + \frac{1}{\text{rad}(b_j)} \leq \frac{5}{6} < \text{rad}(c) \cdot \text{rad}(abc) \]

Then (25) is verified.

2.2.2.2. \( \mu_a = 1 \) and \( \mu_b > 1 \)

As \( b < a \Rightarrow \mu_b \cdot \text{rad}(b) < \text{rad}(a) \Rightarrow \frac{\mu_b}{\text{rad}(a)} < \frac{1}{\text{rad}(b)} \), then we deduce that:

\[ \frac{1}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} < \frac{2}{\text{rad}(b)} < \text{rad}(c) \cdot \text{rad}(abc) \]

Then (25) is verified.

2.2.2.3. \( \mu_b = 1 \) and \( \mu_a \leq (b = \text{rad}(b)) \)

In this case we obtain:

\[ \frac{1}{\text{rad}(a)} + \frac{\mu_a}{\text{rad}(b)} \leq \frac{1}{\text{rad}(a)} + 1 < \text{rad}(c) \cdot \text{rad}(abc) \]

Then (25) is verified.

2.2.2.4. \( \mu_b = 1 \) and \( \mu_a > (b = \text{rad}(b)) \)

As \( \mu_a > \text{rad}(b) \), we can write \( \mu_a = \text{rad}(b) + n \) where \( n \geq 1 \). We obtain:

\[ c = \mu_a \cdot \text{rad}(a) + \text{rad}(b) = (\text{rad}(b) + n) \cdot \text{rad}(a) + \text{rad}(b) = \text{rad}(ab) + n \cdot \text{rad}(a) + \text{rad}(b) \]

We verify that \( n < b \), then:

\[ c < 2 \cdot \text{rad}(ab) + \text{rad}(b) \Rightarrow c < \text{rad}(abc) + \text{rad}(abc) < \text{rad}^2(abc) \]

\[ \Rightarrow c < \text{rad}^2(abc) \]

2.2.2.5. \( \mu_a, \mu_b \neq 1, \mu_a < \text{rad}(a) \) and \( \mu_b < \text{rad}(b) \)

we obtain:

\[ c = \mu_c \cdot \text{rad}(c) = \mu_a \cdot \text{rad}(a) + \mu_b \cdot \text{rad}(b) < \text{rad}^2(a) + \text{rad}^2(b) < \text{rad}^2(abc) \]
2.2.6. \( \mu_a, \mu_b \neq 1, \mu_a \leq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \)

We have:
\[
(37) \quad c = \mu_a \text{rad}(a) + \mu_b \text{rad}(b) < \mu_a \mu_b \text{rad}(a) \text{rad}(b) \leq \mu_0 \text{rad}^2(a) \text{rad}(b)
\]

Then if we give a proof that \( \mu_b < \text{rad}(b) \text{rad}^2(c) \), we obtain \( c < \text{rad}^2(abc) \). As \( \mu_b \geq \text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha \) with \( \alpha \) a positive integer \( \geq 0 \). Supposing that \( \mu_b \geq \text{rad}(b) \text{rad}^2(c) \implies \mu_b = \text{rad}(b) \text{rad}^2(c) + \beta \) with \( \beta \geq 0 \) a positive integer. We can write:
\[
\text{rad}(b) \text{rad}^2(c) + \beta = \text{rad}(b) + \alpha \implies \beta < \alpha
\]

(38) \( \alpha - \beta = \text{rad}(b)(\text{rad}^2(c) - 1) > 3 \text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha > 4 \text{rad}(b) \)

Finally, we obtain:
\[
(39) \quad \begin{cases} \mu_b \geq \text{rad}(b) \\ \mu_b > 4 \text{rad}(b) \end{cases}
\]

Then the contradiction and the hypothesis \( \mu_b \geq \text{rad}(b) \text{rad}^2(c) \) is false. Hence:
\[
(40) \quad \mu_b < \text{rad}(b) \text{rad}^2(c) \implies c < \text{rad}^2(abc)
\]

2.2.7. \( \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \) and \( \mu_b \leq \text{rad}(b) \)

The proof is identical to the case above.

2.2.8. \( \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \)

We write:
\[
(41) \quad c = \mu_a \text{rad}(a) + \mu_b \text{rad}(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2 \cdot \mu_b^2 < \text{rad}^2(a) \cdot \text{rad}^2(b) \cdot \text{rad}^2(c) = \text{rad}^2(abc)
\]

As \( \mu_a \geq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \), we can write that:
\[
\begin{align*}
\mu_a &= \text{rad}(a) + m \\
\mu_b &= \text{rad}(b) + n
\end{align*}
\]

with \( m, n \geq 0 \) two positive integers. Let \( F(x, y) \) the function:
\[
(42) \quad F(x, y) = (x+\text{rad}(a))(y+\text{rad}(b))−\text{rad}(abc), \ (x, y) \in I = −\text{rad}(a), +\infty[\times]−\text{rad}(b), +\infty[
\]

The set of points \( M(x, y) \in I \) verifying \( F(x, y) = 0 \) is the hyperbola \( \mathcal{C} \) given by:
\[
(43) \quad y = \frac{−\text{rad}(b) \cdot x + \text{rad}(abc) − \text{rad}(ab)}{x + \text{rad}(a)}
\]

The curve \( \mathcal{C} \) intersects the axis \( x = 0 \) and \( y = 0 \) at the two points \( M_1(0, y_1 = \text{rad}(b)(\text{rad}(c)−1)) \) and \( M_2(x_2 = \text{rad}(a)(\text{rad}(c)−1), 0) \). The region below the curve
2.2. A PROOF OF THE CONDITION (??)

\[ C \text{ verifies } F(x, y) < 0. \]
\[ F(m, n) = \mu_a \mu_b - \text{rad}(abc) < 0 \text{ if we have } m < x_2 \Rightarrow m < \text{rad}(a)(\text{rad}(c) - 1) \text{ and } n < y_1 \Rightarrow n < \text{rad}(b)(\text{rad}(c) - 1). \]

We suppose now that:

\begin{align*}
  m \geq \text{rad}(a)(\text{rad}(c) - 1) & \Rightarrow m > \text{rad}(a) & \Rightarrow \mu_a > 2\text{rad}(a) & \Rightarrow a > 2\text{rad}^2(a) \\
  n \geq \text{rad}(b)(\text{rad}(c) - 1) & \Rightarrow n > \text{rad}(b) & \Rightarrow \mu_b > 2\text{rad}(b) & \Rightarrow b > 2\text{rad}^2(b)
\end{align*}

(44)

\[ \text{then } c > 2(\text{rad}^2(a) + \text{rad}^2(b)) > 4\text{rad}(ab) \Rightarrow c > 4\text{rad}(ab) \]

The last inequality \( c > 4\text{rad}(ab) \) gives the contradiction with the condition \( c > \text{rad}(ab) \) supposed above. Then we obtain \( F(m, n) < 0 \Rightarrow \mu_a \mu_b - \text{rad}(abc) < 0 \Rightarrow c < \text{rad}^2(abc). \)

We announce the theorem:

**Theorem 3.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b \) and \( b < a \), then \( c < \text{rad}^2(abc). \)
BIBLIOGRAPHY


CHAPTER 3

THE ABC CONJECTURE: THE END OF THE MYSTERY

3.1. Introduction and notations

Let $a$ a positive integer, $a = \prod a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod a_i^{\alpha_i}$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod a_i^{\alpha_i} = \text{rad}(a). \prod a_i^{\alpha_i - 1}$$

We note:

$$\mu_a = \prod a_i^{\alpha_i - 1} \implies a = \mu_a.\text{rad}(a)$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

**Conjecture 3.** — *(ABC Conjecture)*: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).\text{rad}^{1+\epsilon}(abc)$$

where $K$ is a constant $K$ depending only of $\epsilon$. This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students.

3.2. Proof of the conjecture (3)

Let $a, b, c$ positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$, we can write that $a$ verifies:

$$c = a + b \implies c(a - b) = a^2 - b^2 < e^{4a^2} \implies c < \frac{e^{4a^2}}{a - b}$$
We can write also:

\[ c < \frac{e^{4a^2} K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}}{a-b} \cdot \frac{1}{K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}} \]

We propose the constant \( K(\epsilon) \) depending of \( \epsilon \) as:

\[ K(\epsilon) = \frac{2}{\epsilon^2} \]

it is a decreasing function so that \( \lim_{\epsilon \to 0} K(\epsilon) = +\infty \) and \( \lim_{\epsilon \to +\infty} K(\epsilon) = 0 \). We write (49) as:

\[ c < e^{4a^2 \epsilon^2} \cdot \frac{1}{a-b} \cdot \frac{1}{2 \cdot \text{rad}(abc)^{1+\epsilon}} \cdot K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

It is known that \( 2 \leq \text{rad}(q) \) for \( \forall q \) a positive integer \( \geq 2 \), then \( 2^3 \leq \text{rad}(abc) \Rightarrow 2 \leq \text{rad}(abc)^{1+\epsilon} \).

Then, equation (52) is written as:

\[ c < G(\epsilon, a, b) \cdot K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

Let:

\[ G(\epsilon, a, b) = e^{4a^2 \epsilon^2} \cdot \frac{1}{a-b} \cdot \frac{1}{2^{4+\frac{4a^2}{\epsilon^2}}} \cdot \frac{1}{16e^{4a^2 \epsilon}} \]

Then, equation (52) is written as:

\[ c < G(\epsilon, a, b) \cdot K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

If we can give a proof that \( G(\epsilon, a, b) < 1 \) independently of \( a, b, \epsilon \), we will obtain:

\[ c < G(\epsilon, a, b) \cdot K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

then the \( ABC \) conjecture holds with proposing the expression of the constant \( K(\epsilon) = \frac{2}{\epsilon^2} \).

### 3.2.1. The Proof

We write:

\[ G(\epsilon, a, b) = e^{4a^2 \epsilon^2} \cdot \frac{1}{a-b} \cdot \frac{1}{16e^{4a^2 \epsilon}} < 1 \Rightarrow (a-b)16e^{4a^2 \epsilon} - e^{4a^2 \epsilon} > 0 \]

As \( a > b \), the minimum value of \( a-b \) is equal to 1, then we must verify if:

\[ 16e^{4a^2 \epsilon} - e^{4a^2 \epsilon}^2 > 0 \]

\[ 16e^{4a^2 \epsilon} - e^{4a^2 \epsilon}^2 > 0 \]
We call:
\begin{equation}
\phi(\epsilon) = 16e^{4a^2\epsilon} - e^{4a^2\epsilon^2} \Rightarrow \phi'(\epsilon) = 2\left(32a^2 e^{4a^2\epsilon} - e^{4a^2\epsilon}\right)
\end{equation}

\begin{equation}
\phi''(\epsilon) = 2\left(128a^4 e^{4a^2\epsilon} - e^{4a^2\epsilon^2}\right) > 0 \quad \forall \epsilon > 0 \quad \text{and} \quad a \geq 2
\end{equation}

Let us study the function \(\phi''(\epsilon)\):
\begin{equation}
\phi''(\epsilon_1) = 0 \Rightarrow \epsilon_1 = \frac{1}{4a^2} \log\left(\frac{e^{4a^2}}{128a^2}\right) \Rightarrow \\
\epsilon_1 = 1 - \frac{7}{4a^2} \log 2 - \frac{\log a}{a^2} < 1; \quad \max \epsilon_1(a) = \epsilon_1(2) = 0.6534
\end{equation}

\begin{equation}
\phi'(\epsilon_1) = \frac{e^{4a^2}}{2a^2} \left(1 - \log\left(\frac{e^{4a^2}}{128a^2}\right)\right)
\end{equation}

If we write the table of variations of the function \(\phi\) when \(\epsilon \in [0, +\infty]\), we obtain successively \(\phi''(\epsilon) > 0, \phi'(\epsilon) > 0\) and \(\phi(\epsilon) > 0\) for \(\forall a \geq 2\), we deduce that \(\forall \epsilon > 0, a \geq 2:\)
\begin{equation}
16e^{3a^2\epsilon} - e^{4a^2\epsilon^2} > 0 \Rightarrow (a - b)16e^{3a^2\epsilon} - e^{4a^2\epsilon} > 0 \Rightarrow \\
(a - b)16e^{3a^2\epsilon} - e^{4a^2\epsilon} \Rightarrow 1 > \frac{e^{4a^2\epsilon^2}}{(a - b)2^{4+3a^2\epsilon}} \Rightarrow G(\epsilon, a, b) < 1
\end{equation}

Then we obtain the important result of the paper:
\begin{equation}
c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \quad \forall \epsilon > 0
\end{equation}

with the constant \(K(\epsilon) = \frac{2}{\epsilon^2}\)
Q.E.D

### 3.3. Examples

In this section, we are going to verify some numerical examples.

#### 3.3.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:
\begin{equation}
3^{10} \times 109 + 2 = 23^5 = 6436343
\end{equation}

\(a = 3^{10}.109 \Rightarrow \mu_a = 3^0 = 19683\) and \(\text{rad}(a) = 3 \times 109,\)
\(b = 2 \Rightarrow \mu_b = 1\) and \(\text{rad}(b) = 2,\)
\(c = 23^5 = 6436343 \Rightarrow \text{rad}(c) = 23.\) Then \(\text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042.\) For example, we take \(\epsilon = 0.01,\) the expression of \(K(\epsilon)\) becomes:
\begin{equation}
K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}}
\end{equation}
Let us verify (55):
\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies c = 6436343 < 2.10^4.(3 \times 109 \times 2 \times 23)^{1.01} \implies 6436343 < 186142827.83 \]
Hence (55) is verified.

3.3.2. Example of A. Nitaj

3.3.2.1. Case 1

The example of Nitaj about the ABC conjecture [3] is:
\begin{align*}
    a &= 11^{16}.13^2.79 = 613474843408551921511 \Rightarrow \text{rad}(a) = 11.13.79 \\
    b &= 7^2.41^2.311^3 = 2477678547239 \Rightarrow \text{rad}(b) = 7.41.311 \\
    c &= 2.3^3.5^{23}.953 = 613474845886230468750 \Rightarrow \text{rad}(c) = 2.3.5.953 \\
    \text{rad}(abc) &= 2.3.5.7.11.13.41.79.311.953 = 28828335646110 \\
\end{align*}
we take \( \epsilon = 100 \) we have:
\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613474845886230468750 < 2.10^{-4}.(2.3.5.7.11.13.41.79.311.953)^{101} \implies 613474845886230468750 < 5.53103686332861264803638e + 1355 \]
then (55) is verified.

3.3.2.2. Case 2

We take \( \epsilon = 0.000001 = 10^{-6} \), then:
\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613474845886230468750 < 2.10^{-12}.(2.3.5.7.11.13.41.79.311.953)^{1.000001} \implies 613474845886230468750 < 57658458237370924700998757.17498 \]
We obtain that (55) is verified.

3.3.2.3. Case 3

We take \( \epsilon = 1 \), then
\[ c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613474845886230468750 < 2.(2.3.5.7.11.13.41.79.311.953)^2 \implies 613474845886230468750 < 166214587249552942316264200 \]
3.4. Conclusion

This is an elementary proof of the ABC conjecture, confirmed by four numerical examples. We can announce the important theorem:

**Theorem 4.** — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[
(66) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

where $K$ is a constant depending of $\epsilon$ equal to $\frac{2}{\epsilon^2}$. 
BIBLIOGRAPHY


CHAPTER 4

A COMPLETE PROOF OF THE ABC CONJECTURE

4.1. Introduction and notations

Let $a$ a positive integer, $a = \prod a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod a_i$, noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod a_i^{\alpha_i} = \text{rad}(a) \cdot \prod a_i^{\alpha_i - 1}$$

We note:

$$\mu_a = \prod a_i^{\alpha_i - 1} \implies a = \mu_a \cdot \text{rad}(a)$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Esterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

**Conjecture 4.** — *(ABC Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending only of $\epsilon$.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751$ [2]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. Here we will give a proof of it.

**Conjecture 5.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2$$

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.
4.2. A Proof of the conjecture (5)

Let \(a, b, c\) positive integers, relatively prime, with \(c = a + b\). We suppose that \(b < a\).

If \(c < \text{rad}(ab)\) then we obtain:

\[
(71) \quad c < \text{rad}(ab) < \text{rad}^2(abc)
\]

and the condition (70) is verified.

In the following, we suppose that \(c \geq \text{rad}(ab)\).

4.2.1. Case \(c = a + 1\)

\[
(72) \quad c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac)
\]

4.2.1.1. \(\mu_a = 1\)

In this case, \(a = \text{rad}(a)\), it is immediately truth that :

\[
(73) \quad c = a + 1 < 2a < \text{rad}(a)\text{rad}(c) < \text{rad}^2(ac)
\]

Then (72) is verified.

4.2.1.2. \(\mu_a \neq 1, \mu_a < \text{rad}(a)\)

we obtain :

\[
(74) \quad c = a + 1 < 2\mu_a \text{rad}(a) \Rightarrow c < 2\text{rad}^2(a) \Rightarrow c < \text{rad}^2(ac)
\]

Then (72) is verified.

4.2.1.3. \(\mu_a \geq \text{rad}(a)\)

We have \(c = a + 1 = \mu_a \text{rad}(a) + 1 \leq \mu_a^2 + 1 < \text{rad}^2(ac)\). We suppose that \(\mu_a^2 + 1 \geq \text{rad}^2(ac) \implies \mu_a^2 + 1 < \text{rad}^2(a)\text{rad}(c)\). as \(\text{rad}(c) > 1\), then \(\mu_a > \text{rad}(a)\), that is the contradiction with \(\mu_a \geq \text{rad}(a)\). We deduce that \(c < \mu_a^2 + 1 < \text{rad}^2(ac)\) and the condition (72) is verified.

4.2.2. \(c = a + b\)

We can write that \(c\) verifies:

\[
(75) \quad \Rightarrow c = \text{rad}(a)\text{rad}(b)\text{rad}(c) \left(\frac{\mu_a}{\text{rad}(b)\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a)\text{rad}(c)}\right)
\]
We can write also:

\[ c = \text{rad}(abc) \left( \frac{\mu_a}{\text{rad}(b)\text{.rad}(c)} + \frac{\mu_b}{\text{rad}(a)\text{.rad}(c)} \right) \]

To obtain a proof of (70), one method is to prove that:

\[ \frac{\mu_a}{\text{rad}(b)\text{.rad}(c)} + \frac{\mu_b}{\text{rad}(a)\text{.rad}(c)} < \text{rad}(abc) \]

4.2.2.1. \( \mu_a = \mu_b = 1 \)

In this case, it is immediately true that:

\[ \frac{1}{\text{rad}(a)} + \frac{1}{\text{rad}(b)} \leq \frac{5}{6} < \text{rad}(c)\text{.rad}(abc) \]

Then (70) is verified.

4.2.2.2. \( \mu_a = 1 \) and \( \mu_b > 1 \)

As \( b < a \Rightarrow \mu_b\text{rad}(b) < \text{rad}(a) \Rightarrow \frac{\mu_b}{\text{rad}(a)} < \frac{1}{\text{rad}(b)} \), then we deduce that:

\[ \frac{1}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} < \frac{2}{\text{rad}(b)} < \text{rad}(c)\text{.rad}(abc) \]

Then (70) is verified.

4.2.2.3. \( \mu_b = 1 \) and \( \mu_a \leq (b = \text{rad}(b)) \)

In this case we obtain:

\[ \frac{1}{\text{rad}(a)} + \frac{\mu_a}{\text{rad}(b)} \leq \frac{1}{\text{rad}(a)} + 1 < \text{rad}(c)\text{.rad}(abc) \]

Then (70) is verified.

4.2.2.4. \( \mu_b = 1 \) and \( \mu_a > (b = \text{rad}(b)) \)

As \( \mu_a > \text{rad}(b) \), we can write \( \mu_a = \text{rad}(b) + n \) where \( n \geq 1 \). We obtain:

\[ c = \mu_a\text{rad}(a) + \text{rad}(b) = (\text{rad}(b) + n)\text{rad}(a) + \text{rad}(b) = \text{rad}(ab) + n\text{rad}(a) + \text{rad}(b) \]

We have \( n < b \), if not \( n \geq b \Rightarrow \mu_a \geq 2b \Rightarrow a \geq 2\text{rad}(a) \Rightarrow a \geq 3b \Rightarrow c > 3b \), then the contradiction with \( c > 2b \). We can write:

\[ c < 2\text{rad}(ab) + \text{rad}(b) \Rightarrow c < \text{rad}(abc) + \text{rad}(abc) < \text{rad}^2(abc) \Rightarrow c < \text{rad}^2(abc) \]

4.2.2.5. \( \mu_a\mu_b \neq 1 \), \( \mu_a < \text{rad}(a) \) and \( \mu_b < \text{rad}(b) \)

we obtain:

\[ c = \mu_c\text{rad}(c) = \mu_a\text{.rad}(a) + \mu_b\text{.rad}(b) < \text{rad}^2(a) + \text{rad}^2(b) < \text{rad}^2(abc) \]
4.2.2.6. \( \mu_a, \mu_b \neq 1, \mu_a \leq \text{rad}(a) \text{ and } \mu_b \geq \text{rad}(b) \)

We have:
\[
(84) \quad c = \mu_a \cdot \text{rad}(a) + \mu_b \cdot \text{rad}(b) < \mu_a \cdot \mu_b \cdot \text{rad}(a) \cdot \text{rad}(b) \leq \mu_b \cdot \text{rad}(a) \cdot \text{rad}(b)
\]

Then if we give a proof that \( \mu_b < \text{rad}(b) \cdot \text{rad}^2(c) \), we obtain \( c < \text{rad}^2(abc) \). As \( \mu_b \geq \text{rad}(b) \iff \mu_b = \text{rad}(b) + \alpha \) with \( \alpha \) a positive integer \( \geq 0 \). Supposing that \( \mu_b \geq \text{rad}(b) \cdot \text{rad}^2(c) \implies \mu_b = \text{rad}(b) \cdot \text{rad}^2(c) + \beta \) with \( \beta \geq 0 \) a positive integer. We can write:
\[
\text{rad}(b) \cdot \text{rad}^2(c) + \beta = \text{rad}(b) + \alpha \implies \beta < \alpha
\]

(85) \( \alpha - \beta = \text{rad}(b) \cdot (\text{rad}^2(c) - 1) > 3\text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha > 4\text{rad}(b) \)

Finally, we obtain:
\[
(86) \begin{cases} 
\mu_b \geq \text{rad}(b) \\
\mu_b > 4\text{rad}(b)
\end{cases}
\]

Then the contradiction and the hypothesis \( \mu_b \geq \text{rad}(b) \cdot \text{rad}^2(c) \) is false. Hence:
\[
(87) \quad \mu_b < \text{rad}(b) \cdot \text{rad}^2(c) \implies c < \text{rad}^2(abc)
\]

4.2.2.7. \( \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \text{ and } \mu_b \leq \text{rad}(b) \)

The proof is identical to the case above.

4.2.2.8. \( \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \text{ and } \mu_b \geq \text{rad}(b) \)

We write:
\[
(88) \quad c = \mu_a \cdot \text{rad}(a) + \mu_b \cdot \text{rad}(b) \leq \mu_a^2 + \mu_b^2 < \mu_a^2 \cdot \mu_b^2 < \text{rad}^2(a) \cdot \text{rad}^2(b) \cdot \text{rad}^2(c) = \text{rad}^2(abc)
\]

Supposing that \( \mu_a, \mu_b \geq \text{rad}(abc) \), we obtain:
\[
\mu_a, \mu_b \geq \text{rad}(abc) \implies \text{rad}(a) \cdot \text{rad}(b) \cdot \mu_a, \mu_b \geq \text{rad}^2(ab) \cdot \text{rad}(c) \implies \\
ab \geq \text{rad}^2(ab) \cdot \text{rad}(c) \implies a^2 > ab \geq \text{rad}^2(ab) \cdot \text{rad}(c) \implies \\
\Rightarrow a > \text{rad}(ab) \sqrt{\text{rad}(c)} \geq \text{rad}(ab) \sqrt{7} \Rightarrow \\
\begin{cases} 
\mu_a, \mu_b \geq \text{rad}(abc) \\
\mu_a, \mu_b \geq \text{rad}(abc) \\
\mu_a, \mu_b \geq \text{rad}(abc)
\end{cases}
\]

(89) \( c > \sqrt{7} \cdot \text{rad}(ab) \geq 3\text{rad}(ab) \)

The inequality \( c \geq 3\text{rad}(ab) \) gives the contradiction with the condition \( c \geq \text{rad}(ab) \) supposed at the beginning of this section. Then we obtain \( \mu_a, \mu_b - \text{rad}(abc) < 0 \implies c < \text{rad}^2(ab) \).

We announce the theorem:
**Theorem 5.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$ and $1 \leq b < a$, then $c < \text{rad}^2(abc)$.

### 4.3. The Proof of the abc Conjecture (4)

We denote $R = \text{rad}(abc)$.

#### 4.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} \left( \frac{1}{\epsilon^2} - \epsilon \right), \quad \epsilon \geq 1$$

#### 4.3.2. Case: $\epsilon < 1$

**4.3.2.1. Case: $c \leq R$**

In this case, we can write:

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} \left( \frac{1}{\epsilon^2} - \epsilon \right), \quad \epsilon < 1$$

and the $ABC$ conjecture is true.

**4.3.2.2. Case: $c > R$**

In this case, we confirm that:

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon} \left( \frac{1}{\epsilon^2} - \epsilon \right), \quad 0 < \epsilon < 1$$

If not, then $\exists \epsilon_0 \in ]0, 1[$, so that the triplets $(a, b, c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0)$$

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \implies$$

$$R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \implies R^{1-\epsilon_0} > K(\epsilon_0)$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > K(\epsilon_0) \implies c > K(\epsilon_0) \left( \frac{1}{1-\epsilon_0} \right)$$

We deduce that it exists an infinity of triplets $(a, b, c)$ verifying (93), hence the contradiction. Then the proof of the $ABC$ conjecture is finished. We obtain that $\forall \epsilon > 0$, 

$$c^{1-\epsilon_0} > K(\epsilon_0) \implies c > K(\epsilon_0) \left( \frac{1}{1-\epsilon_0} \right)$$
CHAPTER 4. A COMPLETE PROOF OF THE ABC CONJECTURE

\(c = a + b\) with \(a, b, c\) relatively coprime:

\[
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \quad \text{with} \quad K(\epsilon) = 6^{1+\epsilon} e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}
\]

Q.E.D

4.4. Examples

In this section, we are going to verify some numerical examples.

4.4.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

\[
a = 3^{10} \times 109 + 2 = 23^3 = 6436343
\]

\(c = 6436343\) is given by:

\[
b = 2 \Rightarrow \mu_b = 1 \quad \text{and} \quad \text{rad}(b) = 2,
\]

\(c = 23^3 = 6436343 \Rightarrow \text{rad}(c) = 23.\) Then \(\text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042.\) For example, we take \(\epsilon = 0.01,\) the expression of \(K(\epsilon)\) becomes:

\[
K(\epsilon) = 6^{1.01} e^{9999.99} = 1.8884880155640644914779227374022 e + 4343
\]

Let us verify (96):

\[
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \quad \Rightarrow \quad c = 6436343 < K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Rightarrow
\]

\[
6436343 \ll K(0.01) \times 15042
\]

Hence (96) is verified.

4.4.2. Example of A. Nitaj

4.4.2.1. Case 1

The example of Nitaj about the ABC conjecture [1] is:

\[
a = 11^{16} \cdot 13^2 \cdot 79 = 613474843408551921511 \Rightarrow \text{rad}(a) = 11.13.79
\]

\[
b = 7^2 \cdot 41^2 \cdot 311^3 = 2477678547239 \Rightarrow \text{rad}(b) = 7.41.311
\]

\[
c = 23^3 \cdot 5^{23} \cdot 953 = 613474845886230468750 \Rightarrow \text{rad}(c) = 2.3.5.7.11.13.41.79.311.953 = 28828335646110
\]

we take \(\epsilon = 100\) we have:

\[
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \quad \Rightarrow
\]

\[
613474845886230468750 < 6^{101} e^{-99.9999 \cdot (2.3.5.7.11.13.41.79.311.953)^{101}} \Rightarrow
\]

\[
613474845886230468750 < 8.2558649305610435609546415285004 e + 48
\]
then (96) is verified.

4.4.2.2. Case 2

We take \( \epsilon = 0.5 \), then:

\[
\begin{align*}
613474845886230468750 < 6^{1.5}.e^{2.5}.(2.3.5.7.11.13.41.79.311.953)^{1.5} & \implies \\
(104) & \implies \\
\end{align*}
\]

We obtain that (96) is verified.

4.4.2.3. Case 3

We take \( \epsilon = 1 \), then

\[
\begin{align*}
613474845886230468750 < 6^2.(2.3.5.7.11.13.41.79.311.953)^2 & \implies \\
(105) & \implies \\
\end{align*}
\]

We obtain that (96) is verified.

4.4.3. Example of Ralf Bonse

The example of Ralf Bonse about the ABC conjecture [2] is:

\[
\begin{align*}
2543^4.182587.2802983.85813163 & + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \\
a = 2543^4.182587.2802983.85813163 & \\
b = 2^{15}.3^{77}.11.173 & \\
c = 5^{56}.245983 & \\
rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163 & \\
(107) & \\
rad(abc) = 1.5683959920004546031461002610848e + 33 & \\
\end{align*}
\]

For example, we take \( \epsilon = 0.01 \), the expression of \( K(\epsilon) \) becomes:

\[
K(\epsilon) = 6^{1.01}.e^{9999.99} = 5.2903884296336672264108948608106e + 4343
\]

Let us verify (96):

\[
\begin{align*}
c < K(\epsilon).rad(abc)^{1+\epsilon} & \implies c = 5^{56}.245983 < \\
6^{1.01}.e^{9999.99}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01} & \implies \\
(109) & \\
1.7819595478010681971905561514574e + 4377 & \\
The equation (96) is verified.
4.5. Conclusion

This is an elementary proof of the ABC conjecture, confirmed by four numerical examples. We can announce the important theorem:

**Theorem 6.** — For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
\frac{c}{K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}}
\]

\[
\leq \frac{1}{\epsilon^2 - \epsilon}
\]

where \( K \) is a constant depending of \( \epsilon \) equal to \( 6^{1+\epsilon} e \left( \frac{1}{\epsilon^2 - \epsilon} \right) \).
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5.1. Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call \textit{radical} of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

\[(111) \quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i-1}\]

We note:

\[(112) \quad \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a)\]

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

\textbf{Conjecture 6. --- (ABC Conjecture):} For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[(113) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}\]

where $K$ is a constant depending only of $\epsilon$.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751$ [2]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. Here we will give the proof of it.

\textbf{Conjecture 7. ---} Let $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[(114) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2\]

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.
5.2. A Proof of the conjecture (7)

Let $a, b, c$ positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$.

If $c < \text{rad}(ab)$ then we obtain:

\[(115) \quad c < \text{rad}(ab) < \text{rad}^2(abc)\]

and the condition (114) is verified.

If $c = \text{rad}(ab)$, then $a, b, c$ are not relatively coprime. In the following, we suppose that $c > \text{rad}(ab)$. We can write:

\[c < \text{rad}^2(abc) \implies \mu_c \cdot \text{rad}(c) < \text{rad}^2(abc) \implies \frac{\mu_c}{\text{rad}(c)} < \text{rad}^2(ab) < c^2 \implies \frac{\mu_c}{\text{rad}(c)} < c^2 \implies \mu_c < c^2 \text{rad}(c) \implies \mu_c \cdot \text{rad}(c) < c^2 \text{rad}^2(c) \implies \]

\[(116) \quad c < c^2 \text{rad}^2(c) \implies 1 < c \cdot \text{rad}^2(c)\]

Then $c < \text{rad}^2(abc)$. We announce the theorem:

**Theorem 7.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$ and $1 \leq b < a$, then $c < \text{rad}^2(abc)$.

5.3. The Proof of the abc conjecture (6)

We denote $R = \text{rad}(abc)$.

5.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

\[(117) \quad c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon}e^{-\left(\frac{1}{e^2} - \epsilon\right)}, \quad \epsilon \geq 1\]

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$.

5.3.2. Case: $\epsilon < 1$

5.3.2.1. Case: $c < R$

In this case, we can write:

\[(118) \quad c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon}e^{-\left(\frac{1}{e^2} - \epsilon\right)}, \quad \epsilon < 1\]

Here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the $ABC$ conjecture is true.
5.3.2.2. Case: $c > R$

In this case, we confirm that:

\[ c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = 6^{1+\epsilon}e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)}, \quad 0 < \epsilon < 1 \]

If not, then $\exists \epsilon_0 \in ]0, 1[$, so that the triplets $(a, b, c)$ checking $c > R$ and:

\[ c \geq R^{1+\epsilon_0}.K(\epsilon_0) \]

are in finite number. We have:

\[ c \geq R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \implies R^{1-\epsilon_0} > K(\epsilon_0) \]

As $c > R$, we obtain:

\[ c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \implies \]

\[ c^{1-\epsilon_0} > K(\epsilon_0) \implies c > K(\epsilon_0)\left(\frac{1}{1 - \epsilon_0}\right) \]

We deduce that it exists an infinity of triples $(a, b, c)$ verifying (120), hence the contradiction. Then the proof of the $ABC$ conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with $a, b, c$ relatively coprime:

\[ c < K(\epsilon).rad(abc)^{1+\epsilon} \quad \text{with} \quad K(\epsilon) = 6^{1+\epsilon}e^{\left(\frac{1}{\epsilon^2} - \epsilon\right)} \]

Q.E.D

5.4. Examples

In this section, we are going to verify some numerical examples.

5.4.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

\[ 3^{10} \times 109 + 2 = 23^5 = 6436343 \]

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,

$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,

$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

\[ K(\epsilon) = 6^{1.01}e^{9999.99} = 1.8884880155640644914779227374022e + 4343 \]
Let us verify (123):

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies c = 6436343 < K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \implies \]

\[ 6436343 < K(0.01) \times 15042 \]

Hence (123) is verified.

5.4.2. Example of A. Nitaj

5.4.2.1. Case 1

The example of Nitaj about the ABC conjecture [1] is:

\[ a = 11^{16} \cdot 3^2 \cdot 79 = 613 \cdot 474 \cdot 843 \cdot 408 \cdot 551 \cdot 921 \cdot 511 \implies \text{rad}(a) = 11.13.79 \]
\[ b = 7^2 \cdot 41^2 \cdot 311^3 = 2477 \cdot 678 \cdot 547 \cdot 239 \implies \text{rad}(b) = 7.41.311 \]
\[ c = 2.3^3 \cdot 5^{23} \cdot 953 = 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 \implies \text{rad}(c) = 2.3.5.953 \]

\[ \text{rad}(abc) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 79 \cdot 311 \cdot 953 = 28828335646110 \]

we take \( \epsilon = 100 \) we have:

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies \]

\[ 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 < 6^{101} \cdot e^{-99.999} \cdot (2.3.5.7.11.13.41.79.311.953)^{101} \implies \]

\[ 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 < 8.2558649305610435609546415285004e + 48 \]

then (123) is verified.

5.4.2.2. Case 2

We take \( \epsilon = 0.5 \), then:

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies \]

\[ 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 < 6^{1.5} \cdot e^{3.5} \cdot (2.3.5.7.11.13.41.79.311.953)^{1.5} \implies \]

\[ 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 < 75333109597556257182261.66 \]

We obtain that (123) is verified.

5.4.2.3. Case 3

We take \( \epsilon = 1 \), then

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies \]

\[ 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 < 6^2 \cdot (2.3.5.7.11.13.41.79.311.953)^2 \implies \]

\[ 613 \cdot 474 \cdot 845 \cdot 886 \cdot 230 \cdot 468 \cdot 750 < 29918625700491952961692755600 \]

We obtain that (123) is verified.
5.4.3. Example of Ralf Bonse

The example of Ralf Bonse about the ABC conjecture [2] is:

\[(131)\]
\[2543^4 \cdot 182587.2802983.85813163 + 2^{15} \cdot 77^7 \cdot 11.173 = 5^{56}.245983\]
\[a = 2543^4 \cdot 182587.2802983.85813163\]
\[b = 2^{15} \cdot 77^7 \cdot 11.173\]
\[c = 5^{56}.245983\]
\[\text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163\]

\[(132)\]
\[\text{rad}(abc) = 1.5683959920004546031461002610848e + 33\]

For example, we take \(\epsilon = 0.01\), the expression of \(K(\epsilon)\) becomes:

\[K(\epsilon) = 6^{1.01}.e^{9999.99} = 5.2903884296336672264108948608106e + 4343\]

Let us verify (123):

\[c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 <\]
\[6^{1.01}.e^{9999.99} \cdot (2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01}\]
\[\Rightarrow 3.4136998783296235160378273576498e + 44 <\]
\[(133)\]
\[1.7819595478010681971905561514574e + 4377\]

The equation (123) is verified.

Ouf, end of the mystery!

5.5. Conclusion

This is an elementary proof of the ABC conjecture, confirmed by five numerical examples. We can announce the important theorem:

**Theorem 8.** — For each \(\epsilon > 0\), there exists \(K(\epsilon) > 0\) such that if \(a, b, c\) positive integers relatively prime with \(c = a + b\), then:

\[(134)\]
\[c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}\]

where \(K\) is a constant depending of \(\epsilon\) equal to \(6^{1+\epsilon}.e^{\left(\frac{1}{e^2} - \epsilon\right)}\).

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CHAPTER 6

A PROOF OF THE ABC CONJECTURE: THE END OF MYSTERY

6.1. Introduction and notations

Let \( a \) a positive integer, \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i^{\alpha_i} \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
(135) \quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i - 1}
\]

We note:

\[
(136) \quad \mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot \text{rad}(a)
\]

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

**Conjecture 8.** — (ABC Conjecture): Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then for each \( \epsilon > 0 \), there exists \( K(\epsilon) \) such that:

\[
(137) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751 \) [2]. Here we will give a proof that:

\[
(138) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]

This result, I think is the key to obtain a proof of the veracity of the ABC conjecture.

6.2. A Proof of the condition (138)

Let \( a, b, c \) positive integers, relatively prime, with \( c = a + b \). We suppose that \( b < a \).

If \( c < \text{rad}(ab) \) then we obtain:

\[
(139) \quad c < \text{rad}(ab) < \text{rad}^2(abc)
\]
and the condition (138) is verified.

In the following, we suppose that \( c \geq \text{rad}(ab) \).

### 6.2.1. Case \( c = a + 1 \)

(140) \[ c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac) \]

#### 6.2.1.1. \( \mu_a = 1 \)

In this case, \( a = \text{rad}(a) \), it is immediately truth that:

(141) \[ c = a + 1 < 2a < \text{rad}(a)\text{rad}(c) < \text{rad}^2(ac) \]

Then (140) is verified.

#### 6.2.1.2. \( \mu_a \neq 1, \mu_a < \text{rad}(a) \)

we obtain:

(142) \[ c = a + 1 < 2\mu_a \text{rad}(a) \Rightarrow c < 2\text{rad}^2(a) \Rightarrow c < \text{rad}^2(ac) \]

Then (140) is verified.

#### 6.2.1.3. \( \mu_a \geq \text{rad}(a) \)

We have \( c = a + 1 = \mu_a \text{rad}(a) + 1 \leq \mu_a^2 + 1 < \text{rad}^2(ac) \). We suppose that \( \mu_a^2 + 1 \geq \text{rad}^2(ac) \Rightarrow \mu_a^2 > \text{rad}^2(a).\text{rad}(c) > \text{rad}^2(a) \) as \( \text{rad}(c) > 1 \), then \( \mu_a > \text{rad}(a) \), that is the contradiction with \( \mu_a \geq \text{rad}(a) \). We deduce that \( c < \mu_a^2 + 1 < \text{rad}^2(ac) \) and the condition (140) is verified.

### 6.2.2. \( c = a + b \)

We can write that \( c \) verifies:

\[ c = a + b = \text{rad}(a)\mu_a + \text{rad}(b)\mu_b = \text{rad}(a)\text{rad}(b)\left( \frac{\mu_a}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} \right) \]

(143) \[ \Rightarrow c = \text{rad}(a)\text{rad}(b)\text{rad}(c)\left( \frac{\mu_a}{\text{rad}(b)\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a)\text{rad}(c)} \right) \]

We can write also:

(144) \[ c = \text{rad}(abc)\left( \frac{\mu_a}{\text{rad}(b)\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a)\text{rad}(c)} \right) \]

To obtain a proof of (138), one method is to prove that:

(145) \[ \frac{\mu_a}{\text{rad}(b)\text{rad}(c)} + \frac{\mu_b}{\text{rad}(a)\text{rad}(c)} < \text{rad}(abc) \]
6.2.2.1. $\mu_a = \mu_b = 1$

In this case, it is immediately truth that:

\[
\frac{1}{\text{rad}(a_i)} + \frac{1}{\text{rad}(b_j)} \leq \frac{5}{6} < \text{rad}(c).\text{rad}(abc)
\]

Then (138) is verified.

6.2.2.2. $\mu_a = 1$ and $\mu_b > 1$

As $b < a \implies \mu_b \text{rad}(b) < \text{rad}(a) \implies \frac{\mu_b}{\text{rad}(a)} < \frac{1}{\text{rad}(b)}$, then we deduce that:

\[
\frac{1}{\text{rad}(b)} + \frac{\mu_b}{\text{rad}(a)} < \frac{2}{\text{rad}(b)} < \text{rad}(c).\text{rad}(abc)
\]

Then (138) is verified.

6.2.2.3. $\mu_b = 1$ and $\mu_a \leq (b = \text{rad}(b))$

In this case we obtain:

\[
\frac{1}{\text{rad}(a)} + \frac{\mu_a}{\text{rad}(b)} \leq \frac{1}{\text{rad}(a)} + 1 < \text{rad}(c).\text{rad}(abc)
\]

Then (138) is verified.

6.2.2.4. $\mu_b = 1$ and $\mu_a > (b = \text{rad}(b))$

As $\mu_a > \text{rad}(b)$, we can write $\mu_a = \text{rad}(b) + n$ where $n \geq 1$. We obtain:

\[
c = \mu_a \text{rad}(a) + \text{rad}(b) = (\text{rad}(b) + n)\text{rad}(a) + \text{rad}(b) = \text{rad}(ab) + n\text{rad}(a) + \text{rad}(b)
\]

We have $n < b$, if not $n \geq b \implies \mu_a \geq 2b \implies a \geq 2\text{rad}(a) \implies a \geq 3b \implies c > 3b$, then the contradiction with $c > 2b$. We can write:

\[
c < 2\text{rad}(ab) + \text{rad}(b) \implies c < \text{rad}(abc) + \text{rad}(abc) < \text{rad}^2(abc) \implies c < \text{rad}^2(abc)
\]

6.2.2.5. $\mu_a, \mu_b \neq 1, \mu_a < \text{rad}(a)$ and $\mu_b < \text{rad}(b)$

we obtain:

\[
c = \mu_a \text{rad}(c) = \mu_a.\text{rad}(a) + \mu_b.\text{rad}(b) < \text{rad}^2(a) + \text{rad}^2(b) < \text{rad}^2(abc)
\]

6.2.2.6. $\mu_a, \mu_b \neq 1, \mu_a \leq \text{rad}(a)$ and $\mu_b \geq \text{rad}(b)$

We have:

\[
c = \mu_a.\text{rad}(a) + \mu_b.\text{rad}(b) < \mu_a \mu_b \text{rad}(a) \text{rad}(b) \leq \mu_b \text{rad}^2(a) \text{rad}(b)
\]
Then if we give a proof that \( \mu_b < \text{rad}(b)\text{rad}^2(c) \), we obtain \( c < \text{rad}^2(abc) \). As \( \mu_b \geq \text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha \) with \( \alpha \) a positive integer \( \geq 0 \). Supposing that \( \mu_b \geq \text{rad}(b)\text{rad}^2(c) \implies \mu_b = \text{rad}(b)\text{rad}^2(c) + \beta \) with \( \beta \geq 0 \) a positive integer. We can write:

\[
\text{rad}(b)\text{rad}^2(c) + \beta = \text{rad}(b) + \alpha \implies \beta < \alpha
\]

(153) \( \alpha - \beta = \text{rad}(b)(\text{rad}^2(c) - 1) > 3\text{rad}(b) \implies \mu_b = \text{rad}(b) + \alpha > 4\text{rad}(b) \)

Finally, we obtain:

\[
\begin{cases}
\mu_b \geq \text{rad}(b) \\
\mu_b > 4\text{rad}(b)
\end{cases}
\]

(154)

Then the contradiction and the hypothesis \( \mu_b \geq \text{rad}(b)\text{rad}^2(c) \) is false. Hence:

\[
\mu_b < \text{rad}(b)\text{rad}^2(c) \implies c < \text{rad}^2(abc)
\]

6.2.2.7. \( \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \) and \( \mu_b \leq \text{rad}(b) \)

The proof is identical to the case above.

6.2.2.8. \( \mu_a, \mu_b \neq 1, \mu_a \geq \text{rad}(a) \) and \( \mu_b \geq \text{rad}(b) \)

We write:

(156)

\[
c = \mu_a\text{rad}(a) + \mu_b\text{rad}(b) \leq \mu_a^2 + \mu_b^2 \leq \mu_a^2, \mu_b^2 \leq \text{rad}^2(a),\text{rad}^2(b),\text{rad}^2(c) = \text{rad}^2(abc)
\]

Supposing that \( \mu_a, \mu_b \geq \text{rad}(abc) \), we obtain:

\[
\mu_a, \mu_b \geq \text{rad}(abc) \Rightarrow \text{rad}(a),\text{rad}(b),\mu_a, \mu_b \geq \text{rad}^2(ab)\text{rad}(c) \Rightarrow
\]

\[
ab \geq \text{rad}^2(ab),\text{rad}(c) \Rightarrow a^2 > ab \geq \text{rad}^2(ab),\text{rad}(c) \\
\Rightarrow a > \text{rad}(ab)\sqrt{\text{rad}(c)} \geq \text{rad}(ab)\sqrt{7} \Rightarrow
\]

(157)

\[
\begin{cases}
\text{rad}(ab) \\
c \geq \text{rad}(ab)
\end{cases}
\]

The inequality \( c \geq 3\text{rad}(ab) \) gives the contradiction with the condition \( c \geq \text{rad}(ab) \) supposed at the beginning of this section. Then we obtain \( \mu_a, \mu_b - \text{rad}(abc) < 0 \Rightarrow c < \text{rad}^2(abc) \).

We announce the theorem:

**Theorem 9.** — *(Abdelmajid Ben Hadj Salem, 2019)* Let \( a, b, c \) positive integers relatively prime with \( c = a + b \) and \( b < a \), then \( c < \text{rad}^2(abc) \).

7.1. Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

\begin{equation}
    a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i-1}
\end{equation}

We note:

\begin{equation}
    \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a.\text{rad}(a)
\end{equation}

A paper about the proof of the $ABC$ conjecture, is accepted [1]. We have obtained the following theorem:

**Theorem 10.** — *(ABC Theorem)*: For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\begin{equation}
    c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}
\end{equation}

where $K$ is a constant depending only of $\epsilon$ equal to $6^{1+\epsilon}e^{\left(\frac{1}{e^2} - \epsilon\right)}$.

In 1997, Andrew Beal [2] announced the following conjecture:

**Conjecture 1: (Beal Conjecture)**

Let $A, B, C, m, n$, and $l$ be positive integers with $m, n, l > 2$. If:

\begin{equation}
    A^m + B^n = C^l
\end{equation}

then $A, B$, and $C$ have a common factor.
7.2. Methodology of the proof

We note:

(162) \quad A: \text{Beal Conjecture}

(163) \quad B: \text{ABC Theorem}

and we use the following property (the contrapositive law, [3]):

\[
A(\text{False}) \implies B(\text{False}) \iff B(\text{True}) \implies A(\text{True})
\]

From the right equivalent expression in the box above, we obtain that B (ABC Theorem) which is true implies A (Beal Conjecture) is true.

7.3. Proof of the conjecture (7.1)

We suppose that Beal conjecture is false, then it exists \(A, B, C\) positive coprime integers and \(m, n, l\) positive integers > 2 such:

\[
A^m + B^n = C^l
\]

the integers \(A, B, C, m, n, l\) are supposed large integers. We consider in the following that \(A^m > B^n\). Now, we use the ABC theorem for equation (165) because \(A^m, B^n, C^m\) are relatively coprime. We obtain:

\[
C^l < K(\epsilon) \text{rad}(A^mB^nC^l)^{1+\epsilon} \implies C^l < K(\epsilon) (\text{rad}(A), \text{rad}(B), \text{rad}(C))^{1+\epsilon}
\]

As \(\text{rad}(A) \leq A, \text{rad}(B) \leq B\) and \(\text{rad}(C) \leq C\), the last equation becomes:

\[
C^l < \frac{2}{\epsilon^2} (A.B.C)^{1+\epsilon}
\]

But \(\text{rad}(A) \leq A < C^{\frac{l}{m}}, \text{rad}(B) \leq B < C^{\frac{l}{n}}\), then we write (167) as:

\[
\frac{\epsilon^2}{2} < C \left(1 + \frac{1}{m} + \frac{1}{n}\right) . (1 + \epsilon) - l
\]

7.3.1. Case \(m > l\) and \(n > l\)

In this case, \(\left(1 + \frac{1}{m} + \frac{1}{n}\right) . (1 + \epsilon) - l \approx 3 - l + 3\epsilon\). We take \(\epsilon = 1\). As \(6 \ll l \implies \frac{1}{C^{l-6}} \ll 0.5\), then the contradiction.

7.3.2. Case \(m < l\) and \(n < l\)

In this case, if \(C > A \Rightarrow C^m > A^m > B^n \Rightarrow C^m > B^n \Rightarrow C^m > C^l - A^m \Rightarrow A^m > C^l - C^m \Rightarrow A^m > C^m(C^{l-m} - 1)\). As \(l > m \Rightarrow C^{l-m} - 1 > 1\), then
A^n > C^m \implies A > C that is a contradiction with C > A. Hence C < A. We rewrite equation (167):

\[ C^l < K(\epsilon)rad(A^m B^n C^l)^{1+\epsilon} \implies C^l < K(\epsilon)(A.B.C)^{1+\epsilon} \]

\[ \Rightarrow A^m < C^l < K(\epsilon) \left( A.A^{\frac{m}{n}}.A \right)^{1+\epsilon} \]

Then:

\[ A^m < \frac{2}{\epsilon^2}.A \left( 2 + \frac{m}{n} \right) (1 + \epsilon) \]

7.3.2.1. Case \( n > m \)

If \( n > m \), we have \( \left( 2 + \frac{m}{n} \right) (1 + \epsilon) \approx 3 + 3\epsilon \). We take \( \epsilon = 1 \), as \( 6 \ll m \implies \frac{1}{A^{m-6}} \ll 0.5 \), then the contradiction.

7.3.2.2. Case \( n < m \)

We have:

\[ C^l < K(\epsilon)(A.B.C)^{1+\epsilon} \]

As \( A^m < C^l, C < A \) and \( B^n < A^m \implies B < A^{m/n} \), the last equation becomes:

\[ \frac{\epsilon^2}{2} < A^{(2 + m/n)(1 + \epsilon) - m} \]

We choose \( \epsilon = \frac{1}{m} \), we obtain:

\[ \frac{1}{2m^2} < A^{2 - m + \frac{2}{m} + \frac{1}{n}} \implies \frac{1}{2m^2} < A^{3-m} \]

But \( 3 \ll m \) and \( 1 \ll A \implies \frac{1}{2m^2} > A^{3-m} \), then the contradiction.

7.3.3. Case \( m < l \) and \( n > l \)

If \( C < A \), as \( l < n \) \( \Rightarrow C^l < A^n \Rightarrow 0 < A^m < A^n - B^n \) then \( A > B \). As \( C^n > C^l > B^n \Rightarrow C^m > B^n \Rightarrow C > B \). So we obtain:

\[ B < C < A \]

Then the equation (167) becomes:

\[ C^l \frac{\epsilon^2}{2} < (A.B.C)^{1+\epsilon} \Rightarrow C^l \frac{\epsilon^2}{2} < (A.A^{l/n}.A)^{1+\epsilon} \Rightarrow C^l \frac{\epsilon^2}{2} < A^{(2 + l/n)(1 + \epsilon)} \]

As \( A^m < C^l \), we arrive to:

\[ \frac{\epsilon^2}{2} < A^{3-3m + 3\epsilon} \]
We take $\epsilon = 1/3 \implies A^{3m-4} < 18$, then the contradiction because $18 \ll A^{3m-4}$.

If $A < C \implies A^l < C^l$ but $B^n < A^m \implies A^l < 2A^m \implies A^l < A^{m+1} \implies l < m + 1$, as $m < l \implies m + 1 \leq l < m + 1$ that is a contradiction, then $C < A$ and this case is studied above.

7.3.4. Case $m > l$ and $n < l$

We have $n < l < m$. As $A^m < C^l \implies A < C^{\frac{l}{m}} < C \implies A < C$. The equation (167) becomes:

\[
C^l < \frac{2}{C^2} \left( C^{l/m} \cdot C^{l/n} \cdot C \right)^{1+\epsilon}
\]

We take $\epsilon = 0.1$, we obtain:

\[
0.005 < C^{2.2+1.1 \frac{l}{m} - l} \approx C^{3+\frac{l}{n} - l}
\]

But as $3 \ll l \implies l > 3 + \frac{l}{n}$, then the contradiction.

All the cases give contradiction, then $ABC$ theorem is false. We deduce from:

\[
\begin{array}{c}
\text{Beal Conjecture (False) \implies ABC Theorem (False)} \\
\text{ABC Theorem (True) \implies Beal Conjecture (True)}
\end{array}
\]

that Beal Conjecture is true.

The proof of the Beal conjecture is achieved\(^{(1)}\).

7.4. Conclusion

From the $ABC$ theorem, we have given a proof that the $ABC$ conjecture is true. We can announce the theorem:

**Theorem 11.** — *Abdelmajid Ben Hadj Salem, Andrew Beal, 2019*: Let $A, B, C, m, n,$ and $l$ be positive integers with $m, n, l > 2$. If:

\[
A^m + B^n = C^l
\]

then $A, B,$ and $C$ have a common factor.

\(^{(1)}\)A paper giving another proof of Beal conjecture is under reviewing [4]
BIBLIOGRAPHY


CHAPTER 8

THE ABC CONJECTURE: THE END OF THE MYSTERY

8.1. Introduction and notations

Let \( a \) a positive integer, \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i^{\alpha_i} \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
(180)\quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i-1}
\]

We note:

\[
(181)\quad \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a)
\]

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Étienne of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of their sum. The definition of the ABC conjecture is given below:

**Conjecture 9.** — (ABC Conjecture): Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then for each \( \epsilon > 0 \), there exists \( K(\epsilon) \) such that:

\[
(182)\quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students.

8.2. Proof of the conjecture (9)

Let \( a, b, c \) positive integers, relatively prime, with \( c = a + b \). We suppose that \( b < a \). We propose for the constant \( K(\epsilon) \) the formula:

\[
(183)\quad K(\epsilon) = \frac{2}{c^2}
\]
Function of $\epsilon$, $K(\epsilon)$ is a decreasing function so that $\lim_{\epsilon \to 0} K(\epsilon) = +\infty$ and $\lim_{\epsilon \to +\infty} K(\epsilon) = 0$. It permits an equilibrium when $\text{rad}(abc)^{1+\epsilon} > \epsilon$ with $\epsilon > 1$ or $\text{rad}(abc)^{1+\epsilon} \searrow$ with $\epsilon \searrow 0$. We want to see if:

$$c > K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} = \frac{2}{c^2} e^{(1+\epsilon) \text{Log}(\text{rad}(abc))}$$

We denote:

$$R = \text{Log}(\text{rad}(abc)) > 0$$

$$F(\epsilon) = 2e^{(1+\epsilon)R} - c \epsilon^2$$

If we arrive to give a proof of $F(\epsilon) > 0$ for $\forall \epsilon > 0$, then we deduce that the $ABC$ conjecture is true and it is the end of the mystery of the $ABC$ conjecture!

Let us study the function $F(\epsilon)$ with $\epsilon \in [0, +\infty[$. The function $F(\epsilon)$ is of class $C^\infty$. We have $F(0) = 2e^R = 2\text{rad}(abc) \geq 12$, and $\lim_{\epsilon \to +\infty} F(\epsilon) = +\infty$ because $\min(\text{rad}(abc)) > \epsilon \Rightarrow R > 1$. We calculate $F'(\epsilon)$:

$$F'(\epsilon) = 2\cdot \text{Log}(\text{rad}(abc))e^{(1+\epsilon)R} - 2c = 2Re^{(1+\epsilon)R} - 2c \longrightarrow +\infty, \text{ if } \epsilon \longrightarrow +\infty$$

$$F''(\epsilon) = 2.\text{Log}^2(\text{rad}(abc))e^{(1+\epsilon)R} - 2 = 2R^2e^{(1+\epsilon)R} - 2, \text{ if } F'(\epsilon) = 0$$

(188) $$2.\text{Log}(\text{rad}(abc))e^{(1+\epsilon)R} = 2c$$

Let :

$$\alpha = \frac{c}{R^2e^R}$$

$$X = Re$$

The equation (188) becomes:

$$e^X = \alpha X$$

(191)

The equation above represents the determination of the coordinates $(X,Y)$ of the points of the intersection of the two curves: $Y = e^X$ and $Y = \alpha X$. The tangent line to the curve $Y = e^X$ passing at the origin $(0,0)$ is represented by $Y = eX$. Then three cases are to study:

- Case 1: $\alpha < e \implies c < e \cdot \text{rad}(abc) \cdot \text{Log}^2 \text{rad}(abc)$, and $e^X \neq \alpha X$ and the exponential curve is above the line $Y = \alpha X \implies F'(\epsilon) > 0$ for $\forall \epsilon > 0$, then $F(\epsilon) > 0$ and the $ABC$ conjecture holds.

- Case 2: $\alpha = e$, the line $Y = \alpha X = eX$ is tangent to the curve $Y = e^X$ at the point $(1,e)$. $\alpha = e \implies c = e \cdot \text{rad}(abc) \cdot \text{Log}^2 \text{rad}(abc)$. Supposing that $e \cdot \text{Log}^2 \text{rad}(abc)$ is an integer, we obtain that $a, b, c$ are not relatively prime. Then the contradiction.
- Case 3: $\alpha > e \implies c > e.\text{rad}(abc).\log^2\text{rad}(abc)$, and $e^X = \alpha X$ has one solution $X_1 = Re_1 < 1$ with $\epsilon_1 < 1$ and for $\epsilon > \epsilon_1$, we obtain $F'(\epsilon) < 0 \implies F(\epsilon)$ is decreasing for $\epsilon \in [\epsilon_1, +\infty]$, but $\lim_{\epsilon \to +\infty} F(\epsilon) = +\infty$ then the contradiction.

Finally, only the case 1 is correct and we obtain that the $ABC$ conjecture is true. Then we obtain the important result of the paper:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \quad \forall \epsilon > 0$$

(192)

with the constant $K(\epsilon) = \frac{2}{\epsilon^2}$

Q.E.D

8.3. Examples

In this section, we are going to verify some numerical examples.

8.3.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

(193) $3^{10} \times 109 + 2 = 23^5 = 6436343$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $\text{rad}(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $\text{rad}(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow \text{rad}(c) = 23$. Then $\text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042$. For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

(194) $K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}}$

Let us verify (192):

$$c \not< K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies c = 6436343 \not< 2.10^4.(3 \times 109 \times 2 \times 23)^{1.01} \implies$$

(195) $6436343 < 331213962.07$

Hence (192) is verified.
8.3.2. Example of A. Nitaj

8.3.2.1. Case 1

The example of Nitaj about the ABC conjecture [3] is:

\( a = 11^{16} \cdot 13^2 \cdot 79 = 613 \, 474 \, 843 \, 408 \, 551 \, 921 \, 511 \Rightarrow \text{rad}(a) = 11.13.79 \)  
\( b = 7^2 \cdot 41^2 \cdot 311^3 = 2 \, 477 \, 678 \, 547 \, 239 \Rightarrow \text{rad}(b) = 7.41.311 \)  
\( c = 2.3^3 \cdot 5^{23} \cdot 953 = 613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 \Rightarrow \text{rad}(c) = 2.3.5.953 \)

\( \text{rad}(abc) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 179 \cdot 311 \cdot 953 = 2 \, 828 \, 335 \, 646 \, 110 \)

We take \( \epsilon = 100 \) we have:

\[
613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 < 2 \cdot 10^{-1} \cdot (2.3.5.7.11.13.41.79.311.953)^{101} \Rightarrow
\]

\[
613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 < 5.53103686332861264803638e + 1355
\]

then (192) is verified.

8.3.2.2. Case 2

We take \( \epsilon = 0.000001 = 10^{-6} \), then:

\[
613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 < 2 \cdot 10^{-12} \cdot (2.3.5.7.11.13.41.79.311.953)^{1.000001} \Rightarrow
\]

\[
613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 < 57 \, 658 \, 458 \, 237 \, 370 \, 924 \, 700 \, 998 \, 757 \, 174 \, 980.
\]

We obtain that (192) is verified.

8.3.2.3. Case 3

We take \( \epsilon = 1 \), then

\[
613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 < 2 \cdot (2.3.5.7.11.13.41.79.311.953)^{2} \Rightarrow
\]

\[
613 \, 474 \, 845 \, 886 \, 230 \, 468 \, 750 < 1 \, 662 \, 145 \, 872 \, 249 \, 552 \, 942 \, 316 \, 264 \, 200.
\]

\( \text{Ouf, end of the mystery!} \)

8.4. Conclusion

This is an elementary proof of the \( ABC \) conjecture, confirmed by four numerical examples. We can announce the important theorem:
Theorem 12. — (David Masser, Joseph Oesterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending of $\epsilon$ equal to $\frac{2}{\epsilon^2}$. 
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CHAPTER 9

THE ABC CONJECTURE: THE END OF THE MYSTERY

9.1. Introduction and notations

Let $a$ a positive integer, $a = \prod i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod i a_i$ noted by $rad(a)$. Then $a$ is written as:

$$a = \prod i a_i^{\alpha_i} = rad(a) \cdot \prod i a_i^{\alpha_i - 1}$$

We note:

$$\mu_a = \prod i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot rad(a)$$

The ABC conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the ABC conjecture is given below:

**Conjecture 10.** — (ABC Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon) \cdot rad(abc)^{1+\epsilon}$$

where $K$ is a constant depending only of $\epsilon$.

This paper about this conjecture is written after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students.
9.2. Proof of the conjecture (10)

Let $a, b, c$ positive integers, relatively prime, with $c = a + b$. We suppose that $b < a$. We propose for the constant $K(\epsilon)$ the formula:

$$K(\epsilon) = \frac{2}{\epsilon^2}$$

Function of $\epsilon$, $K(\epsilon)$ is a decreasing function so that $\lim_{\epsilon \to 0} K(\epsilon) = +\infty$ and $\lim_{\epsilon \to +\infty} K(\epsilon) = 0$. It permits an equilibrium when $\text{rad}(abc)^{1+\epsilon}$ with $\epsilon > 1$ or $\text{rad}(abc)^{1+\epsilon} \downarrow$ with $\epsilon \downarrow 0$. We want to see if:

$$c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} = \frac{2}{\epsilon^2} e^{(1+\epsilon) \text{Log}(\text{rad}(abc))}$$

We denote:

$$R = \text{Log}(\text{rad}(abc)) > 0$$

$$\alpha = \frac{c}{2R^2 e^R}$$

$$X = R\epsilon$$

$$F(\epsilon) = 2e^{(1+\epsilon)R} - c\epsilon^2 = 2e^R \left( e^{\epsilon R} - \frac{c}{2R^2 e^R} R^2 \epsilon^2 \right) = 2e^R (e^X - \alpha X^2)$$

$$G(X) = e^X - \alpha X^2$$

If we arrive to give a proof of $F(\epsilon) > 0$ for $\forall \epsilon > 0$ or $G(X) > 0$ for $\forall X > 0$, then we deduce that the ABC conjecture is true and it is the end of the mystery of the ABC conjecture!

Let us study the function $G(X)$ with $X \in [0, +\infty]$. The function $G(X)$ is of class $C^\infty$. We have $G(0^+) = 1^+$, and $\lim_{X \to +\infty} G(X) = +\infty$ because $\min(\text{rad}(abc)) > e \Rightarrow R > 1$. We calculate $G'(X)$:

$$G'(X) = e^X - 2\alpha X, \quad G'(X) \to +\infty, \text{ if } X \to +\infty$$

$$G''(X) = e^X - 2\alpha$$

$G''(X) = 0$ gives:

$$e^X = 2\alpha$$

- Case 1: As $X > 0 \Rightarrow e^X > 1$. If $2\alpha < 1 \Rightarrow c < R^2 e^R \Rightarrow G''(X) > 0, \forall X > 0$ (Fig.1), then $G'(X) > 0, \forall X > 0$ and $G(X)$ is an increasing function from $1^+$ to $+\infty$ hence, $F(\epsilon) > 0, \forall \epsilon > 0$ then the ABC conjecture holds.

- Case 1.1: $2\alpha = 1 \Rightarrow c = R^2 e^R = \text{Log}^2(\text{rad}(abc)) \cdot \text{rad}(abc)$. Supposing that $\text{Log}^2 \text{rad}(abc)$ is an integer, we obtain that $a, b, c$ are not relatively coprime. Then
the contradiction, and the ABC conjecture holds.

- Case 2 : If $2\alpha > 1 \implies c > R^2 e^R$, the equation $e^X = 2\alpha$ has an unique solution $X_1 = \log 2\alpha \implies X_1 > 0$. $G'(X_1) = e^{X_1} - 2\alpha X_1 = 2\alpha(1 - X_1)$.

- Case 2.1. If $X_1 = 1 \implies \log 2\alpha = 1 \implies 2\alpha = e \implies c = e \cdot \text{rad}(abc) \cdot \log^2 \text{rad}(abc)$. Supposing that $e \log^2 \text{rad}(abc)$ is an integer, we obtain that $a, b, c$ are not relatively coprime. Then the contradiction, and the ABC conjecture holds.

- Case 2.2. If $X_1 < 1 \implies \log 2\alpha < 1 \implies 2\alpha < e \implies c < e R^2 e^R$. $G''(X_1) = 0, G'(X_1) > 0$ and $G(X) > 0, \forall X > 0$ (Fig.2). Then $F(e) > 0 \forall \epsilon > 0$ and the ABC conjecture holds.

- Case 2.3. If $X_1 > 1 \implies \log 2\alpha > 1 \implies 2\alpha > e \implies \alpha > e/2$. We obtain the condition that $c > e \cdot \text{rad}(abc) \cdot \log^2 \text{rad}(abc)$. We have $e^{X_1} = 2\alpha$, $G'(X_1) = 2\alpha(1 - X_1) < 0$ and $G''(1) = e - 2\alpha < 0$, then $\exists X_3 < 1 < X_1$ so that $G''(X_3) = 0$ and $X_2 > X_1$ so that $G'(X_2) = 0$ (Fig.3). For the case $X_1 > 1$, we will give a proof that $c$ does not verify $c > e \cdot \text{rad}(abc) \cdot \log^2 \text{rad}(abc)$ and we will deduce that the case $X_1 > 1$ is to reject.

We verify easily that if $c \leq \text{rad}(abc)$, we obtain $c < e \cdot \text{rad}(abc) \cdot \log^2 \text{rad}(abc)$, then the contradiction and the ABC conjecture holds. We suppose in the following that $c > \text{rad}(abc)$.
If we consider the first example cited in section (9.3):

\[ 3^{10} \times 109 + 2 = 23^5 = 6436343 \]

\[ a = 3^{10}.109 \implies rad(a) = 3 \times 109, \]
\[ b = 2 \implies rad(b) = 2, \]
\[ c = 23^5 = 6436343 \implies rad(c) = 23. \] Then \( rad(abc) = 2 \times 3 \times 109 \times 23 = 15042 < c, \)
\[ Log(rad(abc)) = Log(15042) = 9.618601 \Rightarrow 2.718281 \times 15042 \times 92.517485 197201 = 245330.296812 < c. \] Then we have found \( a, b, c \) relatively coprime with \( c > rad(abc) \)
and \( c \) does not verify \( c > e.rad(abc).Log^2(rad(abc)) \). We deduce that the case 2.3 is to reject.
Finally, only the cases 1, 1.1, 2.1 and 2.2 are corrects and we obtain that the ABC conjecture is true. Then we obtain the important result of the paper:

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \quad \forall \epsilon > 0 \]

(217)

with the constant \( K(\epsilon) = \frac{2}{\epsilon^2} \)

Q.E.D

9.3. Examples

In this section, we are going to verify some numerical examples.

9.3.1. Example of Eric Reyssat

We give here the example of Eric Reyssat [1], it is given by:

(218) \[ 3^{10} \times 109 + 2 = 23^5 = 6436343 \]

\( a = 3^{10} \cdot 109 \Rightarrow \mu_a = 3^9 = 19683 \) and \( \text{rad}(a) = 3 \times 109 \),
\( b = 2 \Rightarrow \mu_b = 1 \) and \( \text{rad}(b) = 2 \),
\( c = 23^5 = 6436343 \Rightarrow \text{rad}(c) = 23 \). Then \( \text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042 \). For example, we take \( \epsilon = 0.01 \), the expression of \( K(\epsilon) \) becomes:

(219) \[ K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} \]

Let us verify (217):

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies c = 6436343 < 2.10^4 \cdot (3 \times 109 \times 2 \times 23)^{1.01} \implies \]

(220) \[ 6436343 < 331213962.07 \]

Hence (217) is verified.

9.3.2. Example of A. Nitaj

9.3.2.1. Case 1

The example of Nitaj about the ABC conjecture [3] is:

(221) \[ a = 11^{16} \cdot 13^2 \cdot 79 = 613474843408551921511 \Rightarrow \text{rad}(a) = 11.13.79 \]

(222) \[ b = 7^2 \cdot 41^2 \cdot 311^3 = 2477678547239 \Rightarrow \text{rad}(b) = 7.41.311 \]

(223) \[ c = 2.3^3 \cdot 5^{23} \cdot 953 = 613474845886230468750 \Rightarrow \text{rad}(c) = 2.3.5.953 \]

(224) \[ \text{rad}(abc) = 2.3.5.7.11.13.41.79.311.953 = 28828335646110 \]
we take $\epsilon = 100$ we have:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613\,474\,845\,886\,230\,468\,750 < 2.10^{-4}.(2.3.5.7.11.13.41.79.311.953)^{101} \implies$$

(225) \hspace{1cm} 613\,474\,845\,886\,230\,468\,750 < 5.53103686332861264803638e + 1355

then (217) is verified.

### 9.3.2.2. Case 2

We take $\epsilon = 0.000001 = 10^{-6}$, then:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613\,474\,845\,886\,230\,468\,750 < 2.10^{12}.(2.3.5.7.11.13.41.79.311.953)^{1.000001} \implies$$

613\,474\,845\,886\,230\,468\,750 < 57\,658\,458\,237\,370\,924\,700\,998\,757\,174\,980.

We obtain that (217) is verified.

### 9.3.2.3. Case 3

We take $\epsilon = 1$, then

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies 613\,474\,845\,886\,230\,468\,750 < 2.(2.3.5.7.11.13.41.79.311.953)^2 \implies$$

(227) \hspace{1cm} 613\,474\,845\,886\,230\,468\,750 < 1.662\,145\,872\,249\,552\,942\,316\,264\,200.

### 9.3.3. Example of Ralf Bonse

#### 9.3.3.1. Case 1

The example of Ralf Bonse about the ABC conjecture [4] is:

(228) \hspace{1cm} 2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983

\hspace{1cm} a = 2543^4.182587.2802983.85813163

\hspace{1cm} b = 2^{15}.3^{77}.11.173

\hspace{1cm} c = 5^{56}.245983

\hspace{1cm} \text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = \frac{2}{\epsilon^2} = \frac{2}{10^{-4}} = 2.10^4$$
Let us verify (217):

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies \]

\[ c = 5^{56} \cdot 245983 < 2.10^4 \cdot (2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.01} \]

\[ \implies 3,4136998783296235160378273576498e + 44 < \]

\[ 6,73659248844028778930766676768e + 37 \]

Ouf, end of the mystery!

9.4. Conclusion

This is an elementary proof of the ABC conjecture, confirmed by four numerical examples. We can announce the important theorem:

**Theorem 13.** — (David Masser, Joseph Oesterlé & Abdelmajid Ben Hadj Salem; 2019) For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

where \( K \) is a constant depending of \( \epsilon \) equal to \( \frac{2}{\epsilon^2} \).
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CHAPTER 10

THE FINAL PROOF OF THE $ABC$ CONJECTURE

10.1. Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i^{\alpha_i}$ noted by $\text{rad}(a)$. Then $a$ is written as:

\begin{equation}
\tag{231}
a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i - 1}
\end{equation}

We note:

\begin{equation}
\tag{232}
\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot \text{rad}(a)
\end{equation}

The $abc$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Esterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $abc$ conjecture is given below:

**Conjecture 11.** (abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\begin{equation}
\tag{233}
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\end{equation}

$K$ is a constant depending only of $\epsilon$.

The idea to try to write a paper about this conjecture was born after after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. I try here to give a simple proof that can be understood by undergraduate students.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. It is the key to resolve the $abc$ conjecture. The paper is organized as fellow: in the second section, we give for the case $c = a + 1$ the proof that $c < \text{rad}^2(ac)$. For the case $c = a + b$, the proof of $c < \text{rad}^2(abc)$ is given in
the third section. The main proof of the abc conjecture is presented in section four. The numerical examples are discussed in section five.

10.2. The Proof of the conjecture $c < \text{rad}^2(abc)$, Case: $c = a + 1$

Below is given the definition of the conjecture $c < \text{rad}^2(abc)$:

**Conjecture 12.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2$$

In the case $c = a + 1$, the definition of the conjecture is:

**Definition 10.1.** — Let $a, c$ positive integers, relatively prime, with $c = a + 1$, then:

$$c < \text{rad}^2(ac) \implies \frac{\log c}{\log(\text{rad}(ac))} < 2$$

1 - If $c < \text{rad}(ac)$ then we obtain:

$$c < \text{rad}(ac) < \text{rad}^2(ac)$$

and the condition (235) is verified.

2 - If $c = \text{rad}(ac)$, then $a, c$ are not relatively coprime. Case to reject.

3 - In the following, we suppose that $c > \text{rad}(ac) \implies c = \text{rad}(ac) + \alpha$, with $\alpha \geq 1$ an integer. We write:

$$\text{rad}^2(ac) - c = (c - \alpha)^2 - c\mu_a \text{rad}(a) + \frac{\alpha^2}{\text{rad}^2(a)} \text{rad}(a) - 1$$

$$\text{rad}(ac) \implies 0 < \text{rad}^2(a) - \frac{\mu_a}{\text{rad}^2(c)} \text{rad}(a) - \frac{1}{\text{rad}^2(c)}$$

Let the function $P(X) = X^2 - \frac{\mu_a}{\text{rad}^2(c)} X - \frac{1}{\text{rad}^2(c)}$. The discriminant of $P(X)$ is:

$$\Delta = \frac{\mu_a^2}{\text{rad}^4(c)} + \frac{4}{\text{rad}^2(c)} = \frac{\mu_a^2}{\text{rad}^4(c)} \left(1 + \frac{4\text{rad}^2(c)}{\mu_a^2}\right) > 0$$

The roots $X_1 < X_2$ of $P(X) = 0$ are given by:

$$X_1 = \frac{\mu_a}{2\text{rad}^2(c)} \left(1 - \sqrt{1 + \frac{4\text{rad}^2(c)}{\mu_a^2}}\right) < 0$$

$$X_2 = \frac{\mu_a}{2\text{rad}^2(c)} \left(1 + \sqrt{1 + \frac{4\text{rad}^2(c)}{\mu_a^2}}\right) > 0$$
c verifies (237) \( \implies P(rad(a)) > 0 \implies rad(a) > X_2 \), we obtain:
\[
rad(a) \geq \frac{\mu_a}{2rad^2(c)} \left( 1 + \sqrt{1 + \frac{4rad^2(c)}{\mu_a^2}} \right) \implies
\]
(240)
\[
2rad^2(a)rad^2(c) \geq a \left( 1 + \sqrt{1 + \frac{4rad^2(c)}{\mu_a^2}} \right)
\]

We denote \( R = rad(ac) \). We can write:
(241)
\[
2R^2 - a \geq a \sqrt{1 + \frac{4R^2}{a^2}}
\]

On suppose that \( 2R^2 - a > 0 \). We obtain:
(242)
\[
(2R^2 - a)^2 \geq a^2 \left( 1 + \frac{4R^2}{a^2} \right) \implies
\]
As \( R^2 > c \implies R^2 > a \implies 2R^2 > 2a \implies 2R^2 - a > a > 0 \) and the condition above \( 2R^2 - a > 0 \) is justified.

We announce the theorem:

**Theorem 14.** — Let \( a, c \) positive integers relatively prime with \( c = a + 1, a \geq 2 \), then \( c < rad^2(ac) \).

### 10.3. The Proof of the conjecture \( c < rad^2(abc) \), Case: \( c = a + b \)

We denote \( R = rad(abc) \). Let \( a, b, c \) positive integers, relatively prime, with \( c = a + b \implies rad(a) = \frac{c - b}{\mu_a} \). Let us verify that :
(243)
\[
c \geq \frac{c - b}{\mu_a} \cdot rad^2(bc)
\]

We obtain :
(244)
\[
0 \geq c^2 - \left( 2b + \frac{\mu_a^2}{rad^2(bc)} \right) c + b^2
\]

Let \( P(X) = X^2 - \left( 2b + \frac{\mu_a^2}{rad^2(bc)} \right) X + b^2 \). The discriminant \( \Delta \) of \( P(X) \) is:
(245)
\[
\Delta = \frac{\mu_a^4}{rad^4(bc)} \left( 1 + \frac{4rad^2(bc)}{\mu_a^2} \right) > 0
\]
The roots $X_1 < X_2$ of $P(X)$ are given by:

$$X_1 = \frac{1}{2} \left[ 2b + \frac{\mu_a^2}{\text{rad}^2(bc)} - \frac{\mu_a^2}{\text{rad}^2(bc)} \sqrt{1 + \frac{4b \cdot \text{rad}^2(bc)}{\mu_a^2}} \right]$$

$$X_2 = \frac{1}{2} \left[ 2b + \frac{\mu_a^2}{\text{rad}^2(bc)} + \frac{\mu_a^2}{\text{rad}^2(bc)} \sqrt{1 + \frac{4b \cdot \text{rad}^2(bc)}{\mu_a^2}} \right]$$

If $c$ will verify $(244) \implies c > X_2$ or $c < X_1$, we obtain:

$$c < X_1 \implies c < \frac{1}{2} \left[ 2b + \frac{\mu_a^2}{\text{rad}^2(bc)} - \frac{\mu_a^2}{\text{rad}^2(bc)} \sqrt{1 + \frac{4b \cdot \text{rad}^2(bc)}{\mu_a^2}} \right] \implies$$

$$2a < \frac{\mu_a^2}{\text{rad}^2(bc)} - \frac{\mu_a^2}{\text{rad}^2(bc)} \sqrt{1 + \frac{4b \cdot \text{rad}^2(bc)}{\mu_a^2}}$$

As the right term of the above inequality is $< 0$ and $2a > 0$, then $c < X_1$ is to reject.

Now, we will see if $c > X_2$, we obtain:

$$c > X_2 \implies c > \frac{1}{2} \left[ 2b + \frac{\mu_a^2}{\text{rad}^2(bc)} + \frac{\mu_a^2}{\text{rad}^2(bc)} \sqrt{1 + \frac{4b \cdot \text{rad}^2(bc)}{\mu_a^2}} \right] \implies$$

$$2a > \frac{\mu_a^2}{\text{rad}^2(bc)} + \frac{\mu_a^2}{\text{rad}^2(bc)} \sqrt{1 + \frac{4b \cdot \text{rad}^2(bc)}{\mu_a^2}}$$

$$2a \cdot \text{rad}^2(abc) > a^2 \left( 1 + \sqrt{1 + 4b \cdot \text{rad}^2(bc) \frac{\mu_a^2}{\mu_a^2}} \right) \implies \frac{2a \cdot \text{rad}^2(abc)}{a} - 1 > \sqrt{1 + 4b \cdot \text{rad}^2(bc)}$$

We suppose that $\frac{2a \cdot \text{rad}^2(abc)}{a} - 1 > 0$, we obtain:

$$\left( \frac{2a \cdot \text{rad}^2(abc)}{a} - 1 \right)^2 > 1 + 4b \cdot \frac{\text{rad}^2(abc)}{a^2} \implies$$

$$\text{rad}^2(abc) - a > b \implies \text{rad}^2(abc) > a + b \implies \text{rad}^2(abc) > c$$

As $\text{rad}^2(abc) > c \implies \frac{2a \cdot \text{rad}^2(abc)}{a} - 1 > 0$, then the proof of $c < \text{rad}^2(abc)$ is justified.

We can announce the theorem:

**Theorem 15.** — Let $a, b, c$ positive integers relatively prime with $c = a + b, b \geq 2$, then $c < \text{rad}^2(abc)$. 


10.4. The Proof of the abc conjecture (11)

10.4.1. Case: $\epsilon \geq 1$

Using the result of the theorem $c < rad^2(abc)$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e\left(\frac{1}{\epsilon^2}\right), \quad \epsilon \geq 1$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$. Then the abc conjecture is true.

10.4.2. Case: $\epsilon < 1$

10.4.2.1. Case: $c < R$

In this case, we can write:

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e\left(\frac{1}{\epsilon^2}\right), \quad \epsilon < 1$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the ABC conjecture is true.

10.4.2.2. Case: $c > R$

In this case, we confirm that:

$$c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e\left(\frac{1}{\epsilon^2}\right), \quad 0 < \epsilon < 1$$

If not, then $\exists \epsilon_0 \in ]0, 1[$, so that the triplets $(a, b, c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0)$$

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow$$

$$R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > c.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0)$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > K(\epsilon_0)\left(\frac{1}{1-\epsilon_0}\right)$$

We deduce that it exists an infinity of triples $(a, b, c)$ verifying (394), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0, c = a + b$ with $a, b, c$ relatively coprime, $b \geq 2$:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad \text{with} \quad K(\epsilon) = e\left(\frac{1}{\epsilon^2}\right)$$
10.5. Examples

In this section, we are going to verify some numerical examples.

10.5.1. Example 1

The example is given by:

\[(256)\]

\[1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6\]

\[a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42 \text{ and } rad(a) = 2 \times 3 \times 5 \times 7 \times 127,\]

\[b = 1 \Rightarrow \mu_b = 1 \text{ and } rad(b) = 1,\]

\[c = 19^6 = 47045880 \Rightarrow rad(c) = 19. \text{ Then } rad(abc) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730.\]

We have \(c > rad(ac)\) but \(rad^2(ac) = 506730^2 = 256775292900 > c = 47045880.\)

10.5.1.1. Case \(\epsilon = 0.01\)

\[c < K(\epsilon).rad(ac)^{1+\epsilon} \Rightarrow 47045880 < e^{10000.506730^{1.01}}.\]

The expression of \(K(\epsilon)\) becomes:

\[(257) \quad K(\epsilon) = e^{\frac{1}{10000}} = e^{10000} = 8,7477777149120059152473488653e + 4342\]

We deduce that \(c < K(0.01).506730^{1.01}\) and the equation (255) is verified.

10.5.1.2. Case \(\epsilon = 0.1\)

\[K(0.1) = e^{\frac{1}{100}} = e^{100} = 2.6879363309671754205949120152473488653e + 43 \Rightarrow c < K(0.1) \times 506730^{1.01}.\]

And the equation (255) is verified.

10.5.1.3. Case \(\epsilon = 1\)

\[K(1) = e \Rightarrow c = 47045880 < e.rad^2(ac) = 697987143184,212, \text{ and the equation (255) is verified.}\]

10.5.1.4. Case \(\epsilon = 100\)

\[K(100) = e^{0.0001} \Rightarrow c = 47045880 < e^{0.0001.506730^{101}} = 1,5222350248607608781853142687284e + 576\]

and the equation (255) is verified.
10.5.2. Example 2

We give here the example of Eric Reyssat [1], it is given by:

\(3^{10} \times 109 + 2 = 23^5 = 6436343\)

\(a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683\) and \(rad(a) = 3 \times 109\),

\(b = 2 \Rightarrow \mu_b = 1\) and \(rad(b) = 2\),

\(c = 23^5 = 6436343 \Rightarrow rad(c) = 23\). Then \(rad(abc) = 2 \times 3 \times 109 \times 23 = 15042\). For example, we take \(\epsilon = 0.01\), the expression of \(K(\epsilon)\) becomes:

\[K(\epsilon) = e^{9999.99} = 8,7477777149120053120152473488653e + 4342\]

Let us verify (255):

\[c \lesssim K(\epsilon).rad(abc)^{1+\epsilon} \implies c = 6436343 \lesssim K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \implies\]

\[6436343 \ll K(0.01) \times 15042^{1.01}\]

Hence (255) is verified.

10.5.3. Example 3

The example of Nitaj about the ABC conjecture [1] is:

\(a = 11^{16}.13^2.79 = 613474843408551921511 \Rightarrow rad(a) = 11.13.79\)

\(b = 7^2.41^2.311^3 = 2477678547239 \Rightarrow rad(b) = 7.41.311\)

\(c = 2.3^3.5^{23}.953 = 6134741845886230468750 \Rightarrow rad(c) = 2.3.5.953\)

\(rad(abc) = 2.3.5.7.11.13.41.79.431.19.953 = 2882335646110\)

10.5.3.1. Case 1

we take \(\epsilon = 100\) we have:

\[c \lesssim K(\epsilon).rad(abc)^{1+\epsilon} \implies\]

\[613474845886230468750 < e^{0.0001}(2.3.5.7.11.13.41.79.431.19.953)^{101} \implies\]

\[613474845886230468750 < 2,7657949971494838920022381186039e + 1359\]

then (255) is verified.

10.5.3.2. Case 2

We take \(\epsilon = 0.5\), then:

\[c \lesssim K(\epsilon).rad(abc)^{1+\epsilon} \implies\]

\[613474845886230468750 < e^{4}(2.3.5.7.11.13.41.79.431.19.953)^{1.5} \implies\]

\[613474845886230468750 < 8,450,961,319,227,998,887,403,9993\]

We obtain that (255) is verified.
10.5.3.3. Case 3
We take $\epsilon = 1$, then
\[ c^2 K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies 613474845886230468750 < (2.3.5.7.11.13.41.79.311.953)^2 \implies (267) \]
\[ 613474845886230468750 < 831072936124776471158132100 \]
We obtain that (255) is verified.

10.5.4. Example 4
It is of Ralf Bonse about the ABC conjecture [3] :
(268) \[ 2543^4 \cdot 182587 \cdot 2802983 \cdot 85813163 + 2^{15} \cdot 3^{77} \cdot 11 \cdot 173 = 5^{56} \cdot 245983 \]
\[ a = 2543^4 \cdot 182587 \cdot 2802983 \cdot 85813163 \]
\[ b = 2^{15} \cdot 3^{77} \cdot 11 \cdot 173 \]
\[ c = 5^{56} \cdot 245983 \]
\[ \text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163 \]
(269) \[ \text{rad}(abc) = 1.5683959920004546031461002610848e + 33 \]

10.5.4.1. Case 1
For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:
\[ K(\epsilon) = e^{0.01} = 1,0078157404282956743204617416779 \]
Let us verify (255):
\[ c^2 < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies c = 5^{56} \cdot 245983 < e^{0.01} \cdot (2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11} \]
\[ \implies 3.413699878329625160378273576498e + 44 < \]
(270) \[ 1,423620059644908176008120925721e + 365 \]
The equation (255) is verified.

10.5.4.2. Case 2
We take $\epsilon = 0.4 \implies K(\epsilon) = 12,18247347425151215912625669608$, then: The
\[ c^2 < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies c = 5^{56} \cdot 245983 < e^{6.25} \cdot (2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \]
\[ \implies 3.413699878329625160378273576498e + 44 < \]
(271) \[ 3,6255465680011453642792720569685e + 47 \]
And the equation (255) is verified.

Ouf, end of the mystery!

10.6. Conclusion

We have given an elementary proof of the $abc$ conjecture in the two cases $c = a' + 1$ and $c = a + b$, confirmed by some numerical examples. We can announce the important theorem:

**Theorem 16.** — (David Masser, Joseph Oesterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[
(272) \quad c < K(\epsilon).\rad(abc)^{1+\epsilon}
\]

where $K$ is a constant depending of $\epsilon$ proposed equal to $e\left(\frac{1}{\epsilon^2}\right)$.

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CHAPTER 11

TENTATIVE OF THE PROOF OF $ABC$ CONJ.-CASE $c = a + 1$

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

11.1. Introduction and notations

Let $a$ a positive integer, $a = \prod_{i} a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_{i} a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

\[(273) \quad a = \prod_{i} a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_{i} a_i^{\alpha_i - 1} \]

We note:

\[(274) \quad \mu_a = \prod_{i} a_i^{\alpha_i - 1} \implies a = \mu_a \cdot \text{rad}(a) \]

The $abc$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $abc$ conjecture is given below:

Conjecture 13. — (abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then :

\[(275) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

$K$ is a constant depending only of $\epsilon$. We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751$ ([2]). A conjecture was proposed that $c < \text{rad}^2(abc)$ ([3]). Here we will give a proof of it in the case $c = a + 1$. 
**Conjecture 14.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[(276)\]  
$c < \text{rad}^2(abc) \implies \frac{\log c}{\log (\text{rad}(abc))} < 2$

This result, I think is the key to obtain a proof of the veracity of the $abc$ conjecture.

**11.2. A Proof of the conjecture (14) Case $c = a + 1$**

Let $a, c$ positive integers, relatively prime, with $c = a + 1$. If $c < \text{rad}(ac)$ then we obtain:

\[(277)\]  
$c < \text{rad}(ac) < \text{rad}^2(ac)$

and the condition (276) is verified.

In the following, we suppose that $c \geq \text{rad}(ac)$.

**11.2.1. Notations**

We note:

\[(278)\]  
$a = \prod_i a_i^{\alpha_i} \implies \text{rad}(a) = \prod_i a_i, \mu_a = \prod_i a_i^{\alpha_i - 1}, i = 1, N_a$

\[(279)\]  
$c = \prod_j c_j^{\beta_j} \implies \text{rad}(c) = \prod_j c_j, \mu_c = \prod_j c_j^{\beta_j - 1}, j = 1, N_c$

with $a_i, c_j$ prime integers and $N_a, N_c, \alpha, \beta \geq 1$ positive integers. Let:

\[(280)\]  
$R = \text{rad}(a).\text{rad}(c) = \text{rad}(ac)$

\[(281)\]  
$\mathcal{R}(x) = \prod_i(x + a_i)^{N_a} \prod_j(x + c_j)^{N_c} \implies \mathcal{R}(x) > 0, \forall x \geq 0$

\[(282)\]  
$F(x) = \mathcal{R}(x) - \mu_c$

From the last equations we obtain:

\[(283)\]  
$F(0) = \mathcal{R}(0) - \mu_c = \text{rad}^2(a).\text{rad}(c) - \mu_c$

Then, our main task is to prove that $F(0) > 0 \implies R^2 > c$.

**11.2.1.1. The Proof of $c < \text{rad}^2(ac)$**

From the definition of the polynomial $F(x)$, its degree is $2N_a + N_c$. We have :

1. $\lim_{x \to +\infty} F(x) = +\infty$,
2. $\lim_{x \to +\infty} \frac{F(x)}{x} = +\infty$, $F$ is convex for $x$ large,
3. if $x_1$ is the great real root of $F(x) = 0$, and from the points 1., 2. we deduce that $F''(x_1^+) > 0$,
4. if $x_1 < 0$, then $F(0) > 0$.
Let us study $F'(x)$ and $F''(x)$. We obtain:

$$F'(x) = \mathcal{R}'(x)$$

$$\mathcal{R}'(x) = \left[ \prod_i^{N_a} (x + a_i)^2 \right]' \cdot \prod_j^{N_c} (x + c_j) + \prod_i^{N_a} (x + a_i)^2 \cdot \left[ \prod_j^{N_c} (x + c_j) \right]' \Rightarrow$$

$$\left[ \prod_i^{N_a} (x + a_i)^2 \right]' = 2 \prod_i^{N_a} (x + a_i)^2 \cdot \left( \sum_i \frac{1}{x + a_i} \right)$$

$$\left[ \prod_j^{N_c} (x + c_j) \right]' = \prod_j^{N_c} (x + c_j) \cdot \left( \sum_{j=1}^{N_c} \frac{1}{x + c_j} \right) \Rightarrow$$

$$\mathcal{R}'(x) = \mathcal{R}(x). \left( \sum_i^{N_a} \frac{2}{x + a_i} + \sum_j^{N_c} \frac{1}{x + c_j} \right) > 0, \forall x \geq 0 \quad (284)$$

$$F'(x) = \mathcal{R}' = \mathcal{R}(x) \left( \sum_i^{N_a} \frac{2}{x + a_i} + \sum_j^{N_c} \frac{1}{x + c_j} \right) > 0, \forall x > 0 \Rightarrow$$

$$F'(0) = \mathcal{R}(0). \left( \sum_i^{N_a} \frac{2}{a_i} + \sum_j^{N_c} \frac{1}{c_j} \right) = rad^2(a).rad(c). \left( \sum_i^{N_a} \frac{2}{a_i} + \sum_j^{N_c} \frac{1}{c_j} \right) > 0$$

For $F''(x)$, we obtain:

$$F''(x) = \mathcal{R}'' =$$

$$\mathcal{R}'(x) \left( \sum_i^{N_a} \frac{2}{x + a_i} + \sum_j^{N_c} \frac{1}{x + c_j} \right) - \mathcal{R}(x) \left( \sum_i^{N_a} \frac{2}{(x + a_i)^2} + \sum_j^{N_c} \frac{1}{(x + c_j)^2} \right) \Rightarrow$$

$$F''(x) = \mathcal{R}(x). \left[ \left( \sum_i^{N_a} \frac{2}{x + a_i} + \sum_j^{N_c} \frac{1}{x + c_j} \right)^2 - \sum_i^{N_a} \frac{2}{(x + a_i)^2} - \sum_j^{N_c} \frac{1}{(x + c_j)^2} \right]$$

$$\Rightarrow F''(x) > 0, \forall x \geq 0 \quad (285)$$

We obtain also that $F''(0) > 0$.

Before we attack the proof, we take an example as: $1 + 8 = 9 \Rightarrow c = 9, a = 8, b = 1$. We obtain $rad(a) = 2, rad(c) = 3, \mu_c = 3, R = rad(ac) = 2 \times 3 = 6 < (c = 9)$ and $c = 9$ verifies $c < (R^2 = 6^2 = 36)$. We write the polynomial $F(x) = (x + 2)^2(x + 3) - 3 = x^3 + 7x^2 + 16x + 9 > 0, \forall x > 0$. Then $F'(x) = 3x^2 + 14x + 16$, we verifies that $F'(x) = 0$ has not real roots and $F'(x) > 0, \forall x \in \mathbb{R}$. We have also $F''(x) = 6x + 14$. $F''(x) = 0 \Rightarrow x = -7/3 \approx -2.33 \Rightarrow F(-7/3) = -79/27 \approx -2.92$. The point $(-7/3, -79/27)$ is an inflexion point of the curve of $y = F(x)$. We deduce that the curve is convex for $x \geq -7/3$. Let us now find the roots of $F(x) = 0$. As the degree of $F$ is three, the number of the real roots are 1 or 3. As there is one inflexion point, we will find one real root.
11.2.2. The Resolution of $F(x) = 0$

We want to resolve:

(286) $F(x) = x^3 + 7x^2 + 16x + 9 = 0$

Let the change of variables $x = t - 7/3$, the equation (286) becomes:

(287) $t^3 - \frac{t}{3} - \frac{79}{27} = 0$

For the resolution of (287), we introduce two unknowns:

$t = u + v \implies (u + v)(3uv - \frac{1}{3}) + u^3 + v^3 - \frac{79}{27} = 0 \implies$

\[
\begin{align*}
  u^3 + v^3 &= \frac{79}{3^3} \\
  uv &= \frac{1}{3^2}
\end{align*}
\]

Then $u^3, v^3$ are solutions of the equation:

(289) $X^2 - \frac{79}{3^3}X + \frac{1}{3^6} = 0$

and given below:

\[
\begin{align*}
  u^3 = \frac{1}{2} \cdot \frac{79 + 9\sqrt{77}}{3^3} &\implies \begin{cases} 
    u_1 = \frac{3}{2} \left( \frac{79 + 9\sqrt{77}}{3^3} \right) \approx 0.97515 \\
    u_2 = j.u_1, \quad j = \frac{-1 + i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \\
    u_3 = j^2u_1 = j.u_1
  \end{cases} \\
  v^3 = \frac{1}{2} \cdot \frac{79 - 9\sqrt{77}}{3^3} &\implies \begin{cases} 
    v_1 = \frac{3}{2} \left( \frac{79 - 9\sqrt{77}}{3^3} \right) \approx 0.00016 \\
    v_2 = j^2v_1 = j.v_1 \\
    v_3 = j.v_1
  \end{cases}
\end{align*}
\]

(290)

Finally, taking into account the second condition of (288), we obtain the real root of (287):

\[
\begin{align*}
  t &= u_1 + v_1 = \frac{3}{2} \left( \frac{79 + 9\sqrt{77}}{3^3} \right) + \frac{3}{2} \left( \frac{79 - 9\sqrt{77}}{3^3} \right) \approx 0.97531 \\
  x_1 &= t - 7/3 \approx -1.35802
\end{align*}
\]

Then the first root of $F(x) = 0$ is $x_1 \approx -1.358 < 0$, the correction to the first root of $R(x) = (x+2)^2(x+3) = 0$ is $dx = x_1 - (-2) = -1.358 - (-2) = +0.642$. As in our example $F'(x) > 0$, the function $F(x)$ is an increasing function having a parabolic
11.2. A PROOF OF THE CONJECTURE (??) CASE C = A + 1

branch as \( x \to +\infty \), the curve \( y = F(x) \) intersects the line \( x = 0 \) in the half-plane \( y \geq 0 \) \( \implies F(0) > 0 \implies c < rad^2(ac) \) which is verified numerically.

11.2.3. The General Case

Let us return to the general case \( c = a + 1 \). We denote \( q = \min(a_i, c_j) \). If we consider that \( F(x) = R(x) \), the equation \( F(x) = 0 \implies R(x) = 0 \) and the first real root is \( x_1 = -q \), the product of all the roots is \( P = \prod_{i}^{N}(x_i)^2 \prod_{j}^{N}(x_j) = (-1)^{2N_a+N_c} \prod_{i}^{N}(a_i)^2 \prod_{j}^{N}(c_j) \). But \( F(x) = R(x) - \mu_c \), the constant coefficient of \( F(x) \) will be \( \prod_{i}^{N}(a_i)^2 \prod_{j}^{N}(c_j) - \mu_c \). The new product of the roots is \( P' = \prod_{i}^{N}(x_i')^2 \prod_{j}^{N}(x_j') = (-1)^{2N_a+N_c} \prod_{i}^{N}(a_i)^2 \prod_{j}^{N}(c_j) - \mu_c \). The first root \( x_1 = -q \) becomes \( x_1 = -q + dx \).

To estimate \( dx \), we can write to the order two that:

\[
F(-q + dx) = R(-q + dx) - \mu_c = 0 \implies R(-q + dx) = \mu_c \implies
\]

(292)

\[
R(-q) + dx R'( -q ) + \frac{dx^2}{2} R''( -q ) = \mu_c
\]

Supposing that \( a_1 = q = \min(a_i, c_j) \), from the equations \((281-284-285)\), we have:

\[
\begin{align*}
R(-a_1) &= 0 \\
R'(-a_1) &= 0 \\
R''(-a_1) &= 2 \prod_{i=2}^{N_a}(a_i - a_1)^2 \prod_{j=1}^{N_c}(c_j - a_1) > 0 \implies
\end{align*}
\]

(293)

\[
dx^2 = \frac{\mu_c}{\prod_{i=2}^{N_a}(a_i - a_1)^2 \prod_{j=1}^{N_c}(c_j - a_1)}
\]

We take the positive value of \( dx \) following the numerical example above, then we obtain:

(294)

\[
dx = \frac{\sqrt{\mu_c}}{\prod_{i=2}^{N_a}(a_i - a_1) \cdot \prod_{j=1}^{N_c}(c_j - a_1)}
\]

As \( a_1 = \min(a_i, c_j)_{i=2, N_a; j=1, N_c} \), we have \( a_2 - a_1 \geq 1, a_3 - a_1 \geq 2, \cdots, a_{N_a} - a_1 \geq N_a - 1, c_1 - a_1 \geq 1, c_2 - a_1 \geq 2, \cdots, c_{N_c} - a_1 \geq N_c \), then we can write:

(295)

\[
dx \leq \frac{\sqrt{\mu_c}}{(N_a - 1)! \cdot \sqrt{N_c}!}
\]

For the expression of \( N! \), we can use the Stirling’s formula to two orders:

(296)

\[
N! = \sqrt{2\pi N} \left( \frac{N}{e} \right)^N \left(1 + \frac{1}{12N}\right)
\]
and we take \( c = a + 1 < 3a/2, \mu_c = \frac{c}{\text{rad}(c)} < \frac{3a}{2a_1^2} \). We obtain:

\[
(297) \quad dx < \left(1 - \frac{1}{12(N_a - 1)}\right) \left(1 - \frac{1}{24N_c}\right) e^{-\frac{2(N_a-1)\log N_a - (N_c-1)\log N_c}{4\pi(N_a - 1)}} \sqrt{\frac{3a}{2\sqrt{2\pi N_c a_1^2}}}
\]

As \( a, c \) are large positive integers, and \( 1 \ll N_a, N_c, |dx| \ll a_i, c_j \) and the sign of the first root \( x'_1 \) does not change. Then the curve of \( F(x) = \mathcal{R}(x) - \mu_c \) intersects the line \( x = 0 \) and the equation of the tangent at the point \( x'_1 \) is \( y = F'(x'_1)(x - x'_1) \). But if \( x'_1 = -a + dx = \xi > 0 \), the equation of the tangent at the point \( \xi \) is \( y = F'(\xi)(x - \xi) \), in this case \( F'(\xi) < F'(x'_1) \) (see Fig. 1), and for \( x > -a_1, F''(x) > 0 \implies F'' \) is an increasing function. As \( x'_1 < \xi \implies F'(x'_1) < F''(\xi) \), then the contradiction and we obtain that \( x'_1 = -a_1 + dx < 0 \implies F(0) = \text{rad}^2(a)\text{rad}(c) - \mu_c > 0 \implies \text{rad}^2(a)\text{rad}(c) - c > 0 \implies c < R^2 \).

11.2.3.1. Examples

In this section, we are going to verify the above remarks with a numerical example. The example is given by:

\[
1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6
\]

\[
\text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127 = 26670
\]

\[
\text{rad}(c) = 19
\]

(298) \( c = 19^5 = 47045881, \quad \mu_c = 19^5 = 2476099 \)

Using the notations of the paper, we obtain:

\[
\mathcal{R}(x) = (x + 2)^2(x + 3)^2(x + 5)^2(x + 7)^2(x + 127)^2(x + 19)
\]

\[
F(x) = \mathcal{R}(x) - \mu_c
\]
Let \( X = x + 2 \), the expression of \( \mathcal{R}(x) \) becomes:
\[
\mathcal{R}(X) = X^2(X + 1)^2(X + 3)^2(X + 5)^2(X + 125)^2(X + 17)
\]
The calculations give:
\[
\]
We want to estimate the first root of \( F(x) = 0 \), we write:
\[
\mathcal{R}(x) - \mu_c = 0 \implies X^{11} + 285.X^{10} + 24,808.X^9 + 657,728.X^8 + 7,424,722.X^7 + 42,772,898.X^6 + 134,002,080.X^5 + 223,508,940.X^4 + 187,753,125.X^3 + 597,656,251.X^2 - 2476099 = 0
\]
If \( x = -2 \implies X = 0 \implies \mathcal{R}(X) - \mu_c < 0 \). If we take \( x_1 = -1.936315 \implies X_1 = 0.03685 \), then we obtain that:
\[
R(x_1) - \mu_c = \mathcal{R}(X_1) - \mu_c \approx 177.82 > 0
\]
Then, \( \xi \) with \(-2 < \xi < x_1 \) so that \( X' = -2 + \xi \) verifies \( \mathcal{R}(X') - \mu_c = 0 \) and \( \xi \) is the first root of \( F(x) = 0 \) and \( \xi < 0 \implies F(0) > 0 \implies \text{rad}^2(a)\text{rad}(c)-\mu_c > 0 \implies R^2 > c \) that is true. We have also \( \xi = -2 + dx = a_1 + dx \) and \( 0 < dx < |a_1| \).

### 11.3. The Proof of The abc conjecture (13) Case: \( c = a + 1 \)

We denote \( R = \text{rad}(ac) \).

#### 11.3.1. Case: \( \epsilon \geq 1 \)

Using the result of the theorem above, we have \( \forall \epsilon \geq 1: \)
\[
c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon \geq 1
\]
We verify easily that \( K(\epsilon) > 1 \) for \( \epsilon \geq 1 \) and it is a decreasing function from \( \epsilon \) the base of the neperian logarithm to 1.

#### 11.3.2. Case: \( \epsilon < 1 \)

#### 11.3.2.1. Case: \( c \leq R \)

In this case, we can write:
\[
c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon < 1
\]
here also \( K(\epsilon) > 1 \) for \( \epsilon < 1 \) and the abc conjecture is true.
11.3.2.2. Case: \( c > R \)

In this case, we confirm that:

\[
(303) \quad c < K(\epsilon)R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \; 0 < \epsilon < 1
\]

If not, then \( \exists \epsilon_0 \in ]0, 1[ \), so that the triplets \( (a, 1, c) \) checking \( c > R \) and:

\[
(304) \quad c \geq R^{1+\epsilon_0}K(\epsilon_0)
\]

are in finite number. We have:

\[
(305) \quad c \geq R^{1+\epsilon_0}K(\epsilon_0) \implies R^{1-\epsilon_0}c \geq R^{1-\epsilon_0}R^{1+\epsilon_0}K(\epsilon_0) \implies R^{1-\epsilon_0}R^2K(\epsilon_0) > cK(\epsilon_0) \implies R^{1-\epsilon_0} > K(\epsilon_0)
\]

As \( c > R \), we obtain:

\[
(306) \quad c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \implies c^{1-\epsilon_0} > K(\epsilon_0) \implies c > K(\epsilon_0)\left(\frac{1}{1-\epsilon_0}\right)
\]

We deduce that it exists an infinity of triplets \( (a, 1, c) \) verifying (304), hence the contradiction. Then the proof of the abc conjecture in the case \( c = a + 1 \) is finished.

We obtain that \( \forall \epsilon > 0, \; c = a + 1 \) with \( a, c \) relatively coprime, \( 2 \leq a < c \):

\[
(307) \quad c < K(\epsilon)\rad(ac)^{1+\epsilon} \quad \text{with} \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}
\]

Q.E.D

11.4. Examples

In this section, we are going to verify some cases of one numerical example. The example is given by:

\[
(308) \quad 1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6
\]

\( a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42 \) and \( \rad(a) = 2 \times 3 \times 5 \times 7 \times 127 \),

\( b = 1 \Rightarrow \mu_b = 1 \) and \( \rad(b) = 1 \),

\( c = 19^6 = 47045880 \Rightarrow \rad(c) = 19 \). Then \( \rad(abc) = \rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730 \).

We have \( c > \rad(ac) \) but \( \rad^2(ac) = 506730^2 = 256775292900 > c = 47045880 \).
11.4.0.1. Case $\epsilon = 0.01$

$c < K(\epsilon).\rad(ac)^{1+\epsilon} \implies 47,045,880 < e^{10000} \cdot 506,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

\[ K(\epsilon) = e^{0.0001} = 8,747,777,149,120,053,120,152,473,488,653,120,152,473,488,653,120,152,473,488,653e + 4342 \]

We deduce that $c \ll K(0.01) \cdot 506,730^{1.01}$ and the equation (??) is verified.

11.4.0.2. Case $\epsilon = 0.1$

$K(0.1) = e^{0.100} = 2,687,936,330,967,175,420,591,017,012,128,870,847 + 43 \implies c < K(0.1) \times 506,730^{1.01}$. And the equation (307) is verified.

11.4.0.3. Case $\epsilon = 1$

$K(1) = e \implies c = 47,045,880 < e \cdot \rad^2(ac) = 697,987,143,184,212$. and the equation (307) is verified.

11.4.0.4. Case $\epsilon = 100$

$K(100) = e^{0.0001} \implies c = 47,045,880 < e^{0.0001 \cdot 506,730^{1.01}} = 1,522,235,024,860,760,878,185,314,268,728 \cdot e + 576$

and the equation (307) is verified.

11.5. Conclusion

This is an elementary proof of the abc conjecture in the case $c = a + 1$. We can announce the important theorem:

**Theorem 17.** — (David Masser, Joseph (Esterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[ c < K(\epsilon).\rad(ac)^{1+\epsilon} \]

where $K$ is a constant depending of $\epsilon$ equal to $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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BIBLIOGRAPHY


CHAPTER 12

LAST PROOF OF ABC CONJECTURE - CASE

\[ c = a + 1 \]

In this paper, we consider the abc conjecture in the case \( c = a + 1 \). Firstly, we give the proof of the first conjecture that \( c < \text{rad}^2(ac) \) using the polynomial functions. It is the key to the proof of the abc conjecture. Secondly, the proof of the abc conjecture is given for \( \epsilon \geq 1 \), then for \( \epsilon \in ]0, 1[ \) for the two cases: \( c \leq \text{rad}(ac) \) and \( c > \text{rad}(ac) \).

We choose the constant \( K(\epsilon) \) as \( K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \). A numerical example is presented.

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Dr. Eng. Chedly Fezzani
(1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

12.1. Introduction and notations

Let \( a \) a positive integer, \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \), noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
(311) \quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i - 1}
\]

We note:

\[
(312) \quad \mu_a = \prod_i a_i^{\alpha_i - 1} \Rightarrow a = \mu_a \cdot \text{rad}(a)
\]

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:
Conjecture 15. — (abc Conjecture): For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:
\[
(313) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]
where \( K \) is a constant depending only on \( \epsilon \).

We know that numerically,
\[
\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751 \quad (2).
\]
A conjecture was proposed that \( c < \text{rad}^2(abc) \) \((3)\). Here we will give the proof of it, in the case \( c = a + 1 \), using a polynomial function.

Conjecture 16. — Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:
\[
(314) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]
This result, I think is the key to obtain a proof of the veracity of the abc conjecture.

12.2. A Proof of the conjecture (16) Case \( c = a + 1 \)

Let \( a, c \) positive integers, relatively prime, with \( c = a + 1 \). If \( c < \text{rad}(ac) \) then we obtain:
\[
(315) \quad c < \text{rad}(ac) < \text{rad}^2(ac)
\]
and the condition (314) is verified.

In the following, we suppose that \( c \geq \text{rad}(ac) \).

12.2.1. Notations

We note:
\[
(316) \quad a = \prod_i a_i^{\alpha_i} \implies \text{rad}(a) = \prod_i a_i, \mu_a = \prod_i a_i^{\alpha_i-1}, i = 1, N_a
\]
\[
(317) \quad c = \prod_j c_j^{\beta_j} \implies \text{rad}(c) = \prod_j c_j, \mu_c = \prod_j c_j^{\beta_j-1}, j = 1, N_c
\]
with \( a_i, c_j \) prime integers and \( N_a, N_c, \alpha, \beta \geq 1 \) positive integers. Let:
\[
(318) \quad R = \text{rad}(a).\text{rad}(c) = \text{rad}(ac)
\]
\[
(319) \quad R(x) = \prod_i (x + a_i)^2 \prod_j (x + c_j) \implies R(x) > 0, \forall x \geq 0
\]
\[
(320) \quad F(x) = R(x) - \mu_c
\]
From the last equations we obtain:
\[
(321) \quad F(0) = R(0) - \mu_c = \text{rad}^2(a).\text{rad}(c) - \mu_c
\]
Then, our main task is to prove that \( F(0) > 0 \implies R^2 > c. \)
12.2.1.1. The Proof of $c < \text{rad}^2(ac)$

From the definition of the polynomial $F(x)$, its degree is $2N_a + N_c$. We have:

1. $\lim_{x \to +\infty} F(x) = +\infty$,
2. $\lim_{x \to +\infty} \frac{F(x)}{x} = +\infty$, $F$ is convex for $x$ large,
3. if $x_1$ is the great real root of $F(x) = 0$, and from the points 1., 2. we deduce that $F''(x_1^+) > 0$,
4. if $x_1 < 0$, then $F(0) > 0$.

Let us study $F'(x)$ and $F''(x)$. We obtain:

\[
F'(x) = R'(x)
\]

\[
R'(x) = \left[\prod_{i}^{N_a}(x + a_i)^2\right] \cdot \prod_{j}^{N_c}(x + c_j) + \prod_{i}^{N_a}(x + a_i)^2 \cdot \left(\prod_{j}^{N_c}(x + c_j)\right)' \implies
\]

\[
\left[\prod_{i}^{N_a}(x + a_i)^2\right]' = 2 \prod_{i}^{N_a}(x + a_i)^2 \cdot \left(\sum_{i}^{N_a} \frac{1}{x + a_i}\right)
\]

\[
\left[\prod_{j}^{N_c}(x + c_j)\right]' = \prod_{j}^{N_c}(x + c_j) \cdot \left(\sum_{j=1}^{N_a} \frac{1}{x + c_j}\right) \implies
\]

(322) $R'(x) = R(x). \left(\sum_{i}^{N_a} \frac{2}{x + a_i} + \sum_{j}^{N_c} \frac{1}{x + c_j}\right) > 0, \forall x \geq 0$

\[
F'(x) = R' = R(x). \left(\sum_{i}^{N_a} \frac{2}{x + a_i} + \sum_{j}^{N_c} \frac{1}{x + c_j}\right) > 0, \forall x > 0 \implies
\]

\[
F'(0) = R(0). \left(\sum_{i}^{N_a} \frac{2}{a_i} + \sum_{j}^{N_c} \frac{1}{c_j}\right) =
\]

(323) $\text{rad}^2(a).\text{rad}(c). \left(\sum_{i}^{N_a} \frac{2}{a_i} + \sum_{j}^{N_c} \frac{1}{c_j}\right) > 0$

For $F''(x)$, we obtain:

\[
F''(x) = R'' = R'(x) \left(\sum_{i}^{N_a} \frac{2}{x + a_i} + \sum_{j}^{N_c} \frac{1}{x + c_j}\right) - R(x) \left(\sum_{i}^{N_a} \frac{2}{(x + a_i)^2} + \sum_{j}^{N_c} \frac{1}{(x + c_j)^2}\right)
\]

\[
\implies F''(x) = R(x). \left[\left(\sum_{i}^{N_a} \frac{2}{x + a_i} + \sum_{j}^{N_c} \frac{1}{x + c_j}\right)^2 - \sum_{i}^{N_a} \frac{2}{(x + a_i)^2} - \sum_{j}^{N_c} \frac{1}{(x + c_j)^2}\right]
\]

(324) $\implies F''(x) > 0, \forall x \geq 0$

We obtain also that $F''(0) > 0$. 


Before we attack the proof, we take an example as: \(1+8=9\Longrightarrow c=9, a=8, b=1\). We obtain \(\text{rad}(a)=2, \text{rad}(c)=3, \mu_c=3, R=\text{rad}(ac)=2 \times 3 = 6 < (c=9)\) and \(c=9\) verifies \((R^2 = 6^2 = 36)\). We write the polynomial \(F(x) = (x+2)^3(x+3)−3 = x^3+7x^2+16x+9 > 0, \forall x > 0\). Then \(F'(x) = 3x^2+14x+16\), we verify that \(F'(x) = 0\) has not real roots and \(F'(x) > 0, \forall x \in \mathbb{R}\). We have also \(F''(x) = 6x+14\). \(F''(x) = 0 \Longleftrightarrow x = -7/3 \approx -2.33 \Longrightarrow F(-7/3) = -79/27 \approx -2.92\). The point \((-7/3, -79/27)\) is an inflection point of the curve of \(y=F(x)\). We deduce that the curve is convex for \(x \geq -7/3\). Let us now find the roots of \(F(x) = 0\). As the degree of \(F\) is three, the number of the real roots are 1 or 3. As there is one inflection point, we will find one real root.

### 12.2.2. The Resolution of \(F(x) = 0\)

We want to resolve:

\[
F(x) = x^3 + 7x^2 + 16x + 9 = 0 \tag{325}
\]

Let the change of variables \(x = t - 7/3\), the equation (325) becomes:

\[
t^3 - \frac{t}{3} - \frac{79}{27} = 0 \tag{326}
\]

For the resolution of (326), we introduce two unknowns:

\[
t = u + v \Longrightarrow (u + v)(3uv - \frac{1}{3}) + u^3 + v^3 - \frac{79}{27} = 0 \Longrightarrow
\]

\[
\begin{cases}
  u^3 + v^3 = \frac{79}{3^3} \\
  uv = \frac{1}{3^2}
\end{cases} \tag{327}
\]

Then \(u^3, v^3\) are solutions of the equation:

\[
X^2 - \frac{79}{3^3}X + \frac{1}{3^6} = 0 \tag{328}
\]

and given below:

\[
u^3 = \frac{1}{2} \cdot \frac{79 + 9\sqrt{77}}{3^3} \quad \Rightarrow \quad \begin{cases}
  u_1 = \sqrt[3]{\frac{1}{2} \left( \frac{79 + 9\sqrt{77}}{3^3} \right)} \approx 0.97515 \\
  u_2 = j.u_1, \quad j = \frac{-1+i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \\
  u_3 = j^2.u_1 = \overline{j}.u_1
\end{cases}
\]

\[
v^3 = \frac{1}{2} \cdot \frac{79 - 9\sqrt{77}}{3^3} \quad \Rightarrow \quad \begin{cases}
  v_1 = \sqrt[3]{\frac{1}{2} \left( \frac{79 - 9\sqrt{77}}{3^3} \right)} \approx 0.00016 \\
  v_2 = j^2.v_1 = \overline{j}.v_1 \\
  v_3 = j.v_1
\end{cases}
\]

\[
(329)
\]
Finally, taking into account the second condition of (327), we obtain the real root of (326):

\[ t = u_1 + v_1 = \sqrt[3]{\frac{1}{2} \left( \frac{79 + 9\sqrt{77}}{3^3} \right)} + \sqrt[3]{\frac{1}{2} \left( \frac{79 - 9\sqrt{77}}{3^3} \right)} \approx 0.97531 \]

(330)

Then the first root of \( F(x) = 0 \) is \( x_1 \approx -1.358 < 0 \), the correction to the first root of \( R(x) = (x + 2)^2(x + 3) = 0 \) is \( dx = x_1 - ( -2 ) = -1.358 - ( -2 ) = +0.642. \) As in our example \( F'(x) > 0 \), the function \( F(x) \) is an increasing function having a parabolic branch as \( x \to +\infty \), the curve \( y = F(x) \) intersects the line \( x = 0 \) in the half-plane \( y \geq 0 \implies F(0) > 0 \implies c < rad^2(ac) \) which is verified numerically.

12.2.3. The General Case

Let us return to the general case \( c = a + 1 \). We denote \( q = min(a_i, c_j) \). If we consider that \( F(x) = R(x) \), the equation \( F(x) = 0 \implies R(x) = 0 \) and the first real root is \( x_1 = -q \), the product of all the roots is \( P = \prod_i (x_i)^2 \prod_j (c_j) = (-1)^{2N_a + N_c} \prod_i (a_i)^2 \prod_j (c_j) \). But \( F(x) = R(x) - \mu_c \), the constant coefficient of \( F(x) \) will be \( \prod_i (a_i)^2 \prod_j (c_j) - \mu_c \). The new product of the roots is \( P' = \prod_i (x'_i)^2 \prod_j (c'_j) = (-1)^{2N_a + N_c} (\prod_i (a_i)^2 \prod_j (c_j) - \mu_c) \). The first root \( x_1 = -q \) becomes \( x'_1 = -q + dx \).

To estimate \( dx \), we can write to the order two that:

\[ F(-q + dx) = R(-q + dx) - \mu_c = 0 \implies R(-q + dx) = \mu_c \implies R(-q) + dx.R'(-q) + \frac{dx^2}{2} R''(-q) = \mu_c \]

(331)

Supposing that \( a_1 = q = min(a_i, c_j) \), from the equations (319-322-324), we have:

\[ R(-a_1) = 0 \]

\[ R'(-a_1) = 0 \]

\[ R''(-a_1) = 2 \prod_{i=2}^{N_a} (a_i - a_1)^2 \prod_{j=1}^{N_c} (c_j - a_1) > 0 \implies \]

(332)

\[ dx^2 = \frac{\mu_c}{\prod_{i=2}^{N_a} (a_i - a_1)^2 \prod_{j=1}^{N_c} (c_j - a_1)} \]

We suppose that \( c > rad^2(ac) \implies \mu_c > rad^2(a).rad(c) \implies \mu_c > R(0) \). We deduce that \( F(0) < 0 \) and \( x'_1 = -a_1 + dx > 0 \implies dx > 0 \). We take the positive
value of $dx$, then we obtain:

\[(333) \quad dx = \frac{\sqrt{\mu_c}}{\prod_{i=2}^{N_a} (a_i - a_1) \cdot \prod_{j=1}^{N_c} (c_j - a_1)}\]

But $\mu_c = \mathcal{R}(x'_1) = \prod_{i} (x'_1 + a_i)^2 \cdot \prod_{j} (x'_1 + c_j)$, we can write:

$$\mu_c = dx^2 \cdot \prod_{i=2}^{N_a} (dx + a_i - a_1)^2 \cdot \prod_{j} (dx + c_j - a_1) \implies$$

\[(334) \quad \mu_c > dx^2 \cdot \prod_{i=2}^{N_a} (a_i - a_1)^2 \cdot \prod_{j} (c_j - a_1)\]

because all the terms $a_i - a_1$ and $c_j - a_1$ are positive numbers. Using the last inequality and the expression of $dx$ given by the equation (333), we obtain:

$$\mu_c > \frac{\mu_c}{\prod_{i=2}^{N_a} (a_i - a_1)^2 \cdot \prod_{j} (c_j - a_1)} \cdot \prod_{i=2}^{N_a} (a_i - a_1)^2 \cdot \prod_{j} (c_j - a_1) \implies$$

\[(335) \quad 1 > 1 \implies \text{the contradiction} \implies \mu_c < \text{rad}^2(a) \text{rad}(c)\]

So, our supposition that $c > \text{rad}^2(ac)$ is false and we obtain the important result that $c < \text{rad}^2(ac)$ and the conjecture (32) is verified.

### 12.2.3.1. Examples

In this section, we are going to verify the above remarks with a numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6$$

$\text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127 = 26\,670$

$\text{rad}(c) = 19$

\[(336) \quad c = 19^6 = 47\,045\,881, \quad \mu_c = 19^5 = 2\,476\,099\]

Using the notations of the paper, we obtain:

$$\mathcal{R}(x) = (x + 2)^2(x + 3)^2(x + 5)^2(x + 7)^2(x + 127)^2(x + 19)$$

$$F(x) = \mathcal{R}(x) - \mu_c$$

Let $X = x + 2$, the expression of $\mathcal{R}(x)$ becomes:

$$\mathcal{R}(X) = X^2(X + 1)^2(X + 3)^2(X + 5)^2(X + 125)^2(X + 17)$$
12.3. The Proof of the abc Conjecture (??) Case: $c = a + 1$

The calculations gives:

$$\overline{R}(X) = X^{11} + 285.X^{10} + 24808.X^9 + 657728.X^8 + 7424722.X^7 + 42772898.X^6 + 134002080.X^5 + 223508940.X^4 + 187753125.X^3 + 597656251.X^2$$

We want to estimate the first root of $F(x) = 0$, we write:

$$\overline{R}(X) - \mu_c = 0 \implies X^{11} + 285.X^{10} + 24808.X^9 + 657728.X^8 + 7424722.X^7 + 42772898.X^6 + 134002080.X^5 + 223508940.X^4 + 187753125.X^3 + 597656251.X^2 - 2476099 = 0$$

If $x = -2 \implies X = 0 \implies \overline{R}(X) - \mu_c < 0$. If we take $x_1 = -1.936315 \implies X_1 = 0.03685$, then we obtain that:

$$\overline{R}(x_1) - \mu_c = \overline{R}(X_1) - \mu_c \approx 177.82 > 0$$

Then, $\exists \xi$ with $-2 < \xi < x_1$ so that $X' = 2 + \xi$ verifies $\overline{R}(X') - \mu_c = 0$ and $\xi$ is the first root of $F(x) = 0$ and $\xi < 0 \implies F(0) > 0 \implies rad^2(a)rad^2(c) - \mu_c > 0 \implies R^2 > c$ that is true. We have also $\xi = -2 + dx = a_1 + dx$ and $0 < dx < a_1$.

12.3. The Proof of the abc Conjecture (15) Case: $c = a + 1$

We denote $R = rad(ac)$.

12.3.1. Case: $\epsilon \geq 1$

Using the result of the theorem above, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon \geq 1$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$ and it is a decreasing function from $\epsilon$ the base of the neperian logarithm to 1.

12.3.2. Case: $\epsilon < 1$

12.3.2.1. Case: $c \leq R$

In this case, we can write :

$$c \leq R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon < 1$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the abc conjecture is true.
12.3.2.2. Case: $c > R$

In this case, we confirm that:

\[(341) \quad c < K(\epsilon).R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad 0 < \epsilon < 1\]

If not, then $\exists \epsilon_0 \in [0, 1[$, so that the triplets $(a, 1, c)$ checking $c > R$ and:

\[(342) \quad c \geq R^{1+\epsilon_0}.K(\epsilon_0)\]

are in finite number. We have:

\[(343) \quad c \geq R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c > K(\epsilon_0) \implies R^{1-\epsilon_0} > K(\epsilon_0)\]

As $c > R$, we obtain:

\[(344) \quad c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \implies c > K(\epsilon_0)\left(\frac{1}{1 - \epsilon_0}\right)\]

We deduce that it exists an infinity of triples $(a, 1, c)$ verifying (342), hence the contradiction. Then the proof of the $abc$ conjecture in the case $c = a + 1$ is finished. We obtain that $\forall \epsilon > 0$, $c = a + 1$ with $a, c$ relatively coprime, $2 \leq a < c$:

\[(345) \quad c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \quad \text{with} \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}\]

Q.E.D

12.4. Examples

In this section, we are going to verify some cases of one numerical example. The example is given by:

\[(346) \quad 1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6\]

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $\text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127,$

$b = 1 \Rightarrow \mu_b = 1$ and $\text{rad}(b) = 1,$

$c = 19^6 = 47045880 \Rightarrow \text{rad}(c) = 19$. Then $\text{rad}(abc) = \text{rad}(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730.$

We have $c > \text{rad}(ac)$ but $\text{rad}^2(ac) = 506730^2 = 25677529200 > c = 47045880.$
12.4.0.1. Case $\epsilon = 0.01$

\[ c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \quad \Rightarrow \quad 47\,045\,880 < e^{10000}\cdot506\,730^{1.01}. \]

The expression of $K(\epsilon)$ becomes:

\[ (347) \quad K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8,747,777,7149,120053120152473488653e + 4342 \]

We deduce that $c \ll K(0.01)\cdot506\,730^{1.01}$ and the equation (345) is verified.

12.4.0.2. Case $\epsilon = 0.1$

\[ K(0.1) = e^{\frac{1}{0.1}} = e^{100} = 2,687,936,330,967,175,420,591,701,121,288,76e + 43 \quad \Rightarrow \quad c < K(0.1)\times506\,730^{1.01}. \]

And the equation (345) is verified.

12.4.0.3. Case $\epsilon = 1$

\[ K(1) = e \quad \Rightarrow \quad c = 47\,045\,880 < e\cdot\text{rad}^2(ac) = 697,987,143,184,212. \]

And the equation (345) is verified.

12.4.0.4. Case $\epsilon = 100$

\[ K(100) = e^{0.0001} \quad \Rightarrow \quad c = 47\,045\,880 < e^{0.0001}\cdot506\,730^{1.01} = 1,522,235,024,860,760,878,185,314,268,728,4e + 576 \]

And the equation (345) is verified.

12.5. Conclusion

This is an elementary proof of the abc conjecture in the case $c = a + 1$. We can announce the important theorem:

Theorem 18. — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a,c$ positive integers relatively prime with $c = a + 1$, then:

\[ (348) \quad c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \]

where $K$ is a constant depending of $\epsilon$ equal to $e^{\left(\frac{1}{\epsilon^2}\right)}$.

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BIBLIOGRAPHY


CHAPTER 13

A PROOF OF THE ABC CONJECTURE: CASE
\[ c = a + 1 \]

Abstract. — In this paper, we consider the abc conjecture, case \( c = a + 1 \). As the conjecture \( c < \text{rad}^2(abc) \) is less open, we give firstly the proof of a modified conjecture that is \( c \leq 2\text{rad}^2(ac) \). The factor 2 is important for the proof of the new conjecture that represents the key of the proof of the main conjecture. Secondly, the proof of the abc conjecture is given for \( \varepsilon \geq 1 \), then for \( \varepsilon \in [0, 1] \). We choose the constant \( K(\varepsilon) \) as \( K(\varepsilon) = 2e \left( \frac{1}{\varepsilon} \right)^{2} \) for \( \varepsilon \geq 1 \) and \( K(\varepsilon) = e \left( \frac{1}{\varepsilon} \right)^{2} \) for \( \varepsilon \in [0, 1] \). Some numerical examples are presented.

13.1. Introduction and notations

Let a positive integer \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
(349) \quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i-1}
\]

We note:

\[
(350) \quad \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a. \text{rad}(a)
\]

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 17. — (abc Conjecture): For each \( \varepsilon > 0 \), there exists \( K(\varepsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(351) \quad c < K(\varepsilon).\text{rad}(abc)^{1+\varepsilon}
\]

where \( K \) is a constant depending only of \( \varepsilon \).
The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the \(abc\) conjecture is due to the incomprehensibility how the prime factors are organized in \(c\) giving \(a, b\) with \(c = a + b\). So, I will give a simple proof in the case \(c = a + 1\) that can be understood by undergraduate students.

We know that numerically, \(\frac{\log c}{\log(\sqrt{abc})} \leq 1.629912\) [1]. A conjecture was proposed that \(c < \sqrt{abc}\) [2]. It is the key to resolve the \(abc\) conjecture. In my paper, I propose to give the proof that \(c \leq 2\sqrt{ac}\), it facilitates the proof of the \(abc\) conjecture. The paper is organized as fellow: in the second section, we give the proof of \(c \leq 2\sqrt{ac}\). The main proof of the \(abc\) conjecture is presented in section three. The numerical examples are discussed in section four.

13.2. The Proof of the conjecture \(c \leq 2\sqrt{ac}\), Case : \(c = a + 1\)

Below is given the definition of the conjecture \(c \leq 2\sqrt{ac}\):

**Conjecture 18.** — Let \(a, b, c\) positive integers relatively prime with \(c = a + b\), then:

\[
(352) \quad c \leq 2\sqrt{abc} \implies \frac{\log c}{\log(\sqrt{abc})} \leq 2 + \frac{\log 2}{\lograd(abc)}
\]

In the case \(c = a + 1\), the definition of the conjecture is:

**Definition 13.1.** — Let \(a, c\) positive integers, relatively prime, with \(c = a + 1, a \geq 2\) then:

\[
(353) \quad c \leq 2\sqrt{ac} \implies \frac{\log c}{\log(\sqrt{ac})} \leq 2 + \frac{\log 2}{\lograd(ac)}
\]

1 - If \(c < \sqrt{ac}\) then we obtain:

\[
(354) \quad c < \sqrt{ac} < \sqrt{2\sqrt{ac}} < 2\sqrt{ac}
\]

and the condition (353) is verified.

2 - If \(c = \sqrt{ac}\), then \(a, c\) are not relatively coprime. Case to reject.

3 - We suppose that \(c > \sqrt{ac}\) \(\implies \mu_c > \sqrt{a}\), we have also \(a > \sqrt{ac} \implies \mu_a > \sqrt{c}\).

3a - Case \(\mu_a \leq \sqrt{a}\): \(c = 1 + a \leq 1 + \sqrt{2\sqrt{ac}} < \sqrt{2\sqrt{ac}} < 2\sqrt{ac}\).

3b - Case \(\mu_c \leq \sqrt{c}\): \(c = \mu_c\sqrt{c} \leq \sqrt{2\sqrt{ac}} < \sqrt{2\sqrt{ac}} < 2\sqrt{ac}\).
3c - Case $\mu_a > \text{rad}(a)$ and $\mu_c > \text{rad}(c)$. As $\mu_a > \text{rad}(c)$, we can write that $\mu_a = l.\text{rad}(c) + l'$ with $1 \leq l' < \text{rad}(c) \implies \mu_a < (l+1)\text{rad}(c) \implies a < (l+1)\text{rad}(c)$

3c1 - We suppose that $l + 1 \leq \text{rad}(ac) \implies l < \text{rad}(ac)$ then $a < (l+1)\text{rad}(ac) \leq \text{rad}^2(ac) \implies a < \text{rad}^2(ac) \implies c \leq \text{rad}^2(ac)$.

3c2 - We suppose that $l = \text{rad}(ac) \implies \mu_a = \text{rad}(a)\text{rad}^2(c) + l' < \text{rad}(c)(\text{rad}(ac) + 1) \implies a < \text{rad}(ac)(\text{rad}(ac) + 1) < 2\text{rad}^2(ac) \implies a < 2\text{rad}^2(ac) \implies c \leq 2\text{rad}^2(ac)$.

3c3 - Case $l > \text{rad}(ac)$

3c3 - 1 - Case $l = \text{rad}(ac) + l''$ with $1 \leq l'' < \text{rad}(ac)$. Then, we write $\mu_a = l.l\text{rad}(c) + l'' = \text{rad}(c)(\text{rad}(ac) + l'') + l' \implies \mu_a < \text{rad}(c)(\text{rad}(ac) + l'') + \text{rad}(c) \implies \mu_a < \text{rad}(c)(\text{rad}(ac) + l'') + 1 \leq 2\text{rad}(c)\text{rad}(ac) \implies a < 2\text{rad}^2(ac) \implies c \leq 2\text{rad}^2(ac)$.

3c3 - 2 - Case $l = q.\text{rad}(ac) + q'$ with $q, q' \in \mathbb{N}, q \geq 2$ and $0 < q' < \text{rad}(ac)$. From $c = a + 1 \implies c = \mu_a.\text{rad}(a) + 1 = (l\text{rad}(c) + l').\text{rad}(a) + 1 = l\text{rad}(ac) + l'\text{rad}(a) + 1 = \text{rad}(ac)(q.\text{rad}(ac) + q') + l'\text{rad}(a) + 1 = q.\text{rad}^2(ac) + q'\text{rad}(ac) + l'\text{rad}(a) + 1$. Let $R = \text{rad}(ac)$, we can write:

\[ q.R^2 + q'R + l'\text{rad}(a) + 1 - c = 0 \]

Let $P(X)$ the polynomial $P(X) = q.X^2 + q'X + l'\text{rad}(a) + 1 - c = 0$, it has a positive integer root $X_1 = R$, then its discriminant $\Delta = q^2 - 4q(l'\text{rad}(a) + 1 - c) = q^2 + 4q(c - l'\text{rad}(a) - 1)$ is a positive integer. Noting that as $c > R \implies c - l'\text{rad}(a) > 0 \implies c \geq l'\text{rad}(a) + 1$, but $c$ and $a$ are coprime, then $c - l'\text{rad}(a) - 1 > 0$. As the root $X_1 = R \in \mathbb{N}$, we can write that $\Delta = m^2$ with $m > 0$ is an integer. We obtain the equation:

\[ m^2 - q^2 = 4q(c - l'\text{rad}(a) - 1) = N > 0 \]

Hence, $m, q'$ are solutions of the Diophantine equation:

\[ x^2 - y^2 = N \]

The roots of $P(X) = 0$ are:

\[ X_1 = R = \frac{-q' + m}{2q} \implies m - q' = 2Rq \]

\[ X_2 = \frac{-q' - m}{2q} \]

From $m^2 - q'^2 = N$ and $m - q' = 2Rq$, we obtain:

\[ m + q' = 2l, \quad N = 4qlR = 2qR.2l \]
Let $Q(N)$ indicate the number of the solutions of (357) and $\tau(N)$ the number of ways representing $N$ as product of its factors, then we have (see the theorem 27.3 in [4]):
- if $N \equiv 2 \pmod{4}$, then $Q(N) = 0$;
- if $N \equiv 1$ or $N \equiv 3 \pmod{4}$, then $Q(N) = [\tau(N)/2]$;
- if $N \equiv 0 \pmod{4}$, then $Q(N) = [\tau(N)/2]$.

Let $(u, v), u, v \in \mathbb{N}$ another couple of solutions of the equation (357), then $u^2 - v^2 = x^2 - y^2 = N = 4qR$, but $m = x$ and $q' = y$ verify the equation $x - y = 2qR$, it follows that $u, v$ verify also $u - v = 2qR$, that gives $u + v = 2l$, then $u = x = m$ and $v = q'$. If not, we have $u - v = 2qR + \lambda$, with $\lambda \neq 0 \in \mathbb{N} \implies u = v + 2qR + \lambda$, from $u^2 - v^2 = N$, we obtain $v = \frac{N - (2qR + \lambda)^2}{2(2qR + \lambda)} = \frac{N}{2(2qR + \lambda)}$, but $u + v = \frac{N}{2qR + \lambda} \neq 2l \left( = \frac{N}{2qR} \right)$. So, we have given the proof of the uniqueness of the solutions of the equation (357) with the condition $m - q' = 2qR$. As $N = 4qR \equiv 0 \pmod{4} \implies Q(N) = [\tau(N)/2] = [\tau(qR)/2] > 1$, because $a, c$ are not prime integers so we can suppose the case $c > R$, and we have $I + K \geq 4$ the total number of $a_i, c_k$. But $Q(N) = 1$, then the contradiction, the condition $l > \text{rad}(ac)$ is false and we obtain considering the above results $c \leq 2\text{rad}^2(ac)$.

3c3-3 - Case $l = q\text{rad}(ac)$ with $q \in \mathbb{N}, q \geq 2$. From $c = a + 1 \implies c = \mu_a\text{rad}(a) + 1 = (l\text{rad}(c) + l')\text{rad}(a) + 1 = l\text{rad}(ac) + l'\text{rad}(a) + 1 = q\text{rad}^2(ac) + l'\text{rad}(a) + 1$. We can write:

$$q\text{rad}^2(c)\text{rad}(a) + l'\text{rad}(a) + 1 - c = 0$$

Let $Q(Z)$ the polynomial $Q(Z) = q\text{rad}^2(c)Z^2 + l'Z - a = 0$, it has a positive integer root $Z_1 = \text{rad}(a)$, then its discriminant $\Delta = t^2 + 4a\text{rad}^2(c)$ is a positive integer. As the root $Z_1 = \text{rad}(a) \in \mathbb{N}$, we can write that $\Delta = t^2$ with $t > 0$ is an integer. We obtain the equation:

$$t^2 - l'^2 = 4a\text{rad}^2(c) = M > 0$$

Hence, $t, l'$ are solutions of the Diophantine equation:

$$x^2 - y^2 = M$$

The roots of $Q(Z) = 0$ are:

$$Z_1 = \text{rad}(a) = \frac{-l' + t}{2a\text{rad}^2(c)} \implies t - l' = 2a\text{rad}(a)\text{rad}^2(c)$$

$$Z_2 = \frac{-l' - t}{2a\text{rad}^2(c)}$$

From $t^2 - l'^2 = M$ and $t - l' = 2a\text{rad}(a)\text{rad}^2(c)$, we obtain:

$$t + l' = 2\mu_a, \quad M = 4a\text{rad}^2(c)$$
Let $S(M)$ indicate the number of the solutions of (363) and $\tau(M)$ the number of ways representing $M$ as product of its factors, using the theorem 27.3 in [4]:

- if $M \equiv 2(\text{mod } 4)$, then $S(M) = 0$;
- if $M \equiv 1$ or $M \equiv 3(\text{mod } 4)$, then $S(M) = \lceil \tau(M)/2 \rceil$;
- if $M \equiv 0(\text{mod } 4)$, then $S(M) = \lceil \tau(M/4)/2 \rceil$.

As seen in the case 3c3-2, equation (363) has an unique solution $(t, l')$ and $M \equiv 0(\text{mod } 4)$, we find the same contradiction and the hypothesis $l = \text{grad}(ac)$ is false and we obtain considering the above results $c \leq 2\text{rad}^2(ac)$.

We announce the theorem:

**Theorem 19.** — Let $a, c$ positive integers relatively prime with $c = a + 1, a \geq 2$, then $c \leq 2\text{rad}^2(ac)$.

**13.3. The Proof of The abc Conjecture (17) case $c = a + 1$**

**13.3.1. Case: $\epsilon \geq 1$**

Using the result of the theorem $c \leq 2\text{rad}^2(abc)$, we have $\forall \epsilon \geq 1$:

\begin{equation}
(367) \quad c \leq 2R^2 \leq 2R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = 2e^\left(\frac{1}{\epsilon^2}\right), \quad \epsilon \geq 1
\end{equation}

We verify easily that $K(\epsilon) > 2$ for $\epsilon \geq 1$. Then the abc conjecture is true.

**13.3.2. Case: $\epsilon < 1$**

**13.3.2.1. Case: $c < R$**

In this case, we can write :

\begin{equation}
(368) \quad c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^\left(\frac{1}{\epsilon^2}\right), \quad \epsilon < 1
\end{equation}

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the abc conjecture is true.

**13.3.2.2. Case: $c > R$**

In this case, we confirm that :

\begin{equation}
(369) \quad c < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^\left(\frac{1}{\epsilon^2}\right), 0 < \epsilon < 1
\end{equation}

If not, then $\exists \epsilon_0 \in [0, 1[$, so that the couple $(a, c)$ checking $c > R$ and:

\begin{equation}
(370) \quad c \geq R^{1+\epsilon_0}.K(\epsilon_0)
\end{equation}
are in finite number. We have:

\[ c \geq R^{1-\epsilon_0}K(\epsilon_0) \implies R^{1-\epsilon_0}c \geq R^{1-\epsilon_0}R^{1+\epsilon_0}K(\epsilon_0) \implies \]

\[ R^{1-\epsilon_0}c \geq R^2K(\epsilon_0) \implies \frac{c}{2}K(\epsilon_0) \implies R^{1-\epsilon_0} \geq \frac{K(\epsilon_0)}{2} \]

As \( c > R \), we obtain:

\[ c^{1-\epsilon_0} > R^{1-\epsilon_0} \geq \frac{K(\epsilon_0)}{2} \implies \]

\[ c^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \implies c > \left( \frac{1}{1-\epsilon_0} \right) \]

We deduce that it exists an infinity of couples \((a, c)\) verifying (370), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that \( \forall \epsilon > 0 \), \( c = a + 1 \) with \( a, c \) relatively coprime, \( a \geq 2 \):

\[ c < K(\epsilon).\text{rad}(a)^{1+\epsilon} \quad \text{with} \quad \begin{cases} \text{K(\epsilon) = 2e} \left( \frac{1}{\epsilon^2} \right) & \epsilon \geq 1 \\ \text{K(\epsilon) = e} \left( \frac{1}{\epsilon^2} \right) & 0 < \epsilon < 1 \end{cases} \]

Q.E.D

13.4. Examples

In this section, we are going to verify some numerical examples. We find that \( c < \text{rad}^2(ac) \implies c \leq 2\text{rad}^2(ac) \) and our proposed conjecture is true.

13.4.1. Example 1

The example is given by:

\[ 1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \]

\( a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42 \) and \( \text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127 \), in this example, \( \mu_a < \text{rad}(a) \).

\( c = 19^6 = 47045880 \Rightarrow \text{rad}(c) = 19 \). Then \( \text{rad}(ac) = \text{rad}(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730 \).

We have \( c > \text{rad}(ac) \) but \( \text{rad}^2(ac) = 506730^2 = 256775292900 > c = 47045880 \).

13.4.1.1. Case \( \epsilon = 0.01 \)

\[ c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \implies c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \]

\[ \text{The expression of K(\epsilon) becomes:} \]

\[ \text{K(\epsilon) = e}^{10^{0.001}} = e^{10^{0.000}} = 8,7477777149120053120152473488653e + 4342 \]
13.5. Conclusion

We deduce that $c \ll K(0.01).506730^{1.01}$ and the equation (373) is verified.

13.4.1.2. Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{1.01}} = e^{100} = 2,6879363309671754205917012128876e + 43 \implies c < K(0.1) \times 506730^{1.01}$, and the equation (373) is verified.

13.4.1.3. Case $\epsilon = 1$

$K(1) = 2e \implies c = 47045880 < 2e.\text{rad}(ac) = 2 \times 697987143184,212$ and the equation (373) is verified.

13.4.1.4. Case $\epsilon = 100$

$K(100) = 2e^{0.0001} \implies c = 47045880 < 2e^{0.0001}.506730^{101} = 2 \times 1,52223502486076087818531425687284e + 576$

and the equation (373) is verified.

13.4.2. Example 2

We give here the example 2 from https://nitaj.users.lmno.cnrs.fr:

(376) $3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424808316456140799 \Rightarrow \text{rad}(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Rightarrow \mu_a > \text{rad}(a)$,

$b = 1$, $\text{rad}(c) = 2 \times 5 \times 127 \times 353$ Then $\text{rad}(ac) = 849767 \times 448310 = 3,809,590,886,010 < c$. $\text{rad}^2(ac) = 14,512,982,718,770,456,813,720,100 > c$, then $c \leq 2\text{rad}^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

(377) $K(\epsilon) = e^{1/0.25} = e^4 = 54,598,003,130,965,797,890,56$

Let us verify (373):

$c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \implies c = 424808316456140800 < K(0.5) \times (3809590886010)^{1.5} \implies$

(378) $424808316456140800 < 405970304762905606174,98260818045$

Hence (373) is verified.

Ouf, end of the mystery!

13.5. Conclusion

We have given an elementary proof of the abc conjecture in the case $c = a + 1$, confirmed by some numerical examples. We can announce the important theorem:
\textbf{Theorem 20.} — (David Masser, Joseph (Esterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then :

\begin{equation}
    c < K(\epsilon).\text{rad}(ac)^{1+\epsilon}
\end{equation}

where $K$ is a constant depending of $\epsilon$ proposed as :

\begin{equation}
    K(\epsilon) = \begin{cases} 
        2e \left( \frac{1}{\epsilon^2} \right) & \epsilon \geq 1 \\
        e \left( \frac{1}{\epsilon^2} \right) & 0 < \epsilon < 1 
    \end{cases}
\end{equation}

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BIBLIOGRAPHY


CHAPTER 14

A FINAL PROOF OF THE abc CONJECTURE

Abstract. — In this paper, we consider the abc conjecture. As the conjecture 
$c < \text{rad}^2(abc)$ is less open, we give firstly the proof of a modified conjecture that is 
$c < 2\text{rad}^2(abc)$. The factor 2 is important for the proof of the new conjecture that 
represents the key of the proof of the main conjecture. Secondly, the proof of the 
abc conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in ]0,1[$. We choose the constant $K(\epsilon)$ as 
$K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \geq 1$ and $K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}$ for $\epsilon \in ]0,1[$. Some numerical examples 
are presented.

To the memory of my F ather who taught me arithmetic
To the memory of my colleague and friend Jamel Zaiem (1956-2019)

14.1. Introduction and notations

Let a positive integer $a = \prod a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We 
call radical of $a$ the integer $\prod a_i$ noted by $\text{rad}(a)$. Then $a$ is written as :

$\prod a_i^{\alpha_i} = \text{rad}(a) \cdot \prod a_i^{\alpha_i-1}$

We note:

$\prod a_i^{\alpha_i-1} \Rightarrow a = \mu_a \cdot \text{rad}(a)$

The abc conjecture was proposed independently in 1985 by David Masser of the 
University of Basel and Joseph (Esterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those 
of its sum. The definition of the abc conjecture is given below:
**CHAPTER 14. A FINAL PROOF OF THE abc CONJECTURE**

**Conjecture 19.** — (abc Conjecture): For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(382) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

where \( K \) is a constant depending only of \( \epsilon \).

The idea to try to write a paper about this conjecture was born after after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in \( c \) giving \( a, b \) with \( c = a + b \). So, I will give a simple proof in the two cases \( c = a + 1 \) and \( c = a + b \) that can be understood by undergraduate students.

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) [1]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) [?]. It is the key to resolve the abc conjecture. In my paper, I propose to give the proof that \( c < 2\text{rad}^2(abc) \), it facilitates the proof of the abc conjecture. The paper is organized as fellow: in the second and third section, we give successively the proof of \( c < 2\text{rad}^2(ac) \) and \( c < 2\text{rad}^2(abc) \). The main proof of the abc conjecture is presented in section four three for the two cases \( c = a + 1 \) and \( c = a + b \). The numerical examples are discussed in sections five and six.

**14.2. The Proof of the conjecture \( c < 2\text{rad}^2(ac) \), case : \( c = a + 1 \)**

Below is given the definition of the conjecture \( c < 2\text{rad}^2(ac) \):

**Conjecture 20.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b, a > b, b \geq 2 \), then:

\[
(383) \quad c < 2\text{rad}^2(abc) \Rightarrow \frac{\log c}{\log(\text{rad}(abc))} < 2 + \frac{\log 2}{\log(\text{rad}(abc))}
\]

In the case \( c = a + 1 \), the definition of the conjecture is:

**Definition 14.1.** — Let \( a, c \) positive integers, relatively prime, with \( c = a+1, a \geq 2 \) then:

\[
(384) \quad c < 2\text{rad}^2(ac) \Rightarrow \frac{\log c}{\log(\text{rad}(ac))} < 2 + \frac{\log 2}{\log(\text{rad}(ac))}
\]

1 - If \( c < \text{rad}(ac) \) then we obtain:

\[
(385) \quad c < \text{rad}(ac) < 2\text{rad}^2(ac)
\]

and the condition (384) is verified.

2 - If \( c = \text{rad}(ac) \), then \( a, c \) are not relatively coprime. Case to reject.
3 - We suppose that \( c > \text{rad}(ac) \implies \mu_c > \text{rad}(a) \), we have also \( a > \text{rad}(ac) \implies \mu_a > \text{rad}(c) \).

3a - Case \( \mu_a \leq \text{rad}(a) \): \( c = 1 + a \leq 1 + \text{rad}^2(a) < \text{rad}^2(ac) < 2\text{rad}^2(ac) \), and the condition (384) is verified.

3b - Case \( \mu_c \leq \text{rad}(c) \): \( c = \mu_c \text{rad}(c) \leq \text{rad}^2(c) < \text{rad}^2(ac) < 2\text{rad}^2(ac) \), and the condition (384) is verified.

3c - Case \( \mu_a > \text{rad}(a) \) and \( \mu_c > \text{rad}(c) \). As \( \mu_a > \text{rad}(c) \), we can write that \( \mu_a = l \cdot \text{rad}(c) + l' \) with \( 1 \leq l' < \text{rad}(c) \implies \mu_a < (l+1)\text{rad}(c) \implies a < (l+1)\text{rad}(ac) \).

3c1 - We suppose that \( l + 1 \leq \text{rad}(ac) \implies l < \text{rad}(ac) \) then \( a < (l+1)\text{rad}(ac) \leq \text{rad}^2(ac) \implies c < 2\text{rad}^2(ac) \), and the condition (384) is verified.

3c2 - We suppose that \( l = \text{rad}(ac) \implies \mu_a = \text{rad}(a)\text{rad}^2(c) + l' < \text{rad}(c)(\text{rad}(ac) + 1) \implies a < \text{rad}(ac)(\text{rad}(ac) + 1) < 2\text{rad}^2(ac) \implies a < 2\text{rad}^2(ac) \implies c \leq 2\text{rad}^2(ac) \).

As \( c \) can not be equal to \( 2\text{rad}^2(ac) \), we obtain \( c < 2\text{rad}^2(ac) \) and the condition (384) is verified.

3c3 - Case: \( l > \text{rad}(ac) \). As \( \mu_a = l \cdot \text{rad}(c) + l' \implies \mu_a > \text{rad}(a)\text{rad}^2(c) \), we can write that \( \mu_a = m \cdot \text{rad}(a)\text{rad}^2(c) + r \) with \( m, r \in \mathbb{N}, m \geq 1 \) and \( 0 < r < \text{rad}(a)\text{rad}^2(c) \). Then:

\[
\mu_a = m \cdot \text{rad}(a)\text{rad}^2(c) + r \implies a = \mu_a \cdot \text{rad}(a) = m \cdot \text{rad}(a)\text{rad}^2(c) + r \cdot \text{rad}(a) \implies a < m \cdot \text{rad}^2(ac) + \text{rad}^2(ac) \implies a < (m+1)\text{rad}^2(ac) \text{ with } m \geq 1 \implies
\]

(386) \( a < (1+1)\text{rad}^2(ac) \implies a < 2\text{rad}^2(ac) \implies a + 1 = c \leq 2\text{rad}^2(ac) \)

As \( c \) can not be equal to \( 2\text{rad}^2(ac) \), we deduce that \( c < 2\text{rad}^2(ac) \) and the condition (384) is verified.

We announce the theorem:

**Theorem 21.** — Let \( a, c \) positive integers relatively prime with \( c = a + 1, a \geq 2 \), then \( c < 2\text{rad}^2(ac) \).

**14.3. The Proof of the conjecture** \( c < 2\text{rad}^2(abc) \), **case :** \( c = a + b \)

Below is given the definition of the conjecture \( c < 2\text{rad}^2(abc) \):

**Conjecture 21.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b, a > b, b \geq 2 \), then:

\[
c < 2\text{rad}^2(abc) \implies \frac{\text{Log}c}{\text{Log}(\text{rad}(abc))} < 2 + \frac{\text{Log}2}{\text{Log}(\text{rad}(abc))}
\]
4 - If \( c < \text{rad}(abc) \) then we obtain:
\[
(388) \quad c < \text{rad}(abc) < \text{rad}^2(abc) < 2\text{rad}^2(abc)
\]
and the condition (387) is verified.

5 - If \( c = \text{rad}(abc) \), then \( a, b, c \) are not relatively coprime. Case to reject.

6 - We suppose that \( c > \text{rad}(abc) \Rightarrow \mu_c > \text{rad}(ab) \), we can write:
\[
(389) \quad \mu_c = l\text{rad}(ab) + l', \quad \text{with} \quad 0 < l' < \text{rad}(ab) \Rightarrow\]
\[
6a - \text{Case } l + 1 \leq \text{rad}(abc) \Rightarrow l < \text{rad}(abc), \text{ then } c < \text{rad}^2(abc) < 2\text{rad}^2(abc) \Rightarrow \]
\( c < 2\text{rad}^2(abc) \) and the condition (387) is verified.

6b - Case \( l = \text{rad}(abc) \): From \( c < (l + 1)\text{rad}(abc) \Rightarrow c < \text{rad}(abc)(\text{rad}(abc) + 1) < 2\text{rad}^2(abc) \), then \( c < 2\text{rad}^2(abc) \) and the condition (387) is verified.

6c - Case \( l > \text{rad}(abc) \): From \( \mu_c = l\text{rad}(ab) + l' \), we deduce that \( \mu_c > \text{rad}^2(ab)\text{rad}(c) \), so we can write:
\[
(390) \quad \mu_c = m\text{rad}^2(ab)\text{rad}(c) + r \quad m \geq 1, 0 < r < \text{rad}^2(ab)\text{rad}(c) \Rightarrow \]
\[
\mu_c < (m + 1)\text{rad}^2(ab)\text{rad}(c), m \geq 1 \Rightarrow c < (m + 1)\text{rad}^2(abc)
\]
\( \text{Taking } m = 1 \Rightarrow c < 2\text{rad}^2(abc) \)

And the condition (387) is verified.

We announce the theorem:

**Theorem 22.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b, a > b, b \geq 2 \), then \( c < 2\text{rad}^2(abc) \).

### 14.4. The Proof of the abc conjecture

Let \( R = \text{rad}(ac) \) or \( R = \text{rad}(abc) \).

**14.4.1. Case : \( \epsilon \geq 1 \)**

Using the result that \( c < 2\text{rad}^2(ac) \) or \( c < 2\text{rad}^2(abc) \), we have \( \forall \epsilon \geq 1 \):
\[
(391) \quad c < 2R^2 \leq 2R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with} \quad K(\epsilon) = 2e\left(\frac{1}{\epsilon^2}\right), \quad \epsilon \geq 1
\]

We verify easily that \( K(\epsilon) > 2 \) for \( \epsilon \geq 1 \). Then the abc conjecture is true.
14.4.2. Case: $\epsilon < 1$

14.4.2.1. Case: $c < R$

In this case, we can write:

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon < 1$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the abc conjecture is true.

14.4.2.2. Case: $c > R$

In this case, we confirm that:

$$c < K(\epsilon).R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1$$

If not, then $\exists \epsilon_0 \in ]0, 1[,$ so that the triplet $(a, b, c)$ checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0)$$

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \implies R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \implies$$

$$R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > \frac{c}{2}K(\epsilon_0) \implies R^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \implies$$

$$c^{1-\epsilon_0} > \frac{K(\epsilon_0)}{2} \implies c > \left(\frac{K(\epsilon_0)}{2}\right)^{\left(\frac{1}{1-\epsilon_0}\right)}$$

We deduce that it exists an infinity of triplets $(a, b, c)$ verifying (394), hence the contradiction. Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0, c = a + b$ with $a, b, c$ relatively coprime:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \quad \text{with} \quad \begin{cases} \text{if } \epsilon \geq 1, & K(\epsilon) = 2e^{\left(\frac{1}{\epsilon^2}\right)} \\ \text{if } 0 < \epsilon < 1, & K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \end{cases}$$

Q.E.D

In the two following sections, we are going to verify some numerical examples. We find that $c < \text{rad}^2(abc) \implies c < 2\text{rad}^2(abc)$ and our proposed conjecture is true.
14.5. **Examples : Case** $c = a + 1$

14.5.1. **Example 1**

The example is given by:

\[(398) \quad 1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \]

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $\text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < \text{rad}(a)$.

$c = 19^6 = 47045880 \Rightarrow \text{rad}(c) = 19$. Then $\text{rad}(ac) = \text{rad}(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730$.

We have $c > \text{rad}(ac)$ but $\text{rad}^2(ac) = 506730^2 = 256775292900 > c = 47045880$.

14.5.1.1. **Case** $\epsilon = 0.01$

$c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \implies 47045880 < e^{10000 \cdot 0.01} \cdot 506730^{1.01}$. The expression of $K(\epsilon)$ becomes:

\[(399) \quad K(\epsilon) = e^{\frac{1}{10000}} = e^{10000} = 8,7477777149120053120152473488653e + 4342 \]

We deduce that $c < K(0.01) \cdot 506730^{1.01}$ and the equation (397) is verified.

14.5.1.2. **Case** $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{100}} = e^{100} = 2,6879363309671754205917012128876e + 43 \implies c < K(0.1) \times 506730^{1.01}$, and the equation (397) is verified.

14.5.1.3. **Case** $\epsilon = 1$

$K(1) = 2e \implies c = 47045880 < 2e \cdot \text{rad}^2(ac) = 2 \times 697,987,143,184,212$ and the equation (397) is verified.

14.5.1.4. **Case** $\epsilon = 100$

$K(100) = 2e^{0.0001} \implies c = 47045880 < 2e^{0.0001} \cdot 506,730^{101} = 2 \times 1,5222350248607608781853142687284e + 576$

and the equation (397) is verified.

14.5.2. **Example 2**

We give here the example 2 from [https://nitaj.users.imno.cnrs.fr](https://nitaj.users.imno.cnrs.fr):

\[(400) \quad 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^{2} \times 127 \times 353 \]

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424,808,316,456,140,799 \Rightarrow \text{rad}(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Rightarrow \mu_a > \text{rad}(a)$,
$b = 1$, $\text{rad}(c) = 2 \times 5 \times 127 \times 353$ Then $\text{rad}(ac) = 849767 \times 448310 = 3809590886010 < c$. $\text{rad}^2(ac) = 14512982718770456813720100 > c$, then $c \leq 2\text{rad}^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

\begin{equation}
K(\epsilon) = e^{1/0.25} = e^4 = 54,59800313096579789056
\end{equation}

Let us verify (397):

\begin{equation}
c < K(\epsilon) \cdot \text{rad}(ac)^{1+\epsilon} \implies c = 42480831645614080 < K(0.5) \times (3809590886010)^{1.5} \implies 42480831645614080 < 405970304762905691174, 98260818045
\end{equation}

Hence (397) is verified.

### 14.6. Examples : Case $c = a + b$

#### 14.6.1. Example 1

We give here the example of Eric Reyssat [1], it is given by:

\begin{equation}
3^{10} \times 109 + 2 = 23^5 = 6436343
\end{equation}

\begin{align*}
a &= 3^{10} \cdot 109 \Rightarrow \mu_a = 3^9 = 19683 \text{ and } \text{rad}(a) = 3 \times 109, \\
b &= 2 \Rightarrow \mu_b = 1 \text{ and } \text{rad}(b) = 2, \\
c &= 23^5 = 6436343 \Rightarrow \text{rad}(c) = 23. \text{ Then } \text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042. \text{ For example, we take } \epsilon = 0.01, \text{ the expression of } K(\epsilon) \text{ becomes:}
\end{align*}

\begin{equation}
K(\epsilon) = e^{0.9999.99} = 8,7477777149120053120152473488653e + 4342
\end{equation}

Let us verify (397):

\begin{equation}
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies c = 6436343 < K(0.01) \times (3 \times 109 \times 23)^{1.01} \implies 6436343 \ll K(0.01) \times 15042^{1.01}
\end{equation}

Hence (397) is verified.

#### 14.6.2. Example 2

The example of Nitaj about the ABC conjecture [1] is:

\begin{align*}
a &= 11^{16} \cdot 13^{2.79} = 6134748434085519215111 \Rightarrow \text{rad}(a) = 11.13.79 \\
b &= 7^2 \cdot 41^2 \cdot 311^3 = 2477678547239 \Rightarrow \text{rad}(b) = 7.41.311 \\
c &= 2.3^3 \cdot 5^{23} .953 = 613474845886230468750 \Rightarrow \text{rad}(c) = 2.3.5.953 \\
\text{rad}(abc) &= 2.3.5.7.11.13.41.79.311.953 = 28828335646110
\end{align*}
14.6.2.1. Case 1

We take $\epsilon = 100$ we have:

\[
\sqrt{c} < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies \\
613474845886230468750 < 2\epsilon^{0.0001} \cdot (2.3.5.7.11.13.41.79.311.953)^{101} \implies \\
613474845886230468750 < 2 \times 2.7657949971494838920022381186039e + 1359
\]

Then (397) is verified.

14.6.2.2. Case 2

We take $\epsilon = 0.5$, then:

\[(397)\]

\[
613474845886230468750 < 2e^{0.0001} \cdot (2.3.5.7.11.13.41.79.311.953)^{1.5} \implies \\
613474845886230468750 < 8450961319227998887403,9993
\]

We obtain that (397) is verified.

14.6.2.3. Case 3

We take $\epsilon = 1$, then

\[
613474845886230468750 < 2e^{0.0001} \cdot (2.3.5.7.11.13.41.79.311.953)^{2} \implies \\
613474845886230468750 < 831072936124776471158132100 \times 2e
\]

We obtain that (397) is verified.

14.6.3. Example 3

It is of Ralf Bonse about the ABC conjecture [3] :

\[(397)\]

\[
a = 2543^{4} \cdot 1.82587.2802983.85813163 + 2^{15} \cdot 3^{77} \cdot 11.173 = 5^{56}.245983
\]

\[
b = 2^{15} \cdot 3^{77} \cdot 11.173
\]

\[
c = 5^{56}.245983
\]

\[
\text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163
\]

\[(397)\]

\[
\text{rad}(abc) = 1.5683959920004546031461002610848e + 33
\]
14.6.3.1. Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = 2e^{0.01} = 2.015631480856591348640923483354$$

Let us verify (397):

$$c \leq K(\epsilon).\text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 < 2e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}$$

$$\Rightarrow 3.413698783296235160378273576498e + 44 < 2.8472401192989816352016241851442e + 365$$

The equation (397) is verified.

14.6.3.2. Case 2

We take $\epsilon = 0.4 \Rightarrow K(\epsilon) = 12.18247347425151215912625669608$, then: The

$$c \leq K(\epsilon).\text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 < e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4}$$

$$\Rightarrow 3.413698783296235160378273576498e + 44 < 3.6255465680011453642792720569685e + 47$$

And the equation (397) is verified.

Ouf, end of the mystery!

14.7. Conclusion

We have given an elementary proof of the $abc$ conjecture, confirmed by some numerical examples. We can announce the important theorem:

**Theorem 23.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending of $\epsilon$ proposed as :

$$
\begin{cases}
K(\epsilon) = 2e^{\frac{1}{\epsilon^2}} & \epsilon \geq 1 \\
K(\epsilon) = e^{\frac{1}{\epsilon^2}} & 0 < \epsilon < 1
\end{cases}
$$
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BIBLIOGRAPHY


CHAPTER 15

A DEFINITIVE PROOF OF THE ABC CONJECTURE

Abstract. — In this paper, we consider the abc conjecture. Firstly, we give an elementary proof the conjecture $c < \text{rad}^2(abc)$. Secondly, the proof of the abc conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in [0,1]$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\frac{1}{\epsilon^2}}$. Some numerical examples are presented.

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Jamel Zaiem (1956-2019)

15.1. Introduction and notations

Let a positive integer $a = \prod a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of a the integer $\prod a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

\begin{equation}
(418) \quad a = \prod a_i^{\alpha_i} = \text{rad}(a). \prod a_i^{\alpha_i - 1}
\end{equation}

We note:

\begin{equation}
(419) \quad \mu_a = \prod a_i^{\alpha_i - 1} \implies a = \mu_a. \text{rad}(a)
\end{equation}

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Estrelé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 22. — (abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\begin{equation}
(420) \quad c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}
\end{equation}

where $K$ is a constant depending only of $\epsilon$. 

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in $c$ giving $a,b$ with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) [1]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. It is the key to resolve the abc conjecture. In my paper, I propose an elementary proof of it, it facilitates the proof of the abc conjecture. In the second section, we give the proof that $c < \text{rad}^2(abc)$. We present the proof of the abc conjecture in section three. The numerical examples are discussed in sections four and five.

15.2. The Proof of the conjecture $c < \text{rad}^2(abc)$

Below is given the definition of the conjecture $c < \text{rad}^2(abc)$:

**Conjecture 23.** — Let $a, b, c$ positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:

\[
(421) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]

We note $R = \text{rad}(abc)$ in the case $c = a + b$ or $R = \text{rad}(ac)$ in the case $c = a + 1$.

As $c$ is bounded, it exists an unique couple $(m, n) \in \mathbb{Z}^+ \times \mathbb{N}, n \geq m + 1 \geq 1$, so that:

\[
(422) \quad m.R^2 < c < n.R^2
\]

We can write:

\[
(423) \quad c = mR^2 + r, \quad 1 \leq r < R^2
\]
\[
(424) \quad c = nR^2 - r', \quad 1 \leq r' < R^2
\]

But $mR^2 + r = nR^2 - r' \implies 2 \leq r + r' = (n - m)R^2 < 2R^2 \implies n - m < 2$, we deduce $n = m$ or $n = m + 1$. The case $n = m$ presents a contradiction. Hence $n = m + 1$. The equation (422) becomes:

\[
(425) \quad m.R^2 < c < (m + 1).R^2, \quad m \geq 0
\]

15.2.1. Proof that $c < R^2$

** Case $c < R$: $c < R < R^2$ and the condition (421) is verified.
** Case \( c = R \): case to reject.

** Case \( c > R \): from (425), we obtain:

\[
mR^2 < c < (m + 1)R^2
\]

If \( m = 0 \), we deduce that:

\[
0 < c < R^2
\]

and the condition (421) is verified.

We suppose now that \( m > 0 \). Let \( c = a + b \) or \( c = a + 1 \) so that:

\[
mR^2 < c < (m + 1)R^2
\]

As \( c > mR^2 \), we can write:

\[
c = mR^2 + m', \quad m' < R^2
\]

But \( c > R \Rightarrow c^2 > R^2 \), we obtain also:

\[
c^2 = lR^2 + l', \quad l' < R^2
\]

From the above equations, we can write:

\[
(mR^2 + m')^2 = lR^2 + l' \Rightarrow m^2R^4 + (2mm' - l)R^2 + m'^2 - l' = 0
\]

From the last equation above, \( R^2 \) is the positive root of the polynomial of the second degree:

\[
F(T) = m^2T^2 + (2mm' - l)T + m'^2 - l' = 0
\]

The discriminant of \( F(T) \) is:

\[
\Delta = (2mm' - l)^2 - 4m^2(m'^2 - l')
\]

As a real root of \( F(T) \) exists, and it is an integer, \( \Delta \) is written as:

\[
\Delta = t^2 \geq 0, t \in \mathbb{Z}^+
\]

** - Case \( \Delta = 0 \) and \( m'^2 - l' \neq 0 \): Then \( (2mm' - l)^2 = 4m^2(m'^2 - l') \Rightarrow m'^2 - l' = \alpha^2 \), \( \alpha \in \mathbb{N} \). In this case the equation (431) has a double root \( T_1 = T_2 = \frac{l - 2mm'}{2m^2} = R^2 \Rightarrow l - 2mm' = 2m^2R^2 > 0 \). But \( (l - 2mm')^2 = 4m^4R^4 = 4m^2(m'^2 - l') \Rightarrow m'^2 = m^2R^4 + l' > R^4 \Rightarrow m' > R^2 \). Then the contradiction as \( m' < R^2 \). The case \( \Delta = 0 \) and \( m'^2 - l' \neq 0 \) is impossible.

** - Case \( \Delta = 0 \) and \( m'^2 - l' = 0 \): In this case, \( 2mm' - l = 0 \Rightarrow R^2 = 0 \). Then the contradiction as \( R > 0 \). The case \( \Delta = 0 \) and \( m'^2 - l' = 0 \) is impossible.
** - Case $\Delta > 0$ and $m'^2 - l' = 0$: The equation (431) becomes:

\[
F(T) = m'^2T^2 + (2mm' - l)T = 0 \implies \begin{cases} 
T_1 = 0 \\
T_2 = \frac{l - 2mm'}{m^2} = R^2
\end{cases}
\]

Then, we have:

\[
l - 2mm' = m'^2R^2 \implies l = 2mm' + m'^2R^2
\]

As $m' < R^2 \implies l - m'^2R^2 < 2mR^2 \implies l < 2mR^2 + m'^2R^2$, we obtain

\[
lR^2 < m(2 + m)R^4.
\]

We deduce that $c^2 = lR^2 + l' < m(2 + m)R^4 + R^2$. We know that $c < (m + 1)R^2 \implies c^2 < (m + 1)^2R^4$. We verify easily that $m(2 + m)R^4 + R^2 < (m + 1)^2R^4$, then the contradiction with $mR^2 < c < (m + 1)R^2$. Hence, the case $\Delta > 0$ and $m'^2 - l' = 0$ is impossible.

** - Case $\Delta > 0$ and $m'^2 - l' > 0$: We have: $\Delta = (2mm' - l)^2 - 4m^2(m'^2 - l') = l^2 \implies l^2 < (2mm' - l)^2$. Let the case $|2mm' - l| = 2mm' - l \implies t < 2mm' - l$. The expression of the two roots are:

\[
\begin{cases} 
T_1 = \frac{l - 2mm' + t}{2m^2b} < 0 \\
T_2 = \frac{l - 2mm' - t}{2m^2} < 0
\end{cases}
\]

As $R^2 > 0$ is a root of $F(T) = 0$, then the contradiction. Hence, the case $\Delta > 0$ and $m'^2 - l' > 0$ is impossible.

** - Case $\Delta > 0$ and $m'^2 - l' < 0$: From $m'^2 < l' \implies (c - mR^2)^2 < c^2 - lR^2$, it gives $m'^2R^2 + l - 2mc < 0 \implies m'^2R^2 + l < 2mc < 2m(m + 1)R^2$. Then we obtain

\[
l < m'^2R^2 + 2mR^2 \implies lR^2 < m(m + 2)R^4 \implies c^2 = lR^2 + l' < m(m + 2)R^4 + R^2.
\]

We know that $c < (m + 1)R^2 \implies c^2 < (m + 1)^2R^4$. We verify easily that $m(2 + m)R^4 + R^2 < (m + 1)^2R^4$, then the contradiction with $mR^2 < c < (m + 1)R^2$. Hence, the case $\Delta > 0$ and $m'^2 - l' < 0$ is impossible.

All the cases for the resolution of the equation (431) have given contradictions with the hypothesis $c > mR^2, m > 0$. Then we obtain that $m = 0$ and $0 < c < R^2$. Hence the condition (421) is verified.

We announce the theorem:

**Theorem 24.** Let $a, b, c$ positive integers relatively prime with $c = a + b, a > b$, then $c < \text{rad}^2(abc)$. 

15.3. The Proof of the \( abc \) conjecture

15.3.1. Case : \( \epsilon \geq 1 \)

Using the result that \( c < R^2 \), we have \( \forall \epsilon \geq 1 \):

\[
(436) \quad c < R^2 \leq R^{1+\epsilon} < K(\epsilon)R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon \geq 1
\]

We verify easily that \( K(\epsilon) > 1 \) for \( \epsilon \geq 1 \). Then the \( abc \) conjecture is true.

15.3.2. Case: \( \epsilon < 1 \)

15.3.2.1. Case: \( c < R \)

In this case, we can write :

\[
(437) \quad c < R < R^{1+\epsilon} < K(\epsilon)R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad 0 < \epsilon < 1
\]

Here also \( K(\epsilon) > 1 \) for \( \epsilon < 1 \) and the \( abc \) conjecture is true.

15.3.2.2. Case: \( c > R \)

In this case, we confirm that :

\[
(438) \quad c < K(\epsilon)R^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad 0 < \epsilon < 1
\]

If not, then \( \exists \epsilon_0 \in ]0, 1[ \), so that the triple \((a, b, c)\) checking \( c > R \) and:

\[
(439) \quad c \geq R^{1+\epsilon_0}K(\epsilon_0)
\]

are in finite number. We have:

\[
R^{1-\epsilon_0}c \geq R^2.K(\epsilon_0) > cK(\epsilon_0) \implies R^{1-\epsilon_0} > K(\epsilon_0)
\]

As \( c > R \), we obtain:

\[
(440) \quad c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \implies c^{1-\epsilon_0} > K(\epsilon_0) \implies c > (K(\epsilon_0))^{\left(\frac{1}{1-\epsilon_0}\right)}
\]

We deduce that it exists an infinity of triples \((a, b, c)\) verifying (439), hence the contradiction. Then the proof of the \( abc \) conjecture is finished. We obtain that \( \forall \epsilon > 0, c = a + b \) with \( a, b, c \) relatively coprime:

\[
(441) \quad c < K(\epsilon).rad(abc)^{1+\epsilon}, \quad \text{with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \quad \epsilon > 0
\]
Q.E.D

In the two following sections, we are going to verify some numerical examples.

15.4. *Examples*: Case \( c = a + 1 \)

15.4.1. Example 1

The example is given by:

\[
1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6
\]

\( a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47,045,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42 \) and \( \text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127 \), in this example, \( \mu_a < \text{rad}(a) \).

\( c = 19^6 = 47,045,881 \Rightarrow \text{rad}(c) = 19. \) Then \( \text{rad}(ac) = \text{rad}(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506,730. \)

We have \( c > \text{rad}(ac) \) but \( \text{rad}^2(ac) = 506,730^2 = 256,775,292,900 > c = 47,045,881. \)

15.4.1.1. Case \( \epsilon = 0.01 \)

\( c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \Rightarrow 47,045,881 < e^{10000.506,730^{1.01}}. \) The expression of \( K(\epsilon) \) becomes:

\[
K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8.7477777149120053120152473488653e + 4342
\]

We deduce that \( c < K(0.01).506,730^{1.01} \) and the equation (694) is verified.

15.4.1.2. Case \( \epsilon = 0.1 \)

\( K(0.1) = e^{\frac{1}{0.1}} = e^{100} = 2.6879363309671754205917012128876e + 43 \Rightarrow c < K(0.1) \times 506,730^{1.01} \), and the equation (694) is verified.

15.4.1.3. Case \( \epsilon = 1 \)

\( K(1) = e \Rightarrow c = 47,045,881 < e.\text{rad}^2(ac) = 697,987,143,184,212 \) and the equation (694) is verified.

15.4.1.4. Case \( \epsilon = 100 \)

\[
K(100) = e^{0.0001} \Rightarrow c = 47,045,881 < e^{0.0001.506,730^{101}} = 1.5222350248607608781853142687284e + 576
\]

and the equation (694) is verified.
15.5. Examples : Case $C = A + B$

15.4.2. Example 2

We give here the example 2 from https://nitaj.users.lmno.cnrs.fr:

\[(445) \quad 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353\]

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424808316456140799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Rightarrow \mu_a > rad(a)$.

$b = 1$, \(rad(c) = 2 \times 5 \times 127 \times 353\) Then \(rad(ac) = 849767 \times 448310 = 3809590886010 > c\).

\[c = 23^5 = 6436343\]

\(rad(ac) = 15042\). For example, we take \(\epsilon = 0.5\), the expression of \(K(\epsilon)\) becomes:

\[(446) \quad K(\epsilon) = e^{1/0.25} = e^4 = 54.59800313096579789056\]

Let us verify (694):

\[c < K(\epsilon).rad(ac)^{1+\epsilon} \Rightarrow c = 424808316456140800 < K(0.5) \times (3809590886010)^{1.5} \Rightarrow (447) \quad 424808316456140800 < 405970304762905691174.98260818045\]

Hence (694) is verified.

15.5. Examples : Case $c = a + b$

15.5.1. Example 1

We give here the example of Eric Reyssat [1], it is given by:

\[(448) \quad 3^{10} \times 109 + 2 = 29^5 = 6436343\]

\(a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683\) and \(rad(a) = 3 \times 109\),

\(b = 2 \Rightarrow \mu_b = 1\) and \(rad(b) = 2\),

\(c = 23^5 = 6436343 \Rightarrow rad(c) = 23\). Then \(rad(ac) = 2 \times 3 \times 109 \times 23 = 15042\). For example, we take \(\epsilon = 0.01\), the expression of \(K(\epsilon)\) becomes:

\[(449) \quad K(\epsilon) = e^{0.999.99} = 8.7477777149120053120152473488653e + 4342\]

Let us verify (694):

\[c < K(\epsilon).rad(ac)^{1+\epsilon} \Rightarrow c = 6436343 < K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Rightarrow (450) \quad 6436343 < K(0.01) \times 15042^{1.01}\]

Hence (694) is verified.
15.5.2. Example 2

The example of Nitaj about the ABC conjecture [1] is:

\[(451)\] \[a = 11^{16} \cdot 13^2 \cdot 79 = 613474843408551921511 \Rightarrow \text{rad}(a) = 11.13.79\]
\[(452)\] \[b = 7^2 \cdot 41^2 \cdot 311^3 = 2477678547239 \Rightarrow \text{rad}(b) = 7.41.311\]
\[(453)\] \[c = 2.3^3 \cdot 5^{23} \cdot 953 = 613474845886230468750 \Rightarrow \text{rad}(c) = 2.3.5.953\]
\[(454)\] \[\text{rad}(abc) = 2.3.5.7.11.13.41.79.311.953 = 28828335646110\]

15.5.2.1. Case 1

we take \(\epsilon = 100\) we have:

\[c \leq K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies\]
\[613474845886230468750 < e^{0.0001} \cdot (2.3.5.7.11.13.41.79.311.953)^{101} \implies\]
\[613474845886230468750 < 2.7657949971494838920022381186039e + 1359\]

then (694) is verified.

15.5.2.2. Case 2

We take \(\epsilon = 0.5\), then:

\[c \leq K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies\]
\[613474845886230468750 < e^{0.5} \cdot (2.3.5.7.11.13.41.79.311.953)^{1.5} \implies\]
\[613474845886230468750 < 8.4509613192279887403,9993\]

We obtain that (694) is verified.

15.5.2.3. Case 3

We take \(\epsilon = 1\), then

\[c \leq K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies\]
\[613474845886230468750 < e \cdot (2.3.5.7.11.13.41.79.311.953)^2 \implies\]
\[613474845886230468750 < 831072936124776471158132100 \times e\]

We obtain that (694) is verified.
15.5.3. Example 3

It is of Ralf Bonse about the ABC conjecture [3]:

\[(458)\]
\[
2543^4 \cdot 182587 \cdot 2802983.85813163 + 2^{15} \cdot 3^{77} \cdot 11.173 = 5^{56} \cdot 245983
\]
\[
a = 2543^4 \cdot 182587 \cdot 2802983.85813163
\]
\[
b = 2^{15} \cdot 3^{77} \cdot 11.173
\]
\[
c = 5^{56} \cdot 245983
\]
\[
\text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163
\]
\[
(459) \quad \text{rad}(abc) = 1.5683959920004546031461002610848e + 33
\]

15.5.3.1. Case 1

For example, we take \(\epsilon = 10\), the expression of \(K(\epsilon)\) becomes:

\[
K(\epsilon) = e^{0.01} = 1.007815740428295674320461741677
\]

Let us verify (??):

\[
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56} \cdot 245983 < e^{0.01} \cdot (2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11}
\]
\[
\Rightarrow 3.413699873296235160378273576498e + 44 < 1.423620059649490817600812092572e + 365
\]

The equation (694) is verified.

15.5.3.2. Case 2

We take \(\epsilon = 0.4 \Rightarrow K(\epsilon) = 12.18247347425151215912625669608\), then: The

\[
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56} \cdot 245983 < e^{0.25} \cdot (2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4}
\]
\[
\Rightarrow 3.413699873296235160378273576498e + 44 < 3.6255465680011453642792720569685e + 47
\]

And the equation (694) is verified.

\[
\text{Ouf, end of the mystery!}
\]

15.6. Conclusion

We have given an elementary proof of the abc conjecture, confirmed by some numerical examples. We can announce the important theorem:
Theorem 25. — (David Masser, Joseph (Esterlé & Abdelmajid Ben Hadj Salem; 2019) For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\[ c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \]

where $K$ is a constant depending of $\epsilon$ proposed as:

\[ K(\epsilon) = e \left( \frac{1}{\epsilon^2} \right) , \epsilon > 0 \]

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BIBLIOGRAPHY


CHAPTER 16

IF \( c < R.\exp\left(\frac{3\sqrt{2}}{2}\log^{2/3}R\right) \implies \text{THE ABC CONJECTURE TRUE} \)

Assuming \( c < R.\exp\left(\frac{3\sqrt{2}}{2}\log^{2/3}R\right) \) - A New Conjecture - Implies The abc Conjecture True

Abstract. — In this paper about the abc conjecture, we propose a new conjecture about an upper bound for \( c \) as \( c < R.\exp\left(\frac{3\sqrt{2}}{2}\log^{2/3}R\right) \). Assuming the last condition holds, we give the proof of the abc conjecture by proposing the expression of the constant \( K(\epsilon) \), then we approve that \( \forall \epsilon > 0 \), for \( a, b, c \) positive integers relatively prime with \( c = a + b \), we have \( c < K(\epsilon).\rad^{1+\epsilon}(abc) \). Some numerical examples are given.

16.1. Introduction and notations

Let a positive integer \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \) noted by \( \rad(a) \). Then \( a \) is written as :

\[
(463) \quad a = \prod_i a_i^{\alpha_i} = \rad(a). \prod_i a_i^{\alpha_i-1}
\]

We note:

\[
(464) \quad \mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a.\rad(a)
\]

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 24. — (abc Conjecture): For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then :

\[
(465) \quad c < K(\epsilon).\rad^{1+\epsilon}(abc)
\]

where \( K \) is a constant depending only of \( \epsilon \).
The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the \textit{abc} conjecture is due to the incomprehensibility how the prime factors are organized in \( c \) giving \( a, b \) with \( c = a + b \).

We know that numerically, \[ \frac{\text{Log} c}{\text{Log}(\text{rad}(abc))} \leq 1.629912 \] [1]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) [3]. It is the key to resolve the \textit{abc} conjecture. In my paper, I propose the constant \( K(\epsilon) = e \left( \frac{1}{\epsilon^2} \right) \) and assuming that \( c < R.exp(\frac{3^\sqrt{2}}{2} \text{Log}^{2/3} R) \) the new conjecture more stronger than \( c < R^2 \). In my proof of the \textit{abc} conjecture, we will find that \( c \) must verify \( c < R.exp(\frac{3^\sqrt{2}}{2} \text{Log}^{2/3} R) \) so we will obtain that the \textit{abc} conjecture is true. The paper is organized as follows: in the second section, we give the proof of the \textit{abc} conjecture. In sections three and four, we present some numerical examples respectively for the cases \( c = a + 1 \) and \( c = a + b \).

16.2. \textit{The Proof of the \textit{abc} conjecture}

Let \( a, b, c \) (respectively \( a, c \)) positive integers relatively prime with \( c = a + b, a > b, b \geq 2 \) (respectively \( c = a + 1, a \geq 2 \)). We note \( R = \text{rad}(abc) \) in the case \( c = a + b \) or \( R = \text{rad}(ac) \) in the case \( c = a + 1 \). I propose the constant \( K(\epsilon) \) as:

\[ K(\epsilon) = e \left( \frac{1}{\epsilon^2} \right) > 1, \forall \epsilon > 0 \] (466)

16.2.1. Case \( c < R \):

As \( c < R \implies c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0 \) since \( K(\epsilon) > 1 \) and the conjecture (36) is verified.

16.2.2. Case \( c = R \)

Case to reject as \( a, b, c \) (respectively \( a, c \)) are relatively prime.

16.2.3. Case \( R < c \)

In this case, we have \( c/R > 1 \implies \text{Log}(c/R) > 0 \). Let for \( \forall \epsilon > 0 \):

\[ y(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)\text{Log}R - \text{Log}c \] (467)
Our main task is give the proof that \( y(\epsilon) > 0 \implies \frac{1}{\epsilon^2} + (1 + \epsilon)\log R > \log c \), then

\[ \implies \log c < \frac{1}{\epsilon^2} + (1 + \epsilon)\log R \] and we obtain \( c < e^{\left(\frac{1}{\epsilon^2}\right)} R^{1+\epsilon} = K(\epsilon) R^{1+\epsilon}, \forall \epsilon > 0. \)

We have also:

\[ \lim_{\epsilon \to 0} y(\epsilon) = +\infty \tag{468} \]

\[ \lim_{\epsilon \to +\infty} y(\epsilon) = +\infty \tag{469} \]

For \( \epsilon > 0 \), the function derivative \( y'(\epsilon) \) is given by:

\[ y'(\epsilon) = -\frac{2}{\epsilon^3} + \log R = \frac{\epsilon^3 \log R - 2}{\epsilon^3} \]

\( y'(\epsilon) = 0 \) gives:

\[ \epsilon_0 = \sqrt[3]{\frac{2}{\log R}} \leq \sqrt[3]{\frac{2}{\log 6}} \approx 1.03733 \tag{471} \]

If \( R \not\nearrow \), then \( \epsilon_0 \to 0 \). For \( \epsilon = \epsilon_0 \), we obtain:

\[ y(\epsilon_0) = \frac{1}{\epsilon_0^2} + (1 + \epsilon_0)\log R - \log c = \log R + \frac{3}{2} \sqrt[3]{2} \log^{2/3} R - \log c \]

\( y(\epsilon_0) \) is positive if \( \log R + \frac{3}{2} \sqrt[3]{2} \log^{2/3} R - \log c > 0 \). So we assume that :

\[ c < R \exp\left(\frac{3}{2} \sqrt[3]{2} \log^{2/3} R\right) \implies y(\epsilon_0) > 0 \tag{473} \]

Then the new conjecture proposed by us is :

\[ c < R \exp\left(\frac{3}{2} \sqrt[3]{2} \log^{2/3} R\right) \tag{474} \]

From (468-469), the point \( (\epsilon_0, y(\epsilon_0)) \) is the minimum of the curve \( y(\epsilon) \) for all \( \epsilon > 0 \). Then \( y(\epsilon) > 0 \) and the proof of the \( abc \) conjecture is finished. We obtain that \( \forall \epsilon > 0 \),

\[ c = a + b \text{ with } a, b, c \text{ relatively coprime:} \]

\[ c < K(\epsilon) \text{rad}^{1+}(abc) \text{ with } K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon > 0 \tag{475} \]

**Remark 16.1.** — We verify that \( R \exp\left(\frac{3}{2} \sqrt[3]{2} \log^{2/3} R\right) < R^{1+2/3} \) for \( R \) large, \( R > 7830169545 \).

**Remark 16.2.** — Nowadays, we know numerically [1] that \( \frac{\log c}{\log R} \leq 1.629912 < 1 + 2/3 \approx 1.666667 \). All the numerical examples below verify \( c < R^{1+2/3} \), so, I would suggest that \( c < R^{1+2/3} \) as a new open conjecture that it is more difficult than \( c < R^2 \).
In the two following sections, we are going to verify some numerical examples.

16.3. Examples : Case \( c = a + 1 \)

**Example 16.3.** — The example is given by:

\[
1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6
\]

\( a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42 \) and \( \text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127, \) in this example, \( \mu_a < \text{rad}(a). \)

\( c = 19^6 = 47045881 \Rightarrow \text{rad}(c) = 19. \) Then \( R = \text{rad}(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730. \)

We have \( c > R \) and \( R \cdot \exp(\frac{3}{2} \sqrt{2} \log^{2/3} R) = 18800185299.081 > c = 47045881. \)

16.3.0.1. Case \( \epsilon = 0.01 \)

\( c < K(\epsilon) \cdot \text{rad}(ac)^{1+\epsilon} \Leftrightarrow 47045881 < e^{10000.506730^{1.01}}. \) The expression of \( K(\epsilon) \) becomes:

\[
K(\epsilon) = e^{0.0001} = e^{10000} = 8.7477777149120053120152473488653e + 4342
\]

We deduce that \( c \ll K(0.01).506730^{1.01} \) and the equation (475) is verified.

16.3.0.2. Case \( \epsilon = 0.1 \)

\( K(0.1) = e^{0.1} = e^{100} = 2.6879363309671754205917012128876e + 43 \Rightarrow c < K(0.1) \times 506730^{1.01}, \) and the equation (475) is verified.

16.3.0.3. Case \( \epsilon = 1 \)

\[ K(1) = e \Leftrightarrow c = 47045881 < e \cdot \text{rad}^2(ac) = 697987143\text{,}184\text{.}212 \] and the equation (475) is verified.

16.3.0.4. Case \( \epsilon = 100 \)

\[ K(100) = e^{0.0001} \Rightarrow c = 47045881 < e^{0.0001} \cdot 506730^{1.01} = 1.5222350248607608781853142687284e + 576 \]

and the equation (475) is verified.

**Example 16.4.** — We give here the example 2 from https://nitaj.users.lmno.cnrs.fr:

\[
1 + 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 2^{30} \times 5^2 \times 127 \times 353^2
\]

\( a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424,808,316,456,140,799 \Rightarrow \text{rad}(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Rightarrow \mu_a > \text{rad}(a), \)
16.4. EXAMPLES : CASE $C = A + B$

$b = 1$, $c = a + 1 = 424808316456140800 \implies \text{rad}(c) = 2 \times 5 \times 127 \times 353$. Then $R = \text{rad}(ac) = 8497671 \times 448310 = 3809590886010 < c$. We obtain $R.exp(\frac{3}{2}\sqrt{2}\log^{2/3}R) = 210209917628130447085.912 > c$, then $c < R.exp(\frac{3}{2}\sqrt{2}\log^{2/3}R)$.

For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$K(\epsilon) = e^{1/0.25} = e^4 = 54.5980031309657879056$

Let us verify (475):

$c < K(\epsilon).\text{rad}(ac)^{1+\epsilon} \implies c = 424808316456140800 < K(0.5) \times (3809590886010)^{1.5}$

(480)\implies 424808316456140800 < 405970304762905691174.98260818045

Hence (475) is verified.

### 16.4. Examples : Case $c = a + b$

**Example 16.5.** We give here the example of Eric Reyssat [1], it is given by:

(481) \[ 3^{10} \times 109 + 2 = 235 = 6436343 \]

$a = 3^{10} \times 109 = 6436341 \implies \mu_a = 3^9 = 19683$ and $\text{rad}(a) = 3 \times 109 \implies \mu_a > \text{rad}(a)$,

$b = 2 \implies \mu_b = 1$ and $\text{rad}(b) = 2$,

$c = 23^5 = 6436343 \implies \text{rad}(c) = 23$. Then $R = \text{rad}(abc) = 2 \times 3 \times 109 \times 23 = 15042 < c$. Let us verify $c < R.exp(\frac{3}{2}\sqrt{2}\log^{2/3}R)$. We obtain : $c = 6436343 < 77532979.756$.

For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

(482) \[ K(\epsilon) = e^{9999.99} = 8.7477777149120053120152473488653e + 4342 \]

Let us verify (475):

$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \implies c = 6436343 < K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \implies$

(483) \[ 6436343 < K(0.01) \times 15042^{1.01} \]

Hence (475) is verified.

**Example 16.6.** The example of Nitaj about the ABC conjecture [1] is:

$a = 11^{16}13^2.79 = 613474843408551921511 \implies \text{rad}(a) = 11.13.79 = 11297$,

$b = 7^2.41^2.311^3 = 2477678547239 \implies \text{rad}(b) = 7.41.311 = 89257$

$c = 2.3^3.5^3.953 = 613474845886230468750 \implies \text{rad}(c) = 2.3.5.953$

$R = \text{rad}(abc) = 2.3.5.7.11.13.41.79.311.953 = 28828335646110 < c$

We have also $\mu_a > \text{rad}(a), \mu_b > \text{rad}(b) > \text{rad}(a)$ and $\mu_b < \mu_a$. We find $c < R.exp(\frac{3}{2}\sqrt{2}\log^{2/3}R) = 3614932048440771457890.631$. 


16.4.0.1. Case 1

We take \( \epsilon = 100 \) we have:

\[
c^2 < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies
\]

\[
613474845886230468750 < e^{0.0001 \cdot (2.3.5.7.11.13.41.79.311.953)^{101}} \implies
\]

\[
613474845886230468750 < 2.7657949971494838920022381186039e + 1359
\]

then (475) is verified.

16.4.0.2. Case 2

We take \( \epsilon = 0.5 \), then:

\[
c^2 < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies
\]

\[
613474845886230468750 < e^4 \cdot (2.3.5.7.11.13.41.79.311.953)^{1.5} \implies
\]

\[
613474845886230468750 < 8.4509613192279887403,9993
\]

We obtain that (475) is verified.

16.4.0.3. Case 3

We take \( \epsilon = 1 \), then

\[
c^2 < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon} \implies
\]

\[
613474845886230468750 < e \cdot (2.3.5.7.11.13.41.79.311.953)^2 \implies
\]

\[
613474845886230468750 < 831072936124776471158132100 \times e
\]

We obtain that (475) is verified.

**Example 16.7.** — It is of Ralf Bonse about the ABC conjecture [3]:

\[
2543^4 \cdot 182587 \cdot 2802983 \cdot 85813163 + 2^{15} \cdot 37.11.173 = 5^{56} \cdot 245983
\]

\[
a = 2543^4 \cdot 182587 \cdot 2802983 \cdot 85813163
\]

\[
b = 2^{15} \cdot 37,11.173
\]

\[
c = 5^{56} \cdot 245983 = 3.4136998783296235160378273576498e + 44
\]

\[
R = \text{rad}(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163
\]

\[
R = 1,5683959920004546031461002610848e + 33 < c
\]

We have also: \( \mu_a < \text{rad}(a), \mu_b > \text{rad}(b) > \mu_a, \mu_c > \text{rad}(c) \) and \( \mu_b < \mu_c \). The calculate of \( A = R_{exp}(\frac{3}{2} \sqrt{2}\log^{2/3}R) \) gives:

\[
A = 9.5054989139840681669171835013874e + 47 > c
\]
16.4.0.4. Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1.007815740428295674320461741677$$

Let us verify (475):

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 < 1.423620059649490817600812092572e + 365$$

The equation (475) is verified.

16.4.0.5. Case 2

We take $\epsilon = 0.4 \Rightarrow K(\epsilon) = 12.182473474251512159126569608$, then:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 < 3.6255465680011453642792720569685e + 47$$

And the equation (475) is verified.

16.5. Conclusion

Assuming $c < R.\exp\left(\frac{3}{2}\sqrt{2\log^{2/3} R}\right)$, we have given an elementary proof of the $abc$ conjecture, confirmed by some numerical examples. We can announce the theorem:

**Theorem 26.** — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, and assuming $c < R.\exp\left(\frac{3}{2}\sqrt{2\log^{2/3} R}\right)$ is true, then:

$$c < K(\epsilon).\text{rad}^{1+\epsilon}(abc)$$

where $K$ is a constant depending of $\epsilon$ proposed as:

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon > 0$$

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BIBLIOGRAPHY


CHAPTER 17

THE abc CONJECTURE IS FALSE

Abstract. — In this note, I give the proof that the abc conjecture is false because, in the case $c > \text{rad}(abc)$, for $0 < \epsilon < 1$ we can not find the constant $K(\epsilon)$ so that $c < K(\epsilon).\text{rad}^{1+\epsilon}(abc)$ for $c$ very large. A counter-example is given.

17.1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

\begin{equation}
\label{eq:492}
a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i-1}
\end{equation}

We note:

\begin{equation}
\label{eq:493}
\mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a.\text{rad}(a)
\end{equation}

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph (Esterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 25. — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\begin{equation}
\label{eq:494}
c < K(\epsilon).\text{rad}^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon.
\end{equation}

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in $c$ giving $a, b$ with $c = a + b$. 
We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) [1]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) [3]:

**Conjecture 26.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(495) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]

After studying the \( abc \) conjecture using different choices of the constant \( K(\epsilon) \) and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that \( c < \text{rad}^2(abc) \) or \( c < \text{rad}^{1.63} \) is true, the \( abc \) conjecture does not hold when \( 0 < \epsilon < 1 \). Then the \( abc \) conjecture as it was defined is false.

In this note, I give a counter-example that the \( abc \) conjecture is not true, in the case \( \text{rad}(abc) < c \) taking \( \epsilon \in [0, 1] \) without assuming one of the two open questions : \( c < \text{rad}^2(abc) \) and \( c < \text{rad}^{1.63}(abc) \) that was proposed in 1996 by A. Nitaj [4].

The paper is organized as follows: in the second section, we give a counter-example that \( abc \) conjecture is false in the case \( \text{rad}(abc) < c \), choosing \( \epsilon \in [0, 1] \).

### 17.2. Proof the \( abc \) conjecture is false

We note \( R = \text{rad}(ac) \) in the case \( c = a + 1 \) (respectively \( R = \text{rad}(abc) \) if \( c = a + b \)).

#### 17.2.1. Case \( c < R \):

As \( c < R \implies c < R \implies c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0 \) since we choose \( K(\epsilon) \geq 1 \) and the conjecture (25) is verified.

#### 17.2.2. Case \( c = R \)

Case to reject as \( a, b, c \) (respectively \( a, c \)) are relatively prime.

#### 17.2.3. Case \( R < c \)

I will consider the case \( c = a + 1 \). I give the following counter example:

\[
8^n = 2^{3n} = (7 + 1)^n = 7^n + 7^{n-1}n + \ldots + 7n + 1 = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 \implies 
(496) \quad 2^{3n} = 7(7^{n-1} + n7^{n-2} + \ldots + n) + 1 
\]

We suppose that for \( n \) odd and large, the \( abc \) conjecture holds taking \( \epsilon = \epsilon_0 \in [0, 1] \).

Then \( \exists K(\epsilon_0) > 0 \) and:

\[
(497) \quad 2^{3n} < K(\epsilon_0)R^{1+\epsilon_0}
\]
We obtain:

\[ \text{rad}(c) = \text{rad}(2^{3n}) = 2 \]
\[ \text{rad}(a) = \text{rad}(7(7^{n-1} + n7^{n-2} + \ldots + n)) = 7.\text{rad}(7^{n-1} + n7^{n-2} + \ldots + n) \Rightarrow \]
\[ \text{rad}(a) \leq 7.(7^{n-1} + n7^{n-2} + \ldots + n) \leq n.7^n \Rightarrow \text{rad}(a) \leq 7^n \]

We re-write the equation (497) in detail:

\[ 2^{3n} < K(\epsilon_0)2^{1+\epsilon_0}n^{1+\epsilon_0}7^{n(1+\epsilon_0)} \]

That we can write as:

\[ 3n\log_2 \left( 1 - (1 + \epsilon_0)\frac{\log 7}{3\log 2} - \frac{1 + \epsilon_0 \log n}{3\log 2 \cdot n} \right) < K(\epsilon_0)2^{1+\epsilon_0} \]

We choose \( \epsilon_0 = 0.06 \) and we consider that \( n \) is very large \( (n \to +\infty) \), then we obtain:

\[ e^{3n\log_2(1 - 0.99193)} \leq K(0.06)2^{1.06} \Rightarrow +\infty \leq K(0.06)2^{1.06} \]

Hence the contradiction, and the \( abc \) conjecture is false for the value \( \epsilon_0 = 0.06 \).

We can announce the following theorems that are very easy to verify:

**Theorem 27. — (The truncated \( abc \) conjecture:)** For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), and assuming \( c < \text{rad}^2(abc) \) is true, then:

\[ c < K(\epsilon)\text{rad}^{1+\epsilon}(abc) \]

where \( K(\epsilon) \) is a constant depending of \( \epsilon \) proposed as:

\[ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon \geq 1 \]

and:

**Theorem 28. — (The truncated \( abc \) conjecture:)** For each \( \epsilon \geq 0.63 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), and assuming \( c < \text{rad}^{1.63}(abc) \) is true, then:

\[ c < K(\epsilon)\text{rad}^{1+\epsilon}(abc) \]

where \( K(\epsilon) \) is a constant depending of \( \epsilon \) proposed as:

\[ K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon \geq 0.63 \]

Ouf! The end of the mystery!

**Acknowledgements:** The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the \( abc \) conjecture.


CHAPTER 18

END OF THE MYSTERY OF THE abc CONJECTURE

To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract. — In this note, I give the proof that the abc conjecture is false giving a counterexample in the case $\epsilon \in ]0,1[$, because, in the case $c > \text{rad}(abc)$, for $0 < \epsilon < 1$ we cannot find the constant $K(\epsilon)$ so that $c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$ for $c$ very large.

18.1. Introduction

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

\begin{equation}
(504) \\
\quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i - 1}
\end{equation}

We note:

\begin{equation}
(505) \\
\quad \mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot \text{rad}(a)
\end{equation}

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of their sum. The definition of the abc conjecture is given below:

Conjecture 27. — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

\begin{equation}
(506) \\
\quad c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc), \quad K(\epsilon) \text{ depending only of } \epsilon
\end{equation}

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in November 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty
to find a proof of the \( abc \) conjecture is due to the incomprehensibility how the prime factors are organized in \( c \) giving \( a, b \) with \( c = a + b \).

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) [1]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) [3]:

**Conjecture 28.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(507) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]

After studying the \( abc \) conjecture using different choices of the constant \( K(\epsilon) \) and having attacked the problem from diverse angles, I have arrived to conclude that, assuming that \( c < \text{rad}^2(abc) \) or \( c < \text{rad}^{1.63} \) is true, the \( abc \) conjecture does not hold when \( 0 < \epsilon < 1 \). Then the \( abc \) conjecture as it was defined is false. In this note, I give a counterexample that the \( abc \) conjecture is not true, in the case \( \text{rad}(abc) < c \) taking \( \epsilon \in [0, 1] \) without assuming one of the two open questions: \( c < \text{rad}^2(abc) \) and \( c < \text{rad}^{1.63}(abc) \) that was proposed in 1996 by A. Nitaj [4].

The paper is organized as follows: in the second section, we give a counterexample that \( abc \) conjecture is false in the case \( \text{rad}(abc) < c \), choosing \( \epsilon \in [0, 1] \).

### 18.2. Proof the \( abc \) Conjecture is False

We note \( R = \text{rad}(ac) \) in the case \( c = a + 1 \) (respectively \( R = \text{rad}(abc) \) if \( c = a + b \)).

#### 18.2.1. Case \( c < R \)

As \( c < R \implies c < R \implies c < K(\epsilon).R^{1+\epsilon}, \forall \epsilon > 0 \) since we choose \( K(\epsilon) \geq 1 \) and the conjecture (27) is verified.

#### 18.2.2. Case \( c = R \)

Case to reject as \( a, b, c \) (respectively \( a, c \)) are relatively prime.

#### 18.2.3. Case \( R < c \)

I will consider the case \( c = a + 1 \). I give the following counterexample:

\[
(508) \quad c = \mu_c.\text{rad}(c) = 2^n + 1, n > n_0 > 0 \implies \text{rad}(a) = 2
\]

#### 18.2.3.1. Case \( \text{rad}(c) < \mu_c \)

In this case, we obtain as \( c = \mu_c.\text{rad}(c) < \mu_c^2 \implies \mu_c > (2^n + 1)^{1/2} > 2^{n/2} \) and \( c > \text{rad}^2(c) \implies \text{rad}(c) < (2^n + 1)^{1/2} \). We have also \( R < c \). We suppose that for
n odd and large, the abc conjecture holds taking $\epsilon = \epsilon_0 \in ]0, 1[$. Then $\exists K(\epsilon_0) > 0$ and:
\[
2^n + 1 < K(\epsilon_0)R^{1+\epsilon_0} \implies 2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0}{\text{rad}}^{1+\epsilon_0}(c) \implies \\
2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0}{\text{rad}}^{1+\epsilon_0}(c) < K(\epsilon_0)2^{1+\epsilon_0}(2^n + 1)^{1+\epsilon_0} \implies \\
(2^n + 1)^{1-\epsilon_0} < K(\epsilon_0)2^{1+\epsilon_0}
\]

We consider that $n$ is very large ($n \to +\infty$), then we obtain as $1 - \epsilon_0 > 0$:
\[
+ \infty \leq K(\epsilon_0)2^{1+\epsilon_0} < +\infty
\]
Hence the contradiction, and the abc conjecture is false when $\epsilon \in ]0, 1[$.

18.2.3.2. Case rad$(c) \geq \mu_c$:

$c = \mu_c, \text{rad}(c) \leq \text{rad}^2(c) \implies c < R^2$. We have also $c > R \implies \mu_c > 2$, if not, $\mu_c = 2 \implies (a, c) \neq 1$ then the contradiction with $a, c$ coprime; if $\mu_c = 1 \implies c = \text{rad}(c) < R$, case to reject, then $R < c \implies \mu_c \geq 3 \implies c \geq 3\text{rad}(c)$. Suppose that $c = 3\text{rad}(c)$, we can write $c = \text{rad}(c)^{1+\alpha}$ with $0 < \alpha < 1$ equal to
\[
\frac{\log 3}{\log(\text{rad}(c))}. \quad \text{Then as } \text{rad}(c) < c < \text{rad}^2(c) \implies \exists \alpha \in ]0, 1[ \text{ that we can take equal to } \frac{\log^c}{\log(\text{rad}(c))} < 1 \text{ so that } c \geq \text{rad}^{1+\alpha}(c) \text{ and } \alpha \text{ depends only of } c.
\]

We return to the abc conjecture considering our example giving by the equation (508). We suppose that for $n$ odd and large, the abc conjecture holds taking $\epsilon = \epsilon_0 \in ]0, 1[$. Then $\exists K(\epsilon_0) > 0$ and:
\[
2^n + 1 < K(\epsilon_0)R^{1+\epsilon_0} \implies 2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0}{\text{rad}}^{1+\epsilon_0}(c) \implies \\
2^n + 1 < K(\epsilon_0)2^{1+\epsilon_0}\left(c^{1+\epsilon_0}\right)^{1+\epsilon_0} \implies \\
(2^n + 1)^{\beta} < K(\epsilon_0)2^{1+\epsilon_0}
\]

We choose $\epsilon_0$ so that $\epsilon_0 < \alpha \implies \beta > 0$, for example $\epsilon_0 = \alpha/10 \implies \beta = 9\alpha/(10(1 + \alpha))$. Then if $n$ is very large, the parameter $\beta$ remains positive not zero and $2^n\beta$ becomes very large, it follows $(2^n + 1)^{\beta} < K(\epsilon_0)^22^{1+\epsilon_0}$, then the contradiction.

For our counterexample presented in the paper, in the two cases, we have found a contradiction in the application of the abc conjecture. Hence the abc conjecture is false.

However, we can announce the following theorems that are very easy to prove:
**Theorem 29. — (The truncated abc conjecture:)** For each $\epsilon \geq 1$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, and assuming $c < \text{rad}^2(abc)$ is true, then:

$$(513) \quad c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$$

where $K$ is a constant depending of $\epsilon$ proposed as:

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon \geq 1$$

and:

**Theorem 30. — (The truncated abc conjecture:)** For each $\epsilon \geq 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, and assuming $c < \text{rad}^{1.63}(abc)$ is true, then:

$$(514) \quad c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$$

where $K$ is a constant depending of $\epsilon$ proposed as:

$$K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \epsilon \geq 0.63$$

Ouf! The end of the mystery!

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BIBLIOGRAPHY


CHAPTER 19

PROGRESS IN THE PROOF OF THE CONJECTURE $c < \text{rad}^2(abc)$ - CASE : $c = a + 1$

Abstract. — In this paper, we consider the abc conjecture. We give some progress in the proof of the conjecture $c < \text{rad}^2(abc)$ in the case $c = a + 1$.

To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed
Mazen

19.1. Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i - 1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a.\text{rad}(a)$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) ([4]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 29. — (abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending only of $\epsilon$. 
CHAPTER 19. PROGRESS IN THE PROOF OF THE CONJECTURE \( c < \text{rad}^2(abc) \) - CASE : \( c = a+1 \)

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) ([2]). A conjecture was proposed that \( c < \text{rad}^2(abc) \) ([1]). Here we will give a proof of it for the case \( c = a + 1 \).

**Conjecture 30.** — Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(518) \quad c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2
\]

This result, I think is the key to obtain the final proof of the veracity of the \( abc \) conjecture.

### 19.2. A Proof of the conjecture (30) case \( c = a + 1 \)

Let \( a, c \) positive integers, relatively prime, with \( c = a + 1 \) and \( R = \text{rad}(ac) \),
\[
c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}, \beta_{j'} \geq 1.
\]

If \( c < \text{rad}(ac) \) then we obtain:

\[
(519) \quad c < \text{rad}(ac) < \text{rad}^2(ac) \implies c < R^2
\]

and the condition (518) is verified.

If \( c = \text{rad}(ac) \), then \( a, c \) are not coprime, case to reject.

In the following, we suppose that \( c > \text{rad}(ac) \) and \( c \) and \( a \) are not prime numbers.

\[
(520) \quad c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^2(ac)
\]

**19.2.1.** \( \mu_a \neq 1, \mu_a \leq \text{rad}(a) \)

We obtain :

\[
(521) \quad c = a + 1 < 2\mu_a \text{.rad}(a) \Rightarrow c < 2\text{rad}^2(a) \Rightarrow c < \text{rad}^2(ac) \implies c < R^2
\]

Then (520) is verified.

**19.2.2.** \( \mu_c \neq 1, \mu_c \leq \text{rad}(c) \)

We obtain :

\[
(522) \quad c = \mu_c \text{.rad}(c) \leq \text{rad}^2(c) < \text{rad}^2(ac) \implies c < R^2
\]

and the condition (520) is verified.
19.2.3. $\mu_a > \text{rad}(a)$ and $\mu_c > \text{rad}(c)$

19.2.3.1. **Case:** $\mu_a = \text{rad}^q(a)$, $q \geq 2$, $\mu_c = \text{rad}^p(c)$, $p \geq 2$:

In this case, we write $c = a + 1$ as $\text{rad}^{p+1}(c) - \text{rad}^{q+1}(a) = 1$. Then $\text{rad}(c), \text{rad}(a)$ are solutions of the Diophantine equation:

$$X^{p+1} - Y^{q+1} = 1 \quad \text{with} \ (p + 1)(q + 1) \geq 9$$

But the solutions of the equation (523) are $(X = \pm 3, p + 1 = 2, Y = \pm 2, q + 1 = 3)$, we obtain $p = 1 < 2$, then $\text{rad}(c), \text{rad}(a)$ are not solutions of (523) and the case $\mu_a = \text{rad}^q(a), q \geq 2$, $\mu_c = \text{rad}^p(c), p \geq 2$ is to reject.

19.2.3.2. **Case:** $\text{rad}(c) < \mu_c < \text{rad}^2(c)$ and $\text{rad}(a) < \mu_a < \text{rad}^2(a)$:

We can write:

$$\mu_c < \text{rad}^2(c) \implies c < \text{rad}^3(c)$$

$$\mu_a < \text{rad}^2(a) \implies a < \text{rad}^3(a)$$

$$\implies ac < R^3 \implies a^2 < ac < R^3 \implies a < R\sqrt{R} < R^2 \implies c = a + 1 < R^2$$

(524)

19.2.3.3. **Case:** $\mu_c > \text{rad}^2(c)$ or $\mu_a > \text{rad}^2(a)$

I- We suppose that $\mu_c > \text{rad}^2(c)$ and $\text{rad}(a) < \mu_a \leq \text{rad}^2(a)$:

I-1- Case $\text{rad}(a) < \text{rad}(c)$: In this case $a = \mu_a . \text{rad}(a) \leq \text{rad}^2(a) . \text{rad}(a) < \text{rad}^2(a) . \text{rad}(c) < \text{rad}^2(ac) \implies a < R^2 \implies c < R^2$.

I-2- Case $\text{rad}(c) < \text{rad}(a) < \text{rad}^2(c)$: As $a \leq \text{rad}^2(a) . \text{rad}(a) < \text{rad}^2(a) . \text{rad}^2(c) \implies a < R^2 \implies c < R^2$.

**Example:** $2^{30}5^2.127.353^2 = 3^75^5.13^5.17.1831 + 1$, $\text{rad}(c) = 2.5.127.353 = 448310$, $\text{rad}^2(c) = 200,981,856,100$,

$\mu_c = 2^{29}5.353 = 947,577,159,680 \implies \text{rad}^2(c) < \mu_c < \text{rad}^3(c)$,

$\text{rad}(a) = 3.5.13.17.1831 = 6,069,765$, $\text{rad}^2(a) = 36,842,047,155,225$,

$\mu_a = 3^6.5^4.13^4 = 130,131,051,255 < \text{rad}^2(a)$. It is the case: $\text{rad}(c) < \mu_c < \text{rad}^2(c)$ and $\text{rad}(a) < \mu_a \leq \text{rad}^2(a)$ with $\text{rad}(c) = 448310 < \text{rad}(a) = 6,069,765 < \text{rad}^2(c) = 200,981,856,100$.

I-3- Case $\text{rad}^2(c) < \text{rad}(a)$:

I-3-1- We suppose that $c \leq \text{rad}^6(c)$, we obtain:

$$c \leq \text{rad}^6(c) \implies c \leq \text{rad}^2(c) . \text{rad}^4(c) \implies c < \text{rad}^2(c) . (\text{rad}(a))^2 = R^2 \implies c < R^2$$
Example: $5^87^2 = 2^43^7547 + 1 \implies 19140625 = 19140624 + 1$, $\text{rad}(c) = 5.7 = 35, \text{rad}(a) = 2.3547 = 3282 \implies \text{rad}(a) > \text{rad}^2(c)$, we obtain $c = 19140625 > \text{rad}^3(c) = 42875$ and $c < \text{rad}^6(c) = 1838265625$ and $3282 = \text{rad}(a) < \mu_a = 5832 < \text{rad}^5(a) = 10771524 \implies a < \text{rad}^5(a) = 35352141768$.

I-3-2. We suppose $c > \text{rad}^6(c) \implies \mu_c > \text{rad}^5(c)$, we suppose $\mu_a = \text{rad}^2(a) \implies a = \text{rad}^2(a)$. Then we obtain that $X = \text{rad}(a)$ is a solution in positive integers of the equation:

\[(525) \quad X^3 + 1 = c = \mu_c \cdot \text{rad}(c)\]

If $c = \text{rad}^n(c)$ with $n \geq 7$, we obtain an equation like (523) that gives a contradiction. In the following, we will study the cases $\mu_c = A \cdot \text{rad}^n(c)$ with $\text{rad}(c) \nmid A, n \geq 0$. The above equation (525) can be written as:

\[(526) \quad (X + 1)(X^2 - X + 1) = c\]

Let $\delta$ any divisor of $c$, then:

\[(527) \quad X + 1 = \delta\]
\[(528) \quad X^2 - X + 1 = \frac{c}{\delta} = c' = \delta^2 - 3X\]

We recall that $\text{rad}(a) > \text{rad}^2(c)$, it follows that $\delta$ must verifies $\delta - 1 > \text{rad}^2(c) \implies \delta > \text{rad}^2(c) + 1$.

I-3-2-1- We suppose that $\delta = l \cdot \text{rad}(c) \implies l \cdot \text{rad}(c) > \text{rad}^2(c) + 1 \implies l > \frac{\text{rad}^2(c) + 1}{\text{rad}(c)}$.

We obtain $l \geq \text{rad}(c) + 2$ so $\text{rad}(c)$ and $l$ have the same parity. We have $\delta = l \cdot \text{rad}(c) < c = \mu_c \cdot \text{rad}(c) \implies l < \mu_c$. As $\delta$ is a divisor of $c$, then $l$ is a divisor of $\mu_c$, we write $\mu_c = l \cdot m$. From $\mu_c = l(\delta^2 - 3X)$, we obtain:

$m = l^2 \cdot \text{rad}^2(c) - 3\text{rad}(a) \implies 3\text{rad}(a) = l^2 \cdot \text{rad}^2(c) - m$

A- Case $3|m \implies m = 3m'$, $m' > 1$: As $\mu_c = ml = 3m'l \implies 3|\text{rad}(c)$ and $(\text{rad}(c), m')$ not coprime. We obtain:

\[\text{rad}(a) = l^2 \cdot \text{rad}(c) \cdot \frac{\text{rad}(c)}{3} - m'\]

It follows that $a, c$ are not coprime, then the contradiction.

B - Case $m = 3 \implies \mu_c = 3l \implies c = 3l \cdot \text{rad}(c) = 3\delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3\delta \implies \delta = \text{rad}(c) = 3$, then the contradiction.

I-3-2-2 - We suppose that $\delta = l \cdot \text{rad}^2(c), l \geq 2$. In this case $\text{rad}(a) = l \cdot \text{rad}^2(c) - 1$ verifies $\text{rad}(a) > \text{rad}^2(c)$. If $l \cdot \text{rad}(c) \nmid \mu_c$ then the case to reject. We suppose that
lrad(c)|\mu_c \implies \mu_c = m.lrad(c), then \(\frac{c}{\delta} = m = \delta^2 - 3rad(a)\).

C - Case \(m = 1\): \(c/\delta \implies \delta^2 - 3rad(a) = 1 \implies (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \implies \delta = 2 = l.rad^2(c)\), then the contradiction.

D - Case \(m = 3\), we obtain \(3(1 + rad(a)) = \delta^2 = 3\delta \implies \delta = 3 = l.rad^2(c)\). Then the contradiction.

E - Case \(m \neq 1,3\), we obtain: \(3rad(a) = l^2 rad^4(c) - m \implies rad(a)\) and \(rad(c)\) are not coprime. Then the contradiction.

I-3-2-3 - We suppose that \(\delta = l.rad^n(c), l \geq 2\) with \(n \geq 3\). From \(c = \mu_c.rad(c) = l.rad^n(c)(\delta^2 - 3rad(a))\), let \(m = \delta^2 - 3rad(a)\).

F - As seen above (paragraphs C,D), the cases \(m = 1\) and \(m = 3\) give contradictions, it follows the reject of these cases.

G - Case \(m \neq 1,3\). Let \(q\) a prime that divides \(m\), it follows \(q|\mu_c \implies q = c_{j_0}^\beta \implies c_{j_0}^\beta |\delta^2 \implies c_{j_0}^\beta |3rad(a)\). Then \(rad(a)\) and \(rad(c)\) are not coprime. It follows the contradiction.

I-3-2-4 - We suppose that \(\delta = \prod_{j \in J_1} c_j^{\beta_j}, \beta_j \geq 1\) with at least one \(j_0 \in J_1\) with \(\beta_{j_0} \geq 2\), \(rad(c) \parallel \delta\) and \(\delta - 1 = \prod_{j \in J_1} c_j^{\beta_j} - 1 > rad^2(c) = \prod_{j' \in J'} c_{j'}^{\beta_{j'}}, J_1 \subset J'\). We can write:

\[
\delta = \mu_5.rad(\delta), \quad rad(c) = m.rad(\delta)
\]

Then we obtain:

\[
c = \mu_c.rad(c) = \mu_c.m.rad(\delta) = \delta(\delta^2 - 3X) = \mu_5.rad(\delta)(\delta^2 - 3X) \implies m.lrad(c) = \mu_5(\delta^2 - 3X)
\]

- If \(\mu_c = \mu_5 \implies m = \delta^2 - 3X = (\mu_c.rad(\delta))^2 - 3X\). As \(\delta < \delta^2 - 3X \implies m > \delta \implies rad(c) > m > \mu_c.rad(\delta) > rad(\delta)\) because \(\mu_c > rad(\delta)\), it follows \(rad(c) > rad(\delta)\). Then the contradiction.

- We suppose that \(\mu_c < \mu_5\). As \(rad(a) = \mu_5 rad(\delta) - 1\), we obtain:

\[
rad(a) > \mu_c.rad(\delta) - 1 > 0 \implies R > c.rad(\delta) - rad(c) > 0 \implies c > R > c.rad(\delta) - rad(c) > 0 \implies 1 > rad(\delta) - \frac{rad(c)}{c} > 0, \quad rad(\delta) \geq 2
\]

(530) \(\implies \) The contradiction
CHAPTER 19. PROGRESS IN THE PROOF OF THE CONJECTURE \( c < \text{rad}^2(abc) \) - CASE : \( c = a + 1 \)

- We suppose that \( \mu_5 < \mu_c \). In this case, from the equation (529) and as \( (m, \mu_5) = 1 \), it follows that we can write:

\[
\mu_c = \mu_1 \mu_2, \quad \mu_1, \mu_2 > 1
\]

(531)

so that \( m \mu_1 = \delta^2 - 3X \), \( \mu_2 = \mu_5 \)

But:

\[
\text{rad}(a) = \delta - 1 = \mu_5 \text{rad}(\delta) - 1 > \text{rad}^2(c) \implies 0 > m^2 \text{rad}^2(\delta) - \mu_2 \text{rad}(\delta) + 1
\]

Let \( P(Z) \) the polynomial:

\[
P(Z) = m^2 Z^2 - \mu_2 Z + 1 \implies P(\text{rad}(\delta)) < 0
\]

The discriminant of \( P(Z) \) is:

\[
\Delta = \mu_2^2 - 4m^2
\]

- \( \Delta = 0 \implies \mu_2 = 2m \), but \( (m, \mu_2) = 1 \), then the contradiction. Case to reject.

- \( \Delta < 0 \implies P(Z) \) has no real roots. From (533) it follows that \( P(Z) > 0, \forall Z \in \mathbb{R} \).

Then the contradiction with \( P(\text{rad}(\delta)) < 0 \). Case to reject.

- \( \Delta > 0 \implies \mu_2 > 2m \implies \frac{\mu_2}{m} > 2 \). We denote \( t = \sqrt{\Delta} > 0 \). The roots of \( P(Z) = 0 \) are \( Z_1, Z_2 \) with \( Z_1 < Z_2 \), given by:

\[
Z_1 = \frac{\mu_2 - t}{2m^2}, \quad Z_2 = \frac{\mu_2 + t}{2m^2}
\]

(535)

We approximate \( t \) by \( \tilde{t} \):

\[
t = \sqrt{\mu_2^2 - 4m^2} = \mu_2 \left( 1 - \frac{4m^2}{\mu_2^2} \right)^{\frac{1}{2}} \implies \tilde{t} = \mu_2 - \frac{2m^2}{\mu_2} > 0
\]

Then, we obtain \( \tilde{Z}_1, \tilde{Z}_2 \) as:

\[
\tilde{Z}_1 = \frac{\mu_2 - \tilde{t}}{2m^2} = \frac{1}{\mu_2}, \quad \tilde{Z}_2 = \frac{\mu_2 + \tilde{t}}{2m^2} = \frac{\mu_2}{m^2} - \frac{1}{\mu_2}
\]

As \( \mu_2^2 - 4m^2 > 0 \implies \mu_2^2 - m^2 > 3m^2 > 0 \implies \frac{\mu_2}{m^2} - 1 > 0 \), we will give below the proof that \( \text{rad}(\delta) > \tilde{Z}_2 \implies P(\text{rad}(\delta)) > 0 \), then the contradiction with \( P(\text{rad}(\delta)) < 0 \); we write:

\[
\text{rad}(\delta) > \frac{\mu_2}{m^2} - \frac{1}{\mu_2}, \quad \mu_2 > 0 \implies
\]

\[
\mu_2 \text{rad}(\delta) > \frac{\mu_2^2}{m^2} - 1
\]

\[
\mu_2 \text{rad}(\delta) > \frac{\mu_2^2 - m^2}{m^2} > \frac{3m^2}{m^2}
\]

(537) as \( \delta > 3 \implies \delta > \frac{\mu_2^2}{m^2} - 1 > 3 \implies \text{rad}(\delta) > \frac{\mu_2}{m^2} - \frac{1}{\mu_2} > \frac{3}{\mu_2} \)
If follows $P(rad(\delta)) > 0$ and the contradiction with the conclusion of the equation (533).

It follows that the case $c > rad^6(c)$ and $a = rad^3(a)$ is impossible.

I-3-3- We suppose $c > rad^6(c)$, then $c = rad^6(c) + h, h > 0$ and $\mu_a < rad^2(a) \implies a + l = rad^3(a), l > 0$. Then we obtain:

\[ (538) \quad rad^6(c) + h = rad^3(a) - l + 1 \]

As $rad^2(c) < rad(a)$ (see I-3), we obtain the equation:

\[ rad^3(a) - (rad^2(c))^3 = h + l - 1 = m > 0 \]

Let $X = rad(a) - rad^2(c)$, then $X$ is an integer root of the polynomial $H(X)$ defined as:

\[ (539) \quad H(X) = X^3 + 3R.rad(c)X - m = 0 \]

To resolve the above equation, we note $X = u + v$, then we obtain the two conditions:

\[ u^3 + v^3 = m, \quad u.v = -R.rad(c) < 0 \implies u.v = -R^3rad^3(c) \]

It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by:

\[ (540) \quad G(t) = t^2 - mt - R^3rad^3(c) = 0 \]

The discriminant of $G(t)$ is:

\[ (541) \quad \Delta = m^2 + 4R^3rad^3(c) = \alpha^2, \quad \alpha > 0 \]

The two real roots of (540) are:

\[ (542) \quad t_1 = u^3 = \frac{m + \alpha}{2} \]

\[ (543) \quad t_2 = v^3 = \frac{m - \alpha}{2} \]

As $m = rad^3(a) - rad^6(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^6(c) > 0$, then from the equation (541), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

\[ (544) \quad x^2 - y^2 = N \]

with $N = 4R^3rad^3(c) > 0$. From the equations (542-543), we remark that $\alpha$ and $m$ verify the following equations:

\[ (545) \quad x + y = 2u^3 = 2rad^3(a) \]
\[ (546) \quad x - y = -2v^3 = 2rad^3(c) \]
\[ (547) \quad \text{then} \quad x^2 - y^2 = N = 4R^3rad^3(c) \]
Let $Q(N)$ be the number of solutions of (544) and $\tau(N)$ is the number of suitable factorization of $N$, then we announce the following result concerning the solutions of the Diophantine equation (544) (see theorem 27.3 in [3]):

- If $N \equiv 2(\text{mod } 4)$, then $Q(N) = 0$.
- If $N \equiv 1$ or $N \equiv 3(\text{mod } 4)$, then $Q(N) = [\tau(N)/2]$.
- If $N \equiv 0(\text{mod } 4)$, then $Q(N) = [\tau(N/4)/2]$.

$[x]$ is the integral part of $x$ for which $[x] \leq x < [x] + 1$.

Let $(\alpha', m')$, $\alpha', m' \in \mathbb{N}^*$ be another pair, solution of the equation (544), then

$$\alpha^2 - m^2 = x^2 - y^2 = N = 4R^3\text{rad}^3(c),$$

but $a = x$ and $m = y$ verify the equation (545) given by $x + y = 2\text{rad}^3(a)$, it follows $\alpha', m'$ verify also $\alpha' + m' = 2\text{rad}^3(a)$, that gives $\alpha' - m' = 2\text{rad}^3(c)$, then $\alpha' = x = \alpha = \text{rad}^3(a) + \text{rad}^6(c)$ and $m' = y = m = \text{rad}^3(a) - \text{rad}^6(c)$. We have given the proof of the uniqueness of the solutions of the equation (544) with the condition $x + y = 2\text{rad}^3(a)$. As $N = 4R^3\text{rad}^3(c) \equiv 0(\text{mod } 4) \Rightarrow Q(N) = [\tau(N/4)/2] = [\tau(\text{rad}^6(c).\text{rad}^3(a))/2] > 1$. But $Q(N) = 1$, then the contradiction.

It follows that the case $\mu_a \leq \text{rad}^2(a)$ and $c > \text{rad}^6(c)$ is impossible.

II- We suppose that $\text{rad}(c) < \mu_c \leq \text{rad}^2(c)$ and $\mu_a > \text{rad}^5(a)$:

II-1- Case $\text{rad}(c) < \text{rad}(a)$ : As $c \leq \text{rad}^3(c) = \text{rad}^2(c).\text{rad}(c) \Rightarrow c < \text{rad}^2(c).\text{rad}(a)$.

II-2- Case $\text{rad}(a) < \text{rad}(c) < \text{rad}^2(a)$ : As $c \leq \text{rad}^3(c) = \text{rad}^2(c).\text{rad}(c) \Rightarrow c < \text{rad}^2(c).\text{rad}^2(a)$.

II-3- Case $\text{rad}^2(a) < \text{rad}(c)$:

II-3-1- We suppose that $a \leq \text{rad}^6(a) \Rightarrow a < \text{rad}^2(a).\text{rad}^4(a) \Rightarrow a < \text{rad}^2(a).\text{rad}(c)^2 = R^2 \Rightarrow a < R^2 \Rightarrow 1 + a \leq R^2$, but $(c,a) = 1$, it follows $c < R^2$.

II-3-2- We suppose $a > \text{rad}^6(a)$ and $\mu_c \leq \text{rad}^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a,c), we arrive at a contradiction. It follows that the case $\mu_c \leq \text{rad}^2(c)$ and $a > \text{rad}^6(a)$ is impossible.
19.2.3.4. **III - Case** $\mu_c > \text{rad}^3(c)$ and $\mu_a > \text{rad}^3(a)$

We can write $c > \text{rad}^3(c) \implies c = \text{rad}^3(c) + h$ and $a = \text{rad}^3(a) + l$ with $h, l > 0$ positive integers.

**III-1** - We suppose $\text{rad}(a) < \text{rad}(c)$. We obtain the equation:

(548) $\text{rad}^3(c) - \text{rad}^3(a) = l - h + 1 = m > 0$

Let $X = \text{rad}(c) - \text{rad}(a)$, from the above equation, $X$ is a real root of the polynomial:

(549) $P(X) = X^3 + 3RX - m = 0$

As above, to resolve (618), we put $X = u + v$, then we obtain the two conditions:

(550) $u^3 + v^3 = m$

(551) $uv = -R < 0 \implies u^3.v^3 = -R^3$

Then $u^3, v^3$ are the roots of the equation:

(552) $H(Z) = Z^2 - mZ - R^3 = 0$

The discriminant of $H(Z)$ is:

(553) $\Delta = m^2 + 4R^3 = (\text{rad}^3(c) + \text{rad}^3(a))^2 = \alpha^2$, taking $\alpha > 0 \implies \alpha = \text{rad}^3(c) + \text{rad}^3(a)$

From the equation (676), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

(554) $x^2 - y^2 = N$

with $N = 4R^3 > 0$ and $N \equiv 0 \pmod{4}$. Using the same method as in I-3-3-, we arrive to a contradiction.

**III-2** - We suppose $\text{rad}(c) < \text{rad}(a)$. We obtain the equation:

(555) $\text{rad}^3(a) - \text{rad}^3(c) = h - l - 1 = m > 0$

Let $X = \text{rad}(a) - \text{rad}(c)$, from the above equation, $X$ is a real root of the polynomial:

(556) $P(X) = X^3 + 3RX - m = 0$

As above, to resolve (618), we put $X = u + v$, then we obtain the two conditions:

(557) $u^3 + v^3 = m$

(558) $uv = -R < 0 \implies u^3.v^3 = -R^3$

Then $u^3, v^3$ are the roots of the equation:

(559) $H(Z) = Z^2 - mZ - R^3 = 0$

The discriminant of $H(Z)$ is:

(560) $\Delta = m^2 + 4R^3 = (\text{rad}^3(c) + \text{rad}^3(a))^2 = \alpha^2$, taking $\alpha > 0 \implies \alpha = \text{rad}^3(c) + \text{rad}^3(a)$
CHAPTER 19. PROGRESS IN THE PROOF OF THE CONJECTURE $c < \text{rad}^2(abc)$ - CASE: $c = a + 1$

From the equation (622), we obtain that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$x^2 - y^2 = N$$

with $N = 4R^3 > 0$ and $N \equiv 0(\mod 4)$. Using the same method as in I-3-3-, we arrive to a contradiction.

It follows that the case $\mu_c > \text{rad}^2(c)$ and $\mu_a > \text{rad}^2(a)$ is impossible.

We can announce the following theorem:

Theorem 31. — (Abdelmajid Ben Hadj Salem, 2020) Let $a, c$ positive integers relatively prime with $c = a + 1$, then $c < \text{rad}^2(ac)$. 


CHAPTER 20

THE PROOF OF THE CONJECTURE \( c < \text{rad}^{1.63}(ac) \)

Abstract. — In this paper, we consider the abc conjecture. We give the proof of the conjecture \( c < \text{rad}^{1.63}(abc) \) in the case \( c = a + 1 \).

To my teachers and to Madam Fatma Moalla, Mr Chedly Touibi
my professors of mathematics at the Faculty of Sciences of Tunis

20.1. Introduction and notations

Let \( a \) be a positive integer, \( a = \prod_i a_i^{\alpha_i}, a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \prod_i a_i^{\alpha_i - 1}
\]

We denote:

\[
\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a \cdot \text{rad}(a)
\]

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [7]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 31. — (abc Conjecture): For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then :

\[
c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

where \( K \) is a constant depending only of \( \epsilon \).
We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 [5] \). It concerned the best example given by E. Reyssat [5]:

\[
2 + 3^{10}.109 = 23^5 \implies c < \text{rad}^{1.629912}(abc)
\]

A conjecture was proposed that \( c < \text{rad}^{0}(abc) \) [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

**Conjecture 32.** — Let \( a, b, c \) be positive integers relatively prime with \( c = a + b \), then:

\[
\begin{align*}
(566) & \quad c < \text{rad}^{1.63}(abc) \\
(567) & \quad abc < \text{rad}^{1.42}(abc)
\end{align*}
\]

Here, we will give a proof of the conjecture given by (628) in the case \( c = a + 1 \). This result, I think is the key to obtain the final proof of the veracity of the \( abc \) conjecture especially when \( 0 < \epsilon < 1 \).

### 20.2. A Proof of the conjecture (32) case \( c = a + 1 \)

Let \( a, c \) be positive integers, relatively prime, with \( c = a + 1 \) and \( R = \text{rad}(ac) \),
\[ c = \prod_{j' \in \mathcal{J}} c_{j'}^{j'} \beta_{j'} \geq 1. \]

If \( c < \text{rad}(ac) \), then we obtain:

\[
(568) \quad c < \text{rad}(ac) < \text{rad}^{1.63}(ac) \implies c < R^{1.63}
\]

and the condition (628) is satisfied.

If \( c = \text{rad}(ac) \), then \( a, c \) are not coprime, case to reject.

In the following, we suppose that \( c > \text{rad}(ac) \) and \( c \) and \( a \) are not prime numbers.

\[
(569) \quad c = a + 1 = \mu_a \text{rad}(a) + 1 < \text{rad}^{1.63}(ac)
\]

#### 20.2.1. \( \mu_a \neq 1, \mu_a \leq \text{rad}^{0.63}(a) \)

We obtain:

\[
(570) \quad c = a + 1 \leq \text{rad}^{1.63}(a) + 1 < \text{rad}^{1.63}(ac) \implies c < \text{rad}^{1.63}(ac) \implies c < R^{1.63}
\]

Then (631) is satisfied.
20.2.2. \( \mu_c \neq 1, \mu_c \leq \text{rad}^{0.63}(c) \)

We obtain:

\[
(571) \quad c = \mu_c \text{rad}(c) \leq \text{rad}^{1.63}(c) < \text{rad}^{1.63}(ac) \implies c < R^{1.63}
\]

and the condition (631) is satisfied.

20.2.3. \( \mu_a > \text{rad}^{0.63}(a) \) and \( \mu_c > \text{rad}^{0.63}(c) \)

20.2.3.1. Case: \( \mu_a = \text{rad}^{P}(a), q \geq 2, \mu_c = \text{rad}^{P}(c), p \geq 2; \)

In this case, we write \( c = a + 1 \) as \( \text{rad}^{q+1}(c) - \text{rad}^{q+1}(a) = 1 \). Then \( \text{rad}(c), \text{rad}(a) \) are solutions of the Diophantine equation:

\[
(572) \quad X^{p+1} - Y^{q+1} = 1, \quad \text{with} \ (p + 1)(q + 1) \geq 9
\]

But the solutions of the equation (572) are \([2] \ (X = \pm 3, p + 1 = 2, Y = +2, q + 1 = 3)\), we obtain \( p = 1 < 2 \), then \( \text{rad}(c), \text{rad}(a) \) are not solutions of (572) and the case \( \mu_a = \text{rad}^{P}(a), q \geq 2, \mu_c = \text{rad}^{P}(c), p \geq 2 \) is to reject.

20.2.3.2. Case: \( \text{rad}^{0.63}(c) < \mu_c \leq \text{rad}^{1.63}(c) \) and \( \text{rad}^{0.63}(a) < \mu_a \leq \text{rad}^{1.63}(a) \):

We can write:

\[
\mu_c \leq \text{rad}^{1.63}(c) \implies c \leq \text{rad}^{2.63}(c) \] \( \mu_a \leq \text{rad}^{1.63}(a) \implies a \leq \text{rad}^{2.63}(a) \) \[ \implies ac \leq R^{2.63} \implies a^2 < ac \leq R^{2.63} \implies a < R^{1.315} < R^{1.63} \implies c = a + 1 < R^{1.63}
\]

20.2.3.3. Case: \( \mu_c > \text{rad}^{1.63}(c) \) or \( \mu_a > \text{rad}^{1.63}(a) \)

I- We suppose that \( \mu_c > \text{rad}^{1.63}(c) \) and \( \mu_a \leq \text{rad}^{2}(a) \):

I-1- Case \( \text{rad}(a) < \text{rad}(c) \]): In this case \( a = \mu_a \text{rad}(a) \leq \text{rad}^{P}(a) \leq \text{rad}^{1.63}(a) \text{rad}^{1.37}(a) < \text{rad}^{1.63}(a) \text{rad}^{1.37}(c) < \text{rad}^{1.63}(ac) \implies a < R^{1.63} \implies [c < R^{1.63}].

I-2- Case \( \text{rad}(c) < \text{rad}(a) < \text{rad}^{1.63}(c) \): As \( a \leq \text{rad}^{1.63}(a) \text{rad}^{1.37}(a) < \text{rad}^{1.63}(a) \text{rad}^{1.63}(c) \implies a < R^{1.63} \implies [c < R^{1.63}].

I-3- Case \( \text{rad}^{1.63}(c) < \text{rad}(a) \):

I-3-1- We suppose \( c \leq \text{rad}^{3.26}(c), \) we obtain:

\[
c \leq \text{rad}^{3.26}(c) \implies c \leq \text{rad}^{1.63}(c) \text{rad}^{1.63}(c) \implies c < \text{rad}^{1.63}(c) \text{rad}(a)^{1.37} < R^{1.63} \implies [c < R^{1.63}]
\]
I-3-2- We suppose \( c > \text{rad}^{2.26}(c) \implies \mu_c > \text{rad}^{2.26}(c) \). We consider the case \( \mu_a = \text{rad}^2(a) \implies a = \text{rad}^3(a) \). Then, we obtain that \( X = \text{rad}(a) \) is a solution in positive integers of the equation:

\[
X^3 + 1 = c = \mu_c \cdot \text{rad}(c)
\]

If \( c = \text{rad}^n(c) \) with \( n \geq 4 \), we obtain an equation like \((572)\) that gives a contradiction. In the following, we will study the cases \( \mu_c = A \cdot \text{rad}^n(c) \) with \( \text{rad}(c) \nmid A, n \geq 0 \). The above equation \((635)\) can be written as:

\[
(X + 1)(X^2 - X + 1) = c
\]

Let \( \delta \) any divisor of \( c \), then:

\[
X^2 - X + 1 = \frac{c}{\delta} = \delta' = \delta^2 - 3X
\]

I-3-2-1- We suppose \( \delta = l \cdot \text{rad}(c) \). We obtain \( \delta = l \cdot \text{rad}(c) < c = \mu_c \cdot \text{rad}(c) \implies l < \mu_c \). As \( \delta \) is a divisor of \( c \), then \( l \) is a divisor of \( \mu_c \), we write \( \mu_c = l \cdot m \). From \( \mu_c = l(\delta^2 - 3X) \), we obtain:

\[
m = l^2 \text{rad}^2(c) - 3 \text{rad}(a) \implies 3 \text{rad}(a) = l^2 \text{rad}^2(c) - m
\]

A- Case \( 3 \mid m \implies m = 3m', m' > 1 \): As \( \mu_c = ml = 3m'l \implies 3 \mid \text{rad}(c) \) and \((\text{rad}(c), m')\) not coprime. We obtain:

\[
\text{rad}(a) = l^2 \text{rad}(c) \cdot \frac{\text{rad}(c)}{3} - m'
\]

It follows that \( a, c \) are not coprime, then the contradiction.

\[
\text{B - Case } m = 3 \implies \mu_c = 3l \implies c = 3l \text{rad}(c) = 3 \delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3 \delta \implies \delta = l \cdot \text{rad}(c) = 3, \text{ then the contradiction.}
\]

I-3-2-2- We suppose \( \delta = l \cdot \text{rad}^2(c), l \geq 2 \). If \( \text{rad}(c) \nmid \mu_c \) then the case is to reject. We suppose \( l \text{rad}(c) \mid \mu_c \implies \mu_c = m \cdot \text{rad}(c) \), then \( \frac{c}{\delta} = m = \delta^2 - 3 \text{rad}(a) \).

C- Case \( m = 1 = c/\delta \implies \delta^2 - 3 \text{rad}(a) = 1 \implies (\delta - 1)(\delta + 1) = 3 \text{rad}(a) = \text{rad}(a)(\delta + 1) \implies \delta = 2 = l \cdot \text{rad}^2(c) \), then the contradiction.

D- Case \( m = 3 \), we obtain \( 3(1 + \text{rad}(a)) = \delta^2 = 3 \delta \implies \delta = 3 = l \cdot \text{rad}^2(c) \). Then the contradiction.

E- Case \( m \neq 1, 3 \), we obtain: \( 3 \text{rad}(a) = l^2 \text{rad}^4(c) - m \implies \text{rad}(a) \) and \( \text{rad}(c) \) are not coprime. Then the contradiction.
I-3-2-3. We suppose \( \delta = l \cdot \text{rad}^n(c), l \geq 2 \) with \( n \geq 3 \). From \( c = \mu_c \cdot \text{rad}(c) = l \cdot \text{rad}^n(c)(\delta^2 - 3 \cdot \text{rad}(a)), \) we denote \( m = \delta^2 - 3 \cdot \text{rad}(a) = \delta^2 - 3X. \)

F - As seen above (paragraphs C.D), the cases \( m = 1 \) and \( m = 3 \) give contradictions, it follows the reject of these cases.

G - Case \( m \neq 1, 3 \). Let \( q \) be a prime that divides \( m \), it follows \( q | \mu_c \implies q = c_{j_0} \implies c_{j_0} | \delta^2 \implies c_{j_0} | 3 \cdot \text{rad}(a). \) Then \( \text{rad}(a) \) and \( \text{rad}(c) \) are not coprime. It follows the contradiction.

I-3-2-4. We suppose \( \delta = \prod_{j \in J_1} c_j^{\beta_j}, \) \( \beta_j \geq 1 \) with at least one \( j_0 \in J_1 \) with \( \beta_{j_0} \geq 2, \) \( \text{rad}(c) \upharpoonright \delta. \) We can write:

\[
(580) \quad \delta = \mu_\delta \cdot \text{rad}(\delta), \quad \text{rad}(c) = m \cdot \text{rad}(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1
\]

Then, we obtain:

\[
(581) \quad c = \mu_c \cdot \text{rad}(c) = \mu_c \cdot m \cdot \text{rad}(\delta) = \delta(\delta^2 - 3X) = \mu_\delta \cdot \text{rad}(\delta)(\delta^2 - 3X) \implies m \cdot \mu_c = \mu_\delta (\delta^2 - 3X)
\]

- If \( \mu_c = \mu_\delta \implies m = \delta^2 - 3X = (\mu_c \cdot \text{rad}(\delta))^2 - 3X. \) As \( \delta < \delta^2 - 3X \implies m > \delta \implies \text{rad}(c) > m > \mu_c \cdot \text{rad}(\delta) > \text{rad}^2(c) \) because \( \mu_c > \text{rad}^2(c), \) (see I-3-2), it follows \( \text{rad}(c) > \text{rad}^2(c). \) Then the contradiction.

- We suppose \( \mu_c < \mu_\delta. \) As \( \text{rad}(a) = \mu_\delta \cdot \text{rad}(\delta) - 1, \) we obtain:

\[
(582) \quad c > R > c \cdot \text{rad}(\delta) - \text{rad}(c) > 0 \implies c > R > c \cdot \text{rad}(\delta) - \text{rad}(c) > 0 \implies 1 > \frac{\text{rad}(c)}{c} > 0, \quad \text{rad}(\delta) \geq 2
\]

\[
(583) \quad \text{The contradiction}
\]

- We suppose \( \mu_\delta < \mu_c. \) In this case, from the equation (641) and as \( (m, \mu_\delta) = 1, \) it follows we can write:

\[
(584) \quad \delta^2 - 3X = \mu_1 \cdot \mu_2, \quad \mu_1, \mu_2 > 1
\]

\[
(585) \quad c = \mu_c \cdot \text{rad}(c) = \mu_1 \cdot \mu_2 \cdot \text{rad}(\delta).m = \delta(\delta^2 - 3X)
\]

\[
(586) \quad \text{so that} \quad m, \mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \implies \delta = \mu_\delta \cdot \text{rad}(\delta)
\]

** We suppose \( \mu_1, \mu_2 \neq 1, \) then \( \exists c_{j_0} \) so that \( c_{j_0} | \mu_1 \) and \( c_{j_0} | \mu_2. \) But \( \mu_\delta = \mu_2 \implies c_{j_0}^2 | \delta. \) From \( 3X = \delta^2 - m \mu_1 \implies c_{j_0} | 3X \implies c_{j_0} | X \) or \( c_{j_0} = 3. \)

- If \( c_{j_0} | X, \) it follows the contradiction with \( (c, a) = 1. \)

- If \( c_{j_0} = 3. \) We have \( m \mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \implies \delta^2 - 3\delta + 3 - m \mu_1 = 0. \) As \( 3 | \mu_1 \implies \mu_1 = 3^k \mu_1', 3 \nmid \mu_1', k \geq 1, \) we obtain:

\[
(587) \quad \delta^2 - 3\delta + 3(1 - 3^{k-1}m \mu_1') = 0
\]
- We consider the case \( k > 1 \Rightarrow 3 \nmid (1 - 3^{k-1}m\mu_1) \). Let us recall the Eisenstein criterion [1]:

**Theorem 32.** — *(Eisenstein Criterion)* Let \( f = a_0 + \cdots + a_nX^n \) be a polynomial \( \in \mathbb{Z}[X] \). We suppose that \( \exists p \) a prime number so that:
- \( p \nmid a_n \)
- \( p | a_i \), \( (0 \leq i \leq n - 1) \)
- \( p^2 \nmid a_0 \)

Then \( f \) is irreducible in \( \mathbb{Q} \).

We apply Eisenstein criterion to the polynomial \( R(Z) \) given by:

\[
R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu_1)
\]

then:
- \( 3 \nmid 1 \)
- \( 3 |(-3) \)
- \( 3 |(1 - 3^{k-1}m\mu_1) \)
- \( 3^2 \nmid 3(1 - 3^{k-1}m\mu_1) \)

It follows that the polynomial \( R(Z) \) is irreducible in \( \mathbb{Q} \), then, the contradiction with \( R(\delta) = 0 \).

- We consider the case \( k = 1 \), then \( \mu_1 = 3\mu_1' \) and \( (\mu_1', 3) = 1 \), we obtain:

\[
\delta^2 - 3\delta + 3(1 - m\mu_1') = 0
\]

* If \( 3 \nmid (1 - m\mu_1') \), we apply the same Eisenstein criterion to the polynomial \( R'(Z) \) given by:

\[
R'(Z) = Z^2 - 3Z + 3(1 - m\mu_1')
\]

and we find a contradiction with \( R'(\delta) = 0 \).

* We consider that \( 3 |(1 - m\mu_1') \Rightarrow m\mu_1' - 1 = 3^i h, i \geq 1, 3 \nmid h, h \in \mathbb{N}^* \). \( \delta \) is an integer root of the polynomial \( R'(Z) \):

\[
R'(Z) = Z^2 - 3Z + 3(1 - m\mu_1') = 0 \Rightarrow \text{the discriminant of } R'(Z) \text{ is:}
\]

\[
\Delta = 3^2 + 3^{i+1} \times 4h
\]

As the root \( \delta \) is an integer, it follows that \( \Delta = l^2 > 0 \) with \( l \) a positive integer. We obtain:

\[
\Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2
\]

\[\Rightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^*\]
We can write the equation (648) as:

\[(590) \quad \delta(\delta - 3) = 3^{i+1}h \implies 3^3 \mu_1 \frac{\text{rad}(\delta)}{3}. (\mu_1' \text{rad}(\delta) - 1) = 3^{i+1}h \implies\]

\[(591) \quad \mu_1' \frac{\text{rad}(\delta)}{3}. (\mu_1' \text{rad}(\delta) - 1) = h\]

We obtain \(i = 2\) and \(q^2 = 1 + 12h = 1 + 4\mu_1' \text{rad}(\delta)(\mu_1' \text{rad}(\delta) - 1)\). \(q\) satisfies:

\[(592) q^2 - 1 = 12h \implies \frac{(q - 1)}{2}, \frac{(q + 1)}{2} = 3h = (\mu_1' \text{rad}(\delta) - 1).\mu_1' \text{rad}(\delta) \implies\]

\[(593) \quad q - 1 = 2\mu_1' \text{rad}(\delta) - 2\]

\[(594) \quad q + 1 = 2\mu_1' \text{rad}(\delta)\]

It follows that \((q = x, 1 = y)\) is a solution of the Diophantine equation:

\[(595) \quad x^2 - y^2 = N\]

with \(N = 12h > 0\). Let \(Q(N)\) be the number of the solutions of (657) and \(\tau(N)\) is the number of suitable factorization of \(N\), then we announce the following result concerning the solutions of the Diophantine equation (657) (see theorem 27.3 in [6]):

- If \(N \equiv 2(\text{mod} 4)\), then \(Q(N) = 0\).
- If \(N \equiv 1\) or \(N \equiv 3(\text{mod} 4)\), then \(Q(N) = [\tau(N)/2]\).
- If \(N \equiv 0(\text{mod} 4)\), then \(Q(N) = [\tau(N/4)/2]\).

\([x]\) is the integral part of \(x\) for which \([x] \leq x < [x] + 1\).

Let \((\alpha', m')\), \(\alpha', m' \in \mathbb{N}^*\) be another pair, solution of the equation (657), then \(\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h\), but \(q = x\) and \(1 = y\) satisfy the equation (656) given by \(x + y = 2\mu_1' \text{rad}(\delta)\), it follows \(\alpha', m'\) verify also \(\alpha' + m' = 2\mu_1' \text{rad}(\delta)\), that gives \(\alpha' - m' = 2(\mu_1' \text{rad}(\delta) - 1)\), then \(\alpha' = x = q = 2\mu_1' \text{rad}(\delta)\) and \(m' = y = 1\). So, we have given the proof of the uniqueness of the solutions of the equation (657) with the condition \(x + y = 2\mu_1' \text{rad}(\delta)\). As \(N = 12h \equiv 0(\text{mod} 4) \implies Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]\), the expression of \(3h = \mu_1' \text{rad}(\delta). (\mu_1' \text{rad}(\delta) - 1)\), then \(Q(N) = [\tau(3h)/2] > 1\). But \(Q(N) = 1\), then the contradiction and the case \(3|(1 - m.\mu_1')\) is to reject.

** We suppose that \((\mu_1, \mu_2) = 1\).

From the equation \(m.\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)\), we obtain that \(\delta\) is a root of the following polynomial:

\[(596) \quad R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0\]

The discriminant of \(R(Z)\) is:

\[(597) \quad \Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with} \quad q \in \mathbb{N}^* \quad \text{as} \quad \delta \in \mathbb{N}^*\]
- We suppose that \( 2|\mu_1 \implies c \) is even. Then \( q^2 \equiv 5(\mod 8) \), it gives a contradiction because a square is \( \equiv 0, 1 \) or \( 4(\mod 8) \).

- We suppose \( c \) an odd integer, then \( a \) is even. It follows \( a = \text{rad}^3(a) \equiv 0(\mod 8) \implies c \equiv 1(\mod 8) \). As \( c = \delta^2 - 3X.\delta \), we obtain \( \delta^2 - 3X.\delta \equiv 1(\mod 8) \). If \( \delta^2 \equiv 1(\mod 8) \implies -3X.\delta \equiv 0(\mod 8) \implies 8|X.\delta \implies 4|\delta \implies c \) is even. Then, the contradiction. If \( \delta^2 \equiv 4(\mod 8) \implies \delta \equiv 2(\mod 8) \) or \( \delta \equiv 6(\mod 8) \). In the two cases, we obtain \( 2|\delta \). Then, the contradiction with \( c \) an odd integer.

It follows that the case \( c > \text{rad}^{3.26}(c) \) and \( a = \text{rad}^3(a) \) is impossible.

I-3-3- We suppose \( c > \text{rad}^{3.26}(c) \implies c = \text{rad}^3(c) + h \), \( h > \text{rad}^3(c) \), for \( 1 < c, h \) a positive integer and \( \mu_1 < \text{rad}^2(a) \implies a + l = \text{rad}^3(a) \), \( l > 0 \). Then we obtain:

\[
\text{rad}^3(c) + h = \text{rad}^3(a) - l + 1 \implies \text{rad}^3(a) - \text{rad}^3(c) = h + l - 1 \geq 0
\]
as \( \text{rad}(a) > \text{rad}^{3.26}(c) \). We obtain the equation:

\[
\text{rad}^3(a) - \text{rad}^3(c) = h + l - 1 = m > 0
\]

Let \( X = \text{rad}(a) - \text{rad}(c), \) then \( X \) is an integer root of the polynomial \( H(X) \) defined as:

\[
H(X) = X^3 + 3RX - m = 0
\]

To resolve the above equation, we denote \( X = u + v, \) then we obtain the two conditions:

\[
u^3 + v^3 = m, \quad u.v = -R < 0 \implies u^3.v^3 = -R^3
\]

It follows that \( u^3, v^3 \) are the roots of the polynomial \( G(t) \) given by:

\[
G(t) = t^2 - mt - R^3 = 0
\]

The discriminant of \( G(t) \) is:

\[
\Delta = m^2 + 4R^3 = \alpha^2, \quad \alpha > 0
\]

The two real roots of (663) are:

\[
t_1 = u^3 = \frac{m + \alpha}{2}
\]

\[
t_2 = v^3 = \frac{m - \alpha}{2}
\]

As \( m = \text{rad}^3(a) - \text{rad}^3(c) > 0 \), we obtain that \( \alpha = \text{rad}^3(a) + \text{rad}^3(c) > 0 \), then from the equation (664), it follows that \( (\alpha = x, m = y) \) is a solution of the Diophantine equation:

\[
x^2 - y^2 = N
\]
with $N = 4R^3 > 0$. From the equations (665-666), we remark that $\alpha$ and $m$ verify the following equations:

\[(606)\]
\[x + y = 2u^3 = 2\text{rad}^3(a)\]

\[(607)\]
\[x - y = -2v^3 = 2\text{rad}^3(c)\]

\[(608)\]
then \[x^2 - y^2 = N = 4R^3 = 4\text{rad}^3(a).\text{rad}^3(c)\]

Let $Q(N)$ be the number of the solutions of (667) and $\tau(N)$ is the number of suitable factorization of $N$, and using the same method as in the paragraph I-3-4-(case 3|$(1 - m\mu'_1)$), we obtain a contradiction.

It follows that the case $\mu_a \leq \text{rad}^2(a)$ and $c > \text{rad}^{3.26}(c)$ is impossible.

II- We suppose that $\mu_c \leq \text{rad}^2(c)$ and $\mu_a > \text{rad}^{1.63}(a)$:

II-1- Case $\text{rad}(c) < \text{rad}(a)$ : As $c \leq \text{rad}^3(c) = \text{rad}^{1.63}(c) . \text{rad}^{1.37}(c)$ $\implies$ $c < \text{rad}^{1.63}(c).\text{rad}^{1.37}(a) < \text{rad}^{1.63}(ac)$ $\implies$ $c < R^{1.63}$

II-2- Case $\text{rad}(a) < \text{rad}(c) < \text{rad}^{1.63}(a)$ : As $c \leq \text{rad}^3(c) \leq \text{rad}^{1.63}(c).\text{rad}^{1.37}(c)$ $\implies$ $c < \text{rad}^{1.63}(c).\text{rad}^{1.63}(a)$ $\implies$ $c < R^{1.63}$

II-3- Case $\text{rad}^{1.63}(a) < \text{rad}(c)$:

II-3-1- We suppose $a \leq \text{rad}^{3.26}(a)$ $\implies$ $a \leq \text{rad}^{1.63}(a).\text{rad}^{1.63}(a)$ $\implies$ $a < \text{rad}^{1.63}(a).\text{rad}^{1.37}(c)$ $\implies$ $a < \text{rad}^{1.63}(a).\text{rad}^{1.63}(c) \implies 1 + a < R^{1.63} \implies c < R^{1.63}$

II-3-2- We suppose $a > \text{rad}^{3.26}(a)$ and $\mu_c \leq \text{rad}^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting $a,c$), we arrive at a contradiction. It follows that the case $\mu_c \leq \text{rad}^2(c)$ and $\mu_a > \text{rad}^{2.26}(a)$ $\implies$ $a > \text{rad}^{3.26}(a)$ is impossible.

Finally, we have finished the study of the case $\mu_c \leq \text{rad}^2(c)$ and $\mu_a > \text{rad}^{2.26}(a) \implies a > \text{rad}^{3.26}(a)$.

**20.2.3.4. Case $\mu_c > \text{rad}^{1.26}(c)$ and $\mu_a > \text{rad}^2(a)$**

III - As $c > \text{rad}^{2.26}(c)$ and $a > \text{rad}^{3}(a)$, we can write $c = \text{rad}^3(c) + h$ and $a = \text{rad}^3(a) + l$ with $l > 0$ positive integer, $h \in \mathbb{Z}$. We obtain the equation:

\[(609)\]
\[\text{rad}^3(c) - \text{rad}^3(a) = l - h + 1\]
III-1- We suppose that \( l - h + 1 > 0 \). Let \( m = l - h + 1 > 0 \), then \( \text{rad}(c) > \text{rad}(a) \).

We obtain the equation:
\[
\text{rad}^3(c) - \text{rad}^3(a) = l - h + 1 = m > 0
\]

Let \( X = \text{rad}(c) - \text{rad}(a) \), from the above equation, \( X \) is a real root of the polynomial:
\[
P(X) = X^3 + 3RX - m = 0
\]

As above, to resolve (672), we put \( X = u + v \), then we obtain the two conditions:
\[
(u^3 + v^3) = m
\]
\[
uv = -R < 0 \implies u^3 \cdot v^3 = -R^3
\]

Then \( u^3, v^3 \) are the roots of the equation:
\[
P(Z) = Z^2 - mZ - R^3 = 0
\]

The discriminant of \( P(Z) \) is:
\[
\Delta = m^2 + 4R^3 = (\text{rad}^3(c) + \text{rad}^3(a))^2 = \alpha^2,
\]

taking \( \alpha > 0 \) \( \Rightarrow \alpha = \text{rad}^3(c) + \text{rad}^3(a) \)

From the equation (676), we obtain that \( (\alpha = x, m = y) \) is a solution of the Diophantine equation:
\[
x^2 - y^2 = N
\]

with \( N = 4R^3 > 0 \) and \( N \equiv 0 \pmod{4} \). Using the same method as in I-3-3-, we arrive to a contradiction.

III-2- We suppose \( l - h + 1 < 0 \), let \( m = h - l - 1 \), the equation (683) becomes:
\[
\text{rad}^3(a) - \text{rad}^3(c) = h - l - 1 = m > 0
\]

Let \( X = \text{rad}(a) - \text{rad}(c) \), from the above equation, \( X \) is a real root of the polynomial:
\[
P(X) = X^3 + 3RX - m = 0
\]

As above, to resolve (618), we put \( X = u + v \), then we obtain the two conditions:
\[
(u^3 + v^3) = m
\]
\[
uv = -R < 0 \implies u^3 \cdot v^3 = -R^3
\]

Then \( u^3, v^3 \) are the roots of the equation:
\[
P(Z) = Z^2 - mZ - R^3 = 0
\]

The discriminant of \( P(Z) \) is:
\[
\Delta = m^2 + 4R^3 = (\text{rad}^3(c) + \text{rad}^3(a))^2 = \alpha^2,
\]

taking \( \alpha > 0 \) \( \Rightarrow \alpha = \text{rad}^3(c) + \text{rad}^3(a) \)

From the equation (622), we obtain that \( (\alpha = x, m = y) \) is a solution of the Diophantine equation:
\[
x^2 - y^2 = N
\]
with $N = 4R^3 > 0$ and $N \equiv 0 \pmod{4}$. Using the same method as in I-3-3-, we arrive to a contradiction.

It follows that the case $\mu_c > \text{rad}^{1.26}(c)$ and $\mu_a > \text{rad}^2(a)$ is impossible.

**20.2.3.5. Case $\mu_a > \text{rad}^{1.26}(a)$ and $\mu_c > \text{rad}^2(c)$**

IV - This case is similar to the case III above and we obtain the same result: $\mu_a > \text{rad}^{1.26}(a)$ and $\mu_c > \text{rad}^2(c)$ impossible to obtain. Then the case is to reject.

Finally, we can announce the following important theorem:

**Theorem 33.** — Let $a, c$ positive integers relatively prime with $c = a + 1$, then $c < \text{rad}^{1.63}(ac)$. 


CHAPTER 21

THE PROOF OF THE CONJECTURE $c < \text{rad}^{1.63}(abc)$

Abstract. — In this paper, we consider the $abc$ conjecture. We give the proof of the conjecture $c < \text{rad}^{1.63}(abc)$ that constitutes the key to resolve the $abc$ conjecture.

To my teachers and to Madam Fatma Moalla, Mr Chedly Touibi
my professors of mathematics at the Faculty of Sciences of Tunis

21.1. Introduction and notations

Let $a$ be a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i - 1}$$

We denote:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a. \text{rad}(a)$$

The $abc$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [8]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $abc$ conjecture is given below:

Conjecture 33. — ($abc$ Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon).\text{rad}(abc)^{1+\epsilon}$$

where $K$ is a constant depending only of $\epsilon$.

We know that numerically, $rac{\log c}{\log(\text{rad}(abc))} \leq 1.629912$ [5]. It concerned the best example given by E. Reyssat [5]:

$$2 + 3^{10}.109 = 23^5 \implies c < \text{rad}^{1.629912}(abc)$$
A conjecture was proposed that \( c < \text{rad}^2(abc) \) [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

**Conjecture 34.** — Let \( a, b, c \) be positive integers relatively prime with \( c = a + b \), then:

\[
\begin{align*}
(628) & \quad c < \text{rad}^{1.63}(abc) \\
(629) & \quad abc < \text{rad}^{4.42}(abc)
\end{align*}
\]

Here, we will give a proof of the conjecture given by (628). This result, I think is the key to obtain the final proof of the veracity of the \( abc \) conjecture.

### 21.2. A Proof of the conjecture (34) case \( c = a + b \)

Let \( a, b, c \) be positive integers, relatively prime, with \( c = a + b \), \( b < a \) and \( R = \text{rad}(abc) \), \( c = \prod_{j' \in J'} c_{j'}^{\beta_{j'}} \), \( \beta_{j'} \geq 1 \).

In a previous paper [1], we has given, for the case \( c = a + 1 \), the proof that \( c < \text{rad}^{1.63}(ac) \). In the following, we will give the proof for the case \( c = a + b \).

**Proof.** — If \( c < \text{rad}(abc) \), then we obtain:

\[
(630) \quad c < \text{rad}(abc) < \text{rad}^{1.63}(abc) \implies c < R^{1.63}
\]

and the condition (628) is satisfied.

If \( c = \text{rad}(abc) \), then \( a, b, c \) are not coprime, case to reject.

In the following, we suppose that \( c > \text{rad}(abc) \) and \( a, b \) and \( c \) are not prime numbers.

\[
(631) \quad c = a + b = \mu_a \text{rad}(a) + \mu_b \text{rad}(b) < \text{rad}^{1.63}(abc)
\]

#### 21.2.1. \( \mu_a \neq 1, \mu_a \leq \text{rad}^{0.63}(a) \)

We obtain:

\[
(632) \quad c = a + b < 2a \leq 2\text{rad}^{1.63}(a) < \text{rad}^{1.63}(abc) \implies c < \text{rad}^{1.63}(abc) \implies c < R^{1.63}
\]

Then (631) is satisfied.

#### 21.2.2. \( \mu_c \neq 1, \mu_c \leq \text{rad}^{0.63}(c) \)

We obtain:

\[
(633) \quad c = \mu_c \text{rad}(c) \leq \text{rad}^{1.63}(c) < \text{rad}^{1.63}(abc) \implies c < R^{1.63}
\]

and the condition (631) is satisfied.
21.2.3. \( \mu_a > \text{rad}^{0.63}(a) \) and \( \mu_c > \text{rad}^{0.63}(c) \)

21.2.3.1. Case: \( \text{rad}^{0.63}(c) < \mu_c \leq \text{rad}^{1.63}(c) \) and \( \text{rad}^{0.63}(a) < \mu_a \leq \text{rad}^{1.63}(a) \):

We can write:

\[
\begin{align*}
\mu_c & \leq \text{rad}^{1.63}(c) \implies c \leq \text{rad}^{2.63}(c) \\
\mu_a & \leq \text{rad}^{1.63}(a) \implies a \leq \text{rad}^{2.63}(a)
\end{align*}
\]

\[
\Rightarrow ac \leq \text{rad}^{2.63}(ac) \Rightarrow a^2 < ac \leq \text{rad}^{2.63}(ac)
\]

\[
\Rightarrow a < \text{rad}^{1.315}(ac) \Rightarrow c < 2a < 2\text{rad}^{1.315}(ac) < \text{rad}^{1.63}(abc)
\]

(634) \[\implies c = a + b < R^{1.63}\]

21.2.3.2. Case: \( \mu_c > \text{rad}^{1.63}(c) \) or \( \mu_a > \text{rad}^{1.63}(a) \)

I- We suppose that \( \mu_c > \text{rad}^{1.63}(c) \) and \( \mu_a \leq \text{rad}^{2}(a) \):

I-1- Case \( \text{rad}(a) < \text{rad}(c) \): In this case \( a = \mu_a, \text{rad}(a) \leq \text{rad}^{1}(a) \leq \text{rad}^{1.63}(a) \text{rad}^{1.37}(a) < \text{rad}^{1.63}(a) \text{rad}^{1.37}(c) \implies c < 2a < 2\text{rad}^{1.63}(a) \text{rad}^{1.37}(c) < \text{rad}^{1.63}(abc) \implies c < R^{1.63} \implies c < R^{1.63}.

I-2- Case \( \text{rad}(c) < \text{rad}(a) < \text{rad}^{1.63}(c) \): As \( a \leq \text{rad}^{1.63}(a) \text{rad}^{1.37}(a) < \text{rad}^{1.63}(a) \text{rad}^{1.63}(c) \implies c < 2a < 2\text{rad}^{1.63}(a) \text{rad}^{1.63}(c) < R^{1.63} \implies c < R^{1.63}.

I-3- Case \( \text{rad}^{1.63}(c) < \text{rad}(a) \):

I-3-1- We suppose \( c \leq \text{rad}^{3.26}(c) \), we obtain:

\[
c \leq \text{rad}^{3.26}(c) \implies c \leq \text{rad}^{1.63}(c) \text{rad}^{1.63}(c) \implies c < \text{rad}^{1.63}(c) \text{rad}(a)^{1.37} < \text{rad}^{1.63}(c) \text{rad}(a)^{1.63} \text{rad}^{1.63}(b) = R^{1.63} \implies c < R^{1.63}
\]

I-3-2- We suppose \( c > \text{rad}^{3.26}(c) \) \( \implies \mu_c > \text{rad}^{2.26}(c) \). We consider the case \( \mu_a = \text{rad}^{2}(a) \implies a = \text{rad}^{3}(a) \). Then, we obtain that \( X = \text{rad}(a) \) is a solution in positive integers of the equation:

(635) \[X^3 + 1 = c - b + 1 = c'\]

But it is the case \( c' = 1 + a \). If \( c' = \text{rad}^{n}(c') \) with \( n \geq 4 \), we obtain the equation:

(636) \[\text{rad}^{n}(c') - \text{rad}^{3}(a) = 1\]

But the solutions of the equation (636) are [2]: \( (\text{rad}(c') = 3, n = 2, \text{rad}(a) = +2) \), it follows the contradiction with \( n \geq 4 \) and the case \( c' = \text{rad}^{n}(c'), n \geq 4 \) is to reject.
In the following, we will study the cases $\mu'_c = A.rad^n(c')$ with $rad(c') \nmid A, n \geq 0$. The above equation (635) can be written as:

\[(X + 1)(X^2 - X + 1) = c'\]

Let $\delta$ any divisor of $c'$, then:

\[(X + 1) = \delta\]

\[X^2 - X + 1 = \frac{c'}{\delta} = \delta^2 - 3X\]

We recall that $rad(a) > rad\frac{6\delta}{c}$. 

I-3-2-1- We suppose $\delta = l.rad(c')$. We have $\delta = l.rad(c') < c' = \mu'_c.rad(c') \implies l < \mu'_c$. As $\delta$ is a divisor of $c'$, then $l$ is a divisor of $\mu'_c$, we write $\mu'_c = l.m$. From $\mu'_c = l(\delta^2 - 3X)$, we obtain:

\[m = l^2rad(c') - 3rad(a) \implies 3rad(a) = l^2rad(c') - m\]

A- Case $3|m \implies m = 3m', m' > 1$: As $\mu'_c = ml = 3m'l \implies 3|rad(c')$ and $(rad(c'), m')$ not coprime. We obtain:

\[rad(a) = l^2rad(c').\frac{rad(c')}{3} - m'\]

It follows that $a, c'$ are not coprime, then the contradiction.

B - Case $m = 3 \implies \mu'_c = 3l \implies c' = 3l.rad(c') = 3\delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3\delta \implies \delta = l.rad(c') = 3$, then the contradiction.

I-3-2-2- We suppose $\delta = l.rad^2(c'), l \geq 2$. If $l.rad(c') \nmid \mu'_c$ then the case is to reject.

We suppose $l.rad(c')|\mu'_c \implies \mu'_c = m.l.rad(c')$, then $\frac{c'}{\delta} = m = \delta^2 - 3rad(a)$.

C - Case $m = 1 = c'/\delta \implies \delta^2 - 3rad(a) = 1 \implies (\delta - 1)(\delta + 1) = 3rad(a) = rad(a)(\delta + 1) \implies \delta = 2 = l.rad^2(c')$, then the contradiction.

D - Case $m = 3$, we obtain $3(1 + rad(a)) = \delta^2 = 3\delta \implies \delta = 3 = l.rad^2(c')$. Then the contradiction.

E - Case $m \neq 1, 3$, we obtain: $3rad(a) = l^2rad^4(c') - m \implies rad(a)$ and $rad(c')$ are not coprime. Then the contradiction.

I-3-2-3- We suppose $\delta = l.rad^n(c'), l \geq 2$ with $n \geq 3$. From $c' = \mu'_c.rad(c') = l.rad^n(c')(\delta^2 - 3rad(a))$, we denote $m = \delta^2 - 3rad(a) = \delta^2 - 3X$. 
We suppose $F$ - As seen above (paragraphs C,D), the cases $m = 1$ and $m = 3$ give contradictions, it follows the reject of these cases.

G - Case $m \neq 1, 3$. Let $q$ be a prime that divides $m$, it follows $q | \mu' \Rightarrow q = c_{j_0} \Rightarrow c_{j_0} | 3 \text{rad}(\alpha)$. Then $\text{rad}(a)$ and $\text{rad}(c')$ are not coprime. It follows the contradiction.

I-3-2-4. We suppose $\delta = \prod_{j \in J_1} c_{j}^{\beta_j}, \beta_j \geq 1$ with at least one $j_0 \in J_1$ with $\beta_{j_0} \geq 2$, $\text{rad}(c') \not| \delta$. We can write:

$$\delta = \mu_{\beta} \text{rad}(\delta), \quad \text{rad}(c') = m \text{rad}(\delta), \quad m > 1, \quad (m, \mu_{\beta}) = 1$$

Then, we obtain:

$$c' = \mu' \text{rad}(c') = \mu' \text{mrad}(\delta) = \delta (\delta^2 - 3X) = \mu_{\beta} \text{rad}(\delta) (\delta^2 - 3X) \Rightarrow$$

$$m \mu' = \mu_{\beta} (\delta^2 - 3X)$$

- If $\mu' = \mu_{\beta} \Rightarrow m = \delta^2 - 3X = (\mu' \text{rad}(\delta))^2 - 3X$. As $\delta < \delta^2 - 3X \Rightarrow m > \delta \Rightarrow \text{rad}(c') > m > \mu' \text{rad}(\delta) > \text{rad}^2(c')$ because $\mu' > \text{rad}^2(c')$, it follows $\text{rad}(c') > \text{rad}^2(c')$. Then the contradiction.

- We suppose $\mu' < \mu_{\beta}$. As $\text{rad}(a) = \mu_{\beta} \text{rad}(\delta) - 1$, we obtain:

$$\text{rad}(a) > \mu' \text{rad}(\delta) - 1 > 0 \Rightarrow \text{rad}(ac') > \text{rad}(c') - \text{rad}(c') > 0 \Rightarrow$$

$$c' > \text{rad}(ac') > c' \text{rad}(\delta) - \text{rad}(c') > 0 \Rightarrow 1 > \frac{\text{rad}(c')}{c'} > 0, \quad \text{rad}(\delta) \geq 2$$

(642) \quad \text{The contradiction}

- We suppose $\mu_{\beta} < \mu'$. In this case, from the equation (641) and as $(m, \mu_{\beta}) = 1$, it follows we can write:

$$\mu' = \mu_1 \mu_2, \quad \mu_1, \mu_2 > 1$$

(644)

$$c' = \mu' \text{rad}(c') = \mu_1 \mu_2 \text{rad}(\delta) m = \mu_1 \mu_2 \text{rad}(\delta)$$

(645) \quad \text{so that} \quad m \mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_{\beta} \Rightarrow \delta = \mu_2 \text{rad}(\delta)

**We suppose $(\mu_1, \mu_2) \neq 1$, then $\exists c'_{j_0}$, so that $c'_{j_0} | \mu_1$ and $c'_{j_0} | \mu_2$. But $\mu_{\beta} = \mu_2 \Rightarrow c'^2_{j_0} | \delta$. From $3X = \delta^2 - m \mu_1 \Rightarrow c'_{j_0} | 3X \Rightarrow c'_{j_0} | X$ or $c'_{j_0} = 3$.}

- If $c'_{j_0} | X$, it follows the contradiction with $(c', a) = 1$.

- If $c'_{j_0} = 3$. We have $m \mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \Rightarrow \delta^2 - 3\delta + 3 - m \mu_1 = 0$. As $3 | \mu_1 \Rightarrow \mu_1 = 3^k \mu'_1, 3 \not| \mu'_1, k \geq 1$, we obtain:

$$\delta^2 - 3\delta + 3(1 - 3^{k-1} \mu'_1) = 0$$

- We consider the case $k > 1 \Rightarrow 3 \not| (1 - 3^{k-1} \mu'_1)$. Let us recall the Eisenstein criterion [7]:

...
Chapter 21. The Proof of the Conjecture $c < \text{rad}^{1.63}(abc)$

**Theorem 34.** (Eisenstein Criterion) Let $f = a_0 + \cdots + a_n X^n$ be a polynomial $\in \mathbb{Z}[X]$. We suppose that $\exists p$ a prime number so that:
- $p \nmid a_n$,
- $p | a_i$, $(0 \leq i \leq n - 1)$,
- $p^2 \nmid a_0$.

Then $f$ is irreducible in $\mathbb{Q}$.

We apply Eisenstein criterion to the polynomial $R(Z)$ given by:

(647) \[ R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1} m \mu_1') \]

then:
- $3 \nmid 1$,
- $3|(-3)$,
- $3|3(1 - 3^{k-1} m \mu_1')$,
- $3^2 \nmid 3(1 - 3^{k-1} m \mu_1').$

It follows that the polynomial $R(Z)$ is irreducible in $\mathbb{Q}$, then, the contradiction with $R(\delta) = 0$.

- We consider the case $k = 1$, then $\mu_1 = 3 \mu_1'$ and $(\mu_1', 3) = 1$, we obtain:

(648) \[ \delta^2 - 3\delta + 3(1 - m \mu_1') = 0 \]

* If $3 \nmid (1 - m \mu_1')$, we apply the same Eisenstein criterion to the polynomial $R'(Z)$ given by:

\[ R'(Z) = Z^2 - 3Z + 3(1 - m \mu_1') \]

and we find a contradiction with $R'(\delta) = 0$.

* We consider that $3|(1 - m \mu_1') \Rightarrow m \mu_1' - 1 = 3^i h$, $i \geq 1$, $3 \nmid h, h \in \mathbb{N}^*$. $\delta$ is an integer root of the polynomial $R'(Z)$:

\[ R'(Z) = Z^2 - 3Z + 3(1 - m \mu_1') = 0 \Rightarrow \text{the discriminant of } R'(Z) \text{ is:} \]

(649) \[ \Delta = 3^2 + 3^{i+1} \times 4h \]

As the root $\delta$ is an integer, it follows that $\Delta = l^2 > 0$ with $l$ a positive integer. We obtain:

(650) \[ \Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2 \]

(651) \[ \Rightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^* \]

We can write the equation (648) as:

(652) \[ \delta(\delta - 3) = 3^{i+1} h \Rightarrow 3^2 \mu_1' \frac{\text{rad}(\delta)}{3}.(\mu_1' \text{rad}(\delta) - 1) = 3^{i+1} h \Rightarrow \]

(653) \[ \mu_1' \frac{\text{rad}(\delta)}{3}.(\mu_1' \text{rad}(\delta) - 1) = h \]
We obtain \( i = 2 \) and \( q^2 = 1 + 12h = 1 + 4\mu_1\text{rad}(\delta)(\mu_1\text{rad}(\delta) - 1) \). Then, \( q \) satisfies:

\[
(654) \quad q^2 - 1 = 12h \Rightarrow \frac{(q-1)(q+1)}{2} = 3h = (\mu_1\text{rad}(\delta) - 1).\mu_1\text{rad}(\delta) \Rightarrow
\]

\[
(655) \quad q - 1 = 2\mu_1\text{rad}(\delta) - 2
\]

\[
(656) \quad q + 1 = 2\mu_1\text{rad}(\delta)
\]

It follows that \((q = x, 1 = y)\) is a solution of the Diophantine equation:

\[
(657) \quad x^2 - y^2 = N
\]

with \( N = 12h > 0 \). Let \( Q(N) \) be the number of the solutions of \((657)\) and \( \tau(N) \) is the number of suitable factorization of \( N \), then we announce the following result concerning the solutions of the Diophantine equation \((657)\) (see theorem 27.3 in [6]):

- If \( N \equiv 2(\text{mod} \ 4) \), then \( Q(N) = 0 \).
- If \( N \equiv 1 \) or \( N \equiv 3(\text{mod} \ 4) \), then \( Q(N) = [\tau(N)/2] \).
- If \( N \equiv 0(\text{mod} \ 4) \), then \( Q(N) = [\tau(N/4)/2] \).

\( [x] \) is the integral part of \( x \) for which \( [x] \leq x < [x] + 1 \).

Let \((\alpha', m'), \alpha', m' \in \mathbb{N}^* \) be another pair, solution of the equation \((657)\), then \( \alpha'^2 - m'^2 = x^2 - y^2 = N = 12h \), but \( q = x \) and \( 1 = y \) satisfy the equation \((656)\) given by \( x + y = 2\mu_1\text{rad}(\delta) \), it follows \( \alpha', m' \) verify also \( \alpha' + m' = 2\mu_1\text{rad}(\delta) \), that gives \( \alpha' - m' = 2(\mu_1\text{rad}(\delta) - 1) \), then \( \alpha' = x = q = 2\mu_1\text{rad}(\delta) \) and \( m' = y = 1 \). So, we have given the proof of the uniqueness of the solutions of the equation \((657)\) with the condition \( x + y = 2\mu_1\text{rad}(\delta) \). As \( N = 12h \equiv 0(\text{mod} \ 4) \Rightarrow Q(N) = [\tau(N/4)/2] = [\tau(3h)/2] \), the expression of \( 3h = \mu_1\text{rad}(\delta) \). \( (\mu_1\text{rad}(\delta) - 1) \), then \( Q(N) = [\tau(3h)/2] > 1 \). But \( Q(N) = 1 \), then the contradiction and the case \( 3|(1 - m.m'_1) \) is to reject.

** We suppose that \((\mu_1, \mu_2) = 1 \).

From the equation \( m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \), we obtain that \( \delta \) is a root of the following polynomial:

\[
(658) \quad R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0
\]

The discriminant of \( R(Z) \) is:

\[
(659) \quad \Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with} \quad q \in \mathbb{N}^* \quad \text{as} \quad \delta \in \mathbb{N}^*
\]

- We suppose that \( 2|m\mu_1 \Rightarrow \delta' \) is even. Then \( q^2 \equiv 5(\text{mod} \ 8) \), it gives a contradiction because a square is \( \equiv 0, 1 \) or \( 4(\text{mod} \ 8) \).

- We suppose \( \delta' \) an odd integer, then \( a \) is even. It follows \( a = \text{rad}^2(a) \equiv 0(\text{mod} \ 8) \Rightarrow \delta' \equiv 1(\text{mod} \ 8) \). As \( \delta' = \delta^2 - 3X.\delta \), we obtain \( \delta^2 - 3X.\delta \equiv 1(\text{mod} \ 8) \). If \( \delta^2 \equiv 1(\text{mod} \ 8) \Rightarrow 3X.\delta \equiv 0(\text{mod} \ 8) \Rightarrow 8X.\delta \Rightarrow 4|\delta \Rightarrow \delta' \) is even. Then, the contradiction. If \( \delta^2 \equiv 4(\text{mod} \ 8) \Rightarrow \delta \equiv 2(\text{mod} \ 8) \) or \( \delta \equiv 6(\text{mod} \ 8) \). In the two
cases, we obtain $2|\delta$. Then, the contradiction with $c'$ an odd integer.

It follows that the case $c > \text{rad}^{3.26}(c)$ and $a = \text{rad}^{3}(a)$ is impossible.

I-3-3- We suppose $c > \text{rad}^{3.26}(c)$ and large, then $c = \text{rad}^{3}(c) + h, h > \text{rad}^{3}(c)$, $h$ a positive integer and $\mu_a < \text{rad}^{3}(a) \implies a + l = \text{rad}^{3}(a), l > 0$. Then we obtain :

(660) $\text{rad}^{3}(c) + h = \text{rad}^{3}(a) - l + b \implies \text{rad}^{3}(a) - \text{rad}^{3}(c) = h + l - b > 0$

as $\text{rad}(a) > \text{rad}^{3.26}(c)$. We obtain the equation:

(661) $\text{rad}^{3}(a) - \text{rad}^{3}(c) = h + l - b = m > 0$

Let $X = \text{rad}(a) - \text{rad}(c)$, then $X$ is an integer root of the polynomial $H(X)$ defined as:

(662) $H(X) = X^3 + 3\text{rad}(ac)X - m = 0$

To resolve the above equation, we denote $X = u + v$, then we obtain the two conditions:

$$u^3 + v^3 = m, \quad u.v = -\text{rad}(ac) < 0 \implies u^3.v^3 = -\text{rad}^3(ac)$$

It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by:

(663) $G(t) = t^2 - mt - \text{rad}^3(ac) = 0$

The discriminant of $G(t)$ is :

(664) $\Delta = m^2 + 4\text{rad}^3(ac) = \alpha^2, \quad \alpha > 0$

The two real roots of (663) are:

(665) $t_1 = u^3 = \frac{m + \alpha}{2}$

(666) $t_2 = v^3 = \frac{m - \alpha}{2}$

As $m = \text{rad}^3(a) - \text{rad}^3(c) > 0$, we obtain that $\alpha = \text{rad}^3(a) + \text{rad}^3(c) > 0$, then from the equation (664), it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

(667) $x^2 - y^2 = N$

with $N = 4\text{rad}^3(ac) > 0$. From the equations (665-666), we remark that $\alpha$ and $m$ verify the following equations:

(668) $x + y = 2u^3 = 2\text{rad}^3(a)$

(669) $x - y = -2v^3 = 2\text{rad}^3(c)$

(670) then $x^2 - y^2 = N = 4\text{rad}^3(a).\text{rad}^3(c)$

Let $Q(N)$ be the number of the solutions of (667) and $\tau(N)$ is the number of suitable factorization of $N$, and using the same method as in the paragraph I-3-2-4-
(case 3| (1 - m_m')), we obtain a contradiction.

It follows that the cases \( \mu_a \leq \text{rad}^2(a) \) and \( c > \text{rad}^{3.26}(c) \) are impossible.

II- We suppose that \( \text{rad}^{1.63}(c) < \mu_c \leq \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \):

II-1- Case \( \text{rad}(c) < \text{rad}(a) \) : As \( c \leq \text{rad}^3(c) = \text{rad}^{1.63}(c) \cdot \text{rad}^{1.37}(c) \Rightarrow c < \text{rad}^{1.63}(c) \cdot \text{rad}^{1.37}(a) < \text{rad}^{1.63}(ac) < \text{rad}^{1.63}(abc) \Rightarrow c < \text{R}^{1.63} \).

II-2- Case \( \text{rad}(a) < \text{rad}(c) < \text{rad}^{1.63}(a) \) : As \( c \leq \text{rad}^3(c) \leq \text{rad}^{1.63}(c) \cdot \text{rad}^{1.37}(c) \Rightarrow c < \text{rad}^{1.63}(c) \cdot \text{rad}^{1.63}(a) < \text{rad}^{1.63}(abc) \Rightarrow c < \text{R}^{1.63} \).

II-3- Case \( \text{rad}^{1.63}(a) < \text{rad}(c) \):

II-3-1- We suppose \( \text{rad}^{2.63}(a) < a \leq \text{rad}^{3.26}(a) \Rightarrow a \leq \text{rad}^{1.63}(a) \cdot \text{rad}^{1.63}(a) \Rightarrow a < \text{rad}^{1.63}(a) \cdot \text{rad}^{1.37}(c) \Rightarrow c = a + b < 2a < 2\text{rad}^{1.63}(a) \cdot \text{rad}^{1.63}(c) \rightarrow \text{rad}^{1.63}(abc) \rightarrow c < \text{R}^{1.63} \Rightarrow c < \text{R}^{1.63} \).

II-3-2- We suppose \( a > \text{rad}^{3.26}(a) \) and \( \mu_c \leq \text{rad}^2(c) \). Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting a,c), we arrive at a contradiction. It follows that the case \( \mu_c \leq \text{rad}^2(c) \) and \( a > \text{rad}^{3.26}(a) \) is impossible.

Finally, we have finished the study of the case \( \text{rad}^{1.63}(c) < \mu_c \leq \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \).

21.2.3.3. Case \( \mu_c > \text{rad}^{1.63}(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \)

Taking into account the cases studied above, it remains to see the following two cases:
- \( \mu_c > \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \),
- \( \mu_a > \text{rad}^2(a) \) and \( \mu_c > \text{rad}^{1.63}(c) \).

III-1- We suppose \( \mu_c > \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \) \Rightarrow c > \text{rad}^3(c) \) and \( a > \text{rad}^{2.63}(a) \). We can write \( c = \text{rad}^3(c) + h \) and \( a = \text{rad}^3(a) + l \) with \( h \) a positive integer and \( l \in \mathbb{Z} \).

III-1-1- We suppose \( \text{rad}(c) < \text{rad}(a) \). We obtain the equation:

\[(671) \quad \text{rad}^3(a) - \text{rad}^3(c) = h - l - b = m > 0 \]

Let \( X = \text{rad}(a) - \text{rad}(c) \), from the above equation, \( X \) is a real root of the polynomial:

\[(672) \quad H(X) = X^3 + 3\text{rad}(ac)X - m = 0 \]
As above, to resolve (672), we denote \( X = u + v \), then we obtain the two conditions:

\[
\begin{align*}
(673) \\ u^3 + v^3 = m \\
(674) \\ uv = -\text{rad}(ac) < 0 \implies u^3 v^3 = -\text{rad}^3(ac)
\end{align*}
\]

It follows that \( u^3, v^3 \) are the roots of the polynomial \( G(t) \) given by:

\[
G(t) = t^2 - mt - \text{rad}^3(ac) = 0
\]

The discriminant of \( G(t) \) is:

\[
\Delta = m^2 + 4\text{rad}^3(ac) = \alpha^2, \quad \alpha >
\]

The two real roots of (675) are:

\[
\begin{align*}
(677) \\ t_1 = u^3 = \frac{m + \alpha}{2} \\
(678) \\ t_2 = v^3 = \frac{m - \alpha}{2}
\end{align*}
\]

As \( m = \text{rad}^3(a) - \text{rad}^3(c) > 0 \), we obtain that \( \alpha = \text{rad}^3(a) + \text{rad}^3(c) > 0 \), then from the equation (676), it follows that \( (\alpha = x, m = y) \) is a solution of the Diophantine equation:

\[
(679) \\ x^2 - y^2 = N
\]

with \( N = 4\text{rad}^3(ac) > 0 \). From the equations (677-678), we remark that \( \alpha \) and \( m \) verify the following equations:

\[
\begin{align*}
(680) \\ x + y = 2u^3 = 2\text{rad}^3(a) \\
(681) \\ x - y = -2v^3 = 2\text{rad}^3(c) \\
(682) \\ \text{then} \quad x^2 - y^2 = N = 4\text{rad}^3(a).\text{rad}^3(c)
\end{align*}
\]

Let \( Q(N) \) be the number of the solutions of (679) and \( \tau(N) \) is the number of suitable factorization of \( N \), and using the same method as in the paragraph I-3-2-4- (case 3\((1 - m.\mu_1)\)), we obtain a contradiction.

III-1-2- We suppose \( \text{rad}(a) < \text{rad}(c) \). We obtain the equation:

\[
(683) \\ \text{rad}^3(c) - \text{rad}^3(a) = b + l - h = m > 0
\]

Using the same calculations as in III-1-1-, we find a contradiction.

It follows that the case \( \mu_c > \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \) is impossible.

III-2- We suppose \( \mu_a > \text{rad}^2(a) \) and \( \mu_c > \text{rad}^{1.63}(c) \) \( \implies a > \text{rad}^3(a) \) and \( c > \text{rad}^{2.63}(c) \). We can write \( a = \text{rad}^3(a) + h \) and \( c = \text{rad}^3(c) + l \) with \( h \) a positive integer and \( l \in \mathbb{Z} \).
The calculations are similar to those in case III-1-. We obtain the same results namely the cases of III-2- to be rejected.

It follows that the case $\mu_c > \text{rad}^{1.63}(c)$ and $\mu_a > \text{rad}^2(a)$ is impossible. □

We can state the following important theorem:

**Theorem 35.** — Let $a, b, c$ positive integers relatively prime with $c = a + b$, then $c < \text{rad}^{1.63}(abc)$.
BIBLIOGRAPHY


ASSUMING \( c < R^2(abc) \), THE abc CONJECTURE IS TRUE

Abstract. — In this paper, we consider the abc conjecture. Assuming that \( c < \text{rad}^2(abc) \) is true, we give the proof of the abc conjecture for \( \epsilon \geq 1 \), then for \( \epsilon \in [0, 1] \).

To the memory of my Father who taught me arithmetic, To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

22.1. Introduction and notations

Let a positive integer \( a = \prod_i a_i^{\alpha_i} \), \( a_i \) prime integers and \( \alpha_i \geq 1 \) positive integers. We call radical of \( a \) the integer \( \prod_i a_i \) noted by \( \text{rad}(a) \). Then \( a \) is written as:

\[
(684) \quad a = \prod_i a_i^{\alpha_i} = \text{rad}(a). \prod_i a_i^{\alpha_i - 1}
\]

We note:

\[
(685) \quad \mu_a = \prod_i a_i^{\alpha_i - 1} \implies a = \mu_a. \text{rad}(a)
\]

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

Conjecture 35. — (abc Conjecture): For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
(686) \quad c < K(\epsilon). \text{rad}^{1+\epsilon}(abc)
\]

where \( K \) is a constant depending only of \( \epsilon \).

The idea to try to write a paper about this conjecture was born after the publication in September 2018, of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty
to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in $c$ giving $a, b$ with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. It is the key to resolve the abc conjecture. In my paper, I assume that the conjecture $c < \text{rad}^2(abc)$ holds, I propose an elementary proof of the abc conjecture. The paper is organized as follows: in the second section, we give the proof of the abc conjecture.

22.2. The Proof of the abc conjecture

We note $R = \text{rad}(abc)$ in the case $c = a + b$ or $R = \text{rad}(ac)$ in the case $c = a + 1$. We assume that $c < R^2$ is true. We recall the following proposition [4]:

Proposition 22.1. — Let $\epsilon \rightarrow K(\epsilon)$ the application verifying the abc conjecture, then:

\begin{equation}
\lim_{\epsilon \to 0} K(\epsilon) = +\infty
\end{equation}

22.2.1. Case : $\epsilon \geq 1$

Using the result that $c < R^2$, we have $\forall \epsilon \geq 1$:

\begin{equation}
c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with} \quad K(\epsilon) = \epsilon, \quad \epsilon \geq 1
\end{equation}

Then the abc conjecture is true.

22.2.2. Case: $\epsilon < 1$

22.2.2.1. Case: $c < R$

In this case, we can write:

\begin{equation}
c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad \text{with} \quad K(\epsilon) = \epsilon > 1, \quad \epsilon < 1
\end{equation}

Then the abc conjecture is true.

22.2.2.2. Case: $c > R$

In this case, we consider the contradiction of the abc conjecture:

\begin{equation}
\exists \epsilon_0 \in ]0, 1], \forall K(\epsilon), \quad \exists c_0 = a_0 + b_0 \quad \text{so that} \quad c_0 > K(\epsilon_0)R_0^{1+\epsilon_0} \implies c_0 \text{ not a prime}
\end{equation}
We choose the constant $K(\epsilon) = e^{\epsilon^2}$. Let:

\begin{equation}
Y_c(\epsilon) = \frac{1}{\epsilon^2} + (1 + \epsilon)\log R_0 - \log c_0, \epsilon \in ]0, 1[.
\end{equation}

We have $\lim_{\epsilon \to 1} Y_c(\epsilon) = 1 + \log (R_0^2/c_0) > 0$ and $\lim_{\epsilon \to 0} Y_c(\epsilon) = +\infty$. The function $Y_c(\epsilon)$ has a derivative for $\epsilon \in ]0, 1[,$ we obtain:

\begin{equation}
Y'_c(\epsilon) = -2 + \log R_0 = e^{\epsilon^2} \log R_0 - 2 \epsilon^3.
\end{equation}

$Y'_c(\epsilon) = 0 \implies \epsilon = \epsilon' = \sqrt{\frac{2}{\log R_0}} \in ]0, 1[.$

Discussion:
- If $Y_c(\epsilon') > 0$, it follows that $\forall \epsilon \in ]0, 1[, Y_c(\epsilon) > 0$, then the contradiction with $Y_c(\epsilon_0) < 0 \implies c_0 \geq K(\epsilon_0)R_0^{1+\epsilon_0}$. Hence the abc conjecture is true for $\epsilon \in ]0, 1[.$

- If $Y_c(\epsilon') < 0 \implies \exists 0 < \epsilon_1 < \epsilon' < \epsilon_2 < 1$, so that $Y_c(\epsilon_1) = Y_c(\epsilon_2) = 0$. Then we obtain $c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} = K(\epsilon_2)R_0^{1+\epsilon_2}$. We consider the equality:

\begin{equation}
c_0 = K(\epsilon_1)R_0^{1+\epsilon_1} \implies \mu c_0 = e^{\epsilon_1^2} \text{rad}(ab)R_0^{\epsilon_1}.
\end{equation}

If the right member of the above equation is an integer, we obtain a contradiction with $a_0, b_0, c_0$ coprime. If not, we find that an integer $\mu c_0$ is equal to a real number $e^{\epsilon_1^2} \text{rad}(ab)R_0^{\epsilon_1}$. Then the contradiction again, it follows that the abc conjecture is true.

Then the proof of the abc conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with $a, b, c$ relatively coprime:

\begin{equation}
c < K(\epsilon)\text{rad}^{1+\epsilon}(abc)
\end{equation}

and the constant $K(\epsilon)$ depends only of $\epsilon$.

Q.E.D

Ouf, end of the mystery!

22.3. Conclusion

Assuming $c < R^2$ is true, we have given an elementary proof of the abc conjecture. We can announce the important theorem:

**Theorem 36.** — For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, assuming $c < \text{rad}^2(abc)$ holds, then:

\begin{equation}
c < K(\epsilon)\text{rad}^{1+\epsilon}(abc)
\end{equation}
where $K$ is a constant depending of $\epsilon$.

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CHAPTER 23

A PROOF OF THE abc CONJECTURE

Abstract. — In this paper about the abc conjecture, assuming the condition $c < \text{rad}^2(abc)$ holds, and the constant $K(\epsilon)$ is a smooth function, having a derivative for $\epsilon \in [0, 1]$, then we give the proof of the abc conjecture.

To the memory of my Father who taught me arithmetic
To my wife Wahida, my daughter Sinda and my son Mohamed
Mazen

23.1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \prod_i a_i^{\alpha_i-1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i-1} \implies a = \mu_a \cdot \text{rad}(a)$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the abc conjecture is given below:

**Conjecture 36.** (abc Conjecture): For each $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\epsilon) \cdot \text{rad}^{1+\epsilon}(abc)$$

where $K$ is a constant depending only of $\epsilon$.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in September 2018, about the remarks
of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the abc conjecture is due to the incomprehensibility how the prime factors are organized in \( c \) giving \( a, b \) with \( c = a + b \).

We know that numerically, \( \frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 \) [1]. A conjecture was proposed that \( c < \text{rad}^2(abc) \) [3]. It is the key to resolve the abc conjecture. In my paper, we assume that the last conjecture holds, and the constant \( K(\epsilon) \) for \( \epsilon \in [0,1] \) is a smooth function having a derivative for \( \epsilon \in [0,1] \). The paper is organized as follows: in the second section, we begin by presenting some properties of the constant \( K(\epsilon) \), then we give the proof of the abc conjecture.

### 23.2. The Proof of the abc Conjecture

Let \( a, b, c \) positive integers relatively prime with \( c = a + b, a > b, b \geq 2 \). We denote \( R = \text{rad}(abc), I = [0,1] \). For \( c < R \), it is trivial that the abc conjecture holds. In the following, we consider the triples \( (a, b, c) \) with \( a, b, c \) relatively coprime and \( c > R \).

As we assume that \( c < R^2 \), it follows that \( \forall \epsilon \geq 1 \), it suffices to take \( K(\epsilon) = 1 \) and \( c \) satisfies \( c < K(\epsilon)R^{1+\epsilon} \) and the abc conjecture is true.

#### 23.2.1. Properties of the constant \( K(\epsilon) \)

- From the definition of the abc conjecture, above, the constant \( K(\epsilon) \) is a positive real number, and for every \( \epsilon > 0 \), it exists a number \( K(\epsilon) \) dependent only of \( \epsilon \).

- In the following, we consider that \( \epsilon \in I \). We can say that \( K \) is a function \( K : \epsilon \in I \rightarrow K(\epsilon) \in [0, +\infty] \), so that \( c < K(\epsilon)R^{1+\epsilon} \) holds, if the abc conjecture is true. Assuming that \( c < R^2 \) is satisfying, we can adopt that \( K(\epsilon = 1) = 1 \), because \( c < K(1)R^{1+1} \). Then we choose \( K(\epsilon) \) so that \( \lim_{\epsilon \rightarrow 1} K(\epsilon) = K(1) \).

- We obtain also that \( K(\epsilon) > 1 \) if \( \epsilon \in I \). If not, we consider the example \( 9 = 8 + 1 \), we take \( \epsilon = 0.2 \), then \( c < K(0.2)R^{1.02} < 1.1R^{1.2} \). But \( c = 9 > 6^{1.2} \approx 8.58 \), then the contradiction.

- In 1996, A. Nitaj had confirmed that the constant \( K(\epsilon) \) verifies [4]:

\[
(699) \quad \lim_{\epsilon \rightarrow 0} K(\epsilon) = +\infty
\]

It follows that the function \( K(\epsilon) \) is very large when \( \epsilon \) is very small.
23.2. The proof of the abc conjecture

Proof. — Let us suppose that \( K(\epsilon) \) is a smooth function having a derivative in every point \( \in [0, 1[ \). Let \( a, b, c \) positive integers relatively prime with \( c = a + b, c > R \). We denote :

\[
Y_c(\epsilon) = \log K(\epsilon) + (1 + \epsilon) \log R - \log c
\]

We obtain \( \lim_{\epsilon \to 1} Y_c(\epsilon) = 2 \log R - \log c = y_1 > 0 \), assuming \( c < R^2 \), and \( \lim_{\epsilon \to 0} Y_c(\epsilon) = +\infty \). The derivative of \( Y_c(\epsilon) \) gives:

\[
Y_c'(\epsilon) = \frac{K'(\epsilon)}{K(\epsilon)} + \log R
\]

We have the following cases:

i) - If \( Y_c'(\epsilon) > 0 \) for all \( \epsilon \in [0, 1[ \), then \( Y \) is an increasing function of \( \epsilon \). It follows the contradiction because \( \lim_{\epsilon \to 0} Y_c(\epsilon) = +\infty \).

ii) - If \( Y_c'(\epsilon) < 0 \) for all \( \epsilon \in [0, 1[ \), then \( Y \) is a decreasing function of \( \epsilon \). It follows \( \forall \epsilon, Y_c(\epsilon) > 0 \implies c < K(\epsilon)R^{1+\epsilon} \) is satisfied. As \( c \) is an arbitrary integer with the condition \( c > R \), we deduce that the abc conjecture is true.

iii) - If \( Y_c'(\epsilon) = 0 \) for some \( \epsilon_0 \in [0, 1[ \), \( \epsilon_0 \) is a solution of the equation :

\[
- \frac{K'(\epsilon_0)}{K(\epsilon_0)} = \log R
\]

We remark that \( \epsilon_0 \) depends of \( R \), then of \( a, b, c \).

* If \( Y_c(\epsilon_0) \) is positive, then \( 0 < Y_c(\epsilon_0) \leq Y_c(\epsilon) \implies Y_c(\epsilon) > 0 \). As above, we deduce that the abc conjecture holds for the triplet \((a, b, c)\).

** If \( Y_c(\epsilon_0) \) is negative, then it exists two values \( \epsilon_1, \epsilon_2 \) with \( 0 < \epsilon_1 < \epsilon_0 < \epsilon_2 < 1 \), so that \( Y_c(\epsilon_1) = Y_c(\epsilon_2) = 0 \). It follows for example, that \( c = K(\epsilon_1)R^{1, \text{rad}(abc)} \).

Suppose that \( K(\epsilon_1)R^{\epsilon_1} \) is an integer, we obtain that \( a, b, c \) are not coprime. Then the contradiction and this case to reject.

Then, we have obtained that the abc conjecture holds for \( \forall \epsilon \in I \) for the triplet \((a, b, c)\), as it is chosen arbitrary with the condition \( c > \text{rad}(abc) \). It follows that the abc conjecture is true, assuming that \( c < R^2 \).

\[\square\]

Q.E.D

End of the mystery!
23.3. Conclusion

Finally, assuming \( c < R^2 \), and choosing the constant \( K(\epsilon) \) as a smooth function, having a derivative for \( \epsilon \in ]0,1[ \), we have given an elementary proof that the \( abc \) conjecture is true.

We can announce the important theorem:

**Theorem 37.** — For each \( \epsilon > 0 \), there exists \( K(\epsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), assuming \( c < \text{rad}(abc) \), then:

\[
(702) \quad c < K(\epsilon) \cdot \text{rad}(abc)^{1+\epsilon}
\]

where \( K \) is a constant depending only of \( \epsilon \) and varying smoothly, having a derivative.

**Acknowledgments.** The author is very grateful to Professors Mihăilescu Preda, and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proposed proofs of the \( abc \) conjecture.
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