Right-handed and left-handed circularly polarized light derived from projection operators in Clifford algebras, Stokes parameters and Mueller matrix in four dimensions

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Abstract

The hermitian polarization matrix is written, using the Pauli matrices and the identity matrix as a basis, with four real coefficients, the four Stokes vector parameters; the interaction of light with matter is described as the modification of these 4 parameters by a 4x4 matrix, the Mueller matrix. Pauli matrices form a Clifford algebra, the projection operators R and L define the two Stokes vectors for right-handed and left-handed circularly polarized light.

In four dimensions, the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ leads to the Clifford algebra C(1,3) Dirac matrices, the 16 Dirac matrices form a basis for the polarization matrix, now with 16 Stokes parameters the interaction of light with matter is described by a 16x16 Mueller matrix, the projection operators R and L in this algebra define the right-handed and left-handed circularly polarized light.

Circularly polarized light derived from projection operators, Stokes parameters and Mueller matrix

The hermitian polarization matrix P is written, using the Pauli matrices σ and the identity matrix as a basis, with four real coefficients, the four Stokes parameters s_j, j = 0, 1, 2, 3 [1][2][3][4][5][6][7][8]

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$2 \times P = s_j \sigma^j$

$s_j = Tr(P\sigma^j)$

$s_0 = Tr(P\sigma^0)$ is the intensity of light

The interaction of light with matter is described as the modification of the Stoke vector by a 4x4 matrix, the Mueller matrix M

$s'_i = M_{ij} s_j$

Pauli matrices σ form a Clifford algebra, γ_p is defined as $\sigma^1\sigma^3$ for the C(2,0) signature, $\gamma_p = -i\sigma^2, \gamma_p\gamma_p = -\mathbb{I}$ [9]

In Clifford algebras the projection operators R and L are defined depending on the value of $\gamma_p\gamma_p$ [9]

$L = 1/2(\mathbb{I} - i\gamma_p), R = 1/2(\mathbb{I} + i\gamma_p), \gamma_p\gamma_p = -\mathbb{I}$

$L = 1/2(\mathbb{I} - \gamma_p), R = 1/2(\mathbb{I} + \gamma_p), \gamma_p\gamma_p = +\mathbb{I}$

$L = 1/2(\mathbb{I} - \sigma^2), R = 1/2(\mathbb{I} + \sigma^2)$

$L = 1/2 \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \psi_L = a \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$R = 1/2 \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \psi_R = a \begin{pmatrix} 1 \\ i \end{pmatrix}$

R corresponds to the Stokes parameters $s_0 = 1/2, s_1 = 0, s_2 = 1/2, s_3 = 0$, the right-handed circularly polarized light

L corresponds to the Stokes parameters $s_0 = 1/2, s_1 = 0, s_2 = -1/2, s_3 = 0$, the left-handed circularly polarized light

Circularly polarized light derived from projection operators, Stokes parameters and Mueller matrix in four dimensions

In four dimensions, the Minkowski metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ leads to the Clifford algebra C(1,3) [9], $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \times \mathbb{I}_{4\times4}$, Dirac matrices $\gamma^0 = \sigma^3 \otimes I$, $\gamma^j = i\sigma^2 \otimes \sigma^j, j = 1, 2, 3$; $\gamma_p = \gamma^{14} = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^1 \otimes I, \gamma_p\gamma_p = +\mathbb{I}$

$\gamma^4 = \gamma^0\gamma^1, \gamma^5 = \gamma^0\gamma^2, \gamma^6 = \gamma^0\gamma^3, \gamma^7 = \gamma^1\gamma^2, \gamma^8 = \gamma^1\gamma^3, \gamma^9 = \gamma^2\gamma^3$
$\gamma^{10} = \gamma^0\gamma^1\gamma^2, \gamma^{11} = \gamma^0\gamma^1\gamma^3, \gamma^{12} = \gamma^0\gamma^2\gamma^3, \gamma^{13} = \gamma^1\gamma^2\gamma^3, \gamma^{14} = i\gamma^0\gamma^1\gamma^2\gamma^3$

The polarization matrix P is written, using the 15 Dirac matrices γ and the identity matrix as a basis, with 16 coefficients, the 16 Stokes parameters

$2 \times P = s_0 \times \mathbb{I}_{4\times4} + s_j\gamma^{j-1}(j = 1, , , 15)$, s_0 is the intensity of light

The interaction of light with matter is described as the modification of the Stoke parameters by a 16x16 matrix, the Mueller matrix M

$s'_i = M_{ij}s_j$

$L = 1/2(\mathbb{I} - \gamma_p), R = 1/2(\mathbb{I} + \gamma_p)$, $\gamma_p = \gamma^{14} = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^1 \otimes I, \gamma_p\gamma_p = +\mathbb{I}$

$$L = 1/2 \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \psi_L = \begin{pmatrix} a \\ b \\ -b \\ -a \end{pmatrix}$$

$$R = 1/2 \begin{pmatrix} 1 & 0 & +1 & 0 \\ 0 & 1 & 0 & +1 \\ +1 & 0 & 1 & 0 \\ 0 & +1 & 0 & 1 \end{pmatrix}, \psi_R = \begin{pmatrix} a \\ b \\ +b \\ +a \end{pmatrix}$$

R corresponds to the Stokes parameters $s_0 = 1/2, s_1 = 0, .., s_{14} = 0, s_{15} = 1/2$, the right-handed circularly polarized light

L corresponds to the Stokes parameters $s_0 = 1/2, s_1 = 0, .., s_{14} = 0, s_{15} = -1/2$, the left-handed circularly polarized light

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[9] See Chapter 22 in: M. Fecko, “Differential Geometry and Lie Groups for Physicists”, Cambridge Univesrity Press, 2006