Operational Definition of Electric Charge and Derivation of Coulomb's Law

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Abstract

The paper focuses on the part of the Coulomb's Law that is just a definition and provides one possible mechanism for operationally defining electric charge based on the concept of force(action at a distance). Then a derivation of Coulomb's law from the definition is presented and the sign of the charges are defined. Finally, the paper concludes with a discussion on the conservation of charge.

1   Introduction

Electric Charge can be defined for classical objects as the signed difference of the number of protons and the number of electrons.[4][5][9][12] However, this definition does not enable us to know the charge of an object experimentally in the domain of classical electromagnetism. We have to use Coulomb's Law and its consequences for knowing the charge which means that we don't have any information about charge without appealing to the law.[9, see p. 9] We will show in this paper how one can define and measure electric charge operationally and derive the Coulomb's Law from it. It should be emphasized that in this paper we will not adopt the *field viewpoint*[8][11] or the *Principle of Contiguous Action*[8, see p. 1], neither we will deal with *special theory of relativity.*

Now, we understand by *a quantity defined operationally* that the quantity is defined using its properties and a procedure for at least theoretically measuring the quantity is given in the definition.[1][7](for the usage of operational definition in mechanics see [6][7][10]) Therefore, we will define electric charge using its properties whose verification lies upon experiment and outline a procedure for measuring electric charge quantitatively. It should be remarked here that in the following we will ignore the gravitational force between two particles. However, we will not ignore their weights.

2   Charged Particle

Because of the force between charged and induced particle[4][5][8][9][11][12], we need some device to distinguish between charged and induced particle. The simplest way to do this is to use a third object and calculate the force of the test particles on this object. It is easy to see that the third object can not have a net charge but it will be induced when a charged body is nearby with sufficient
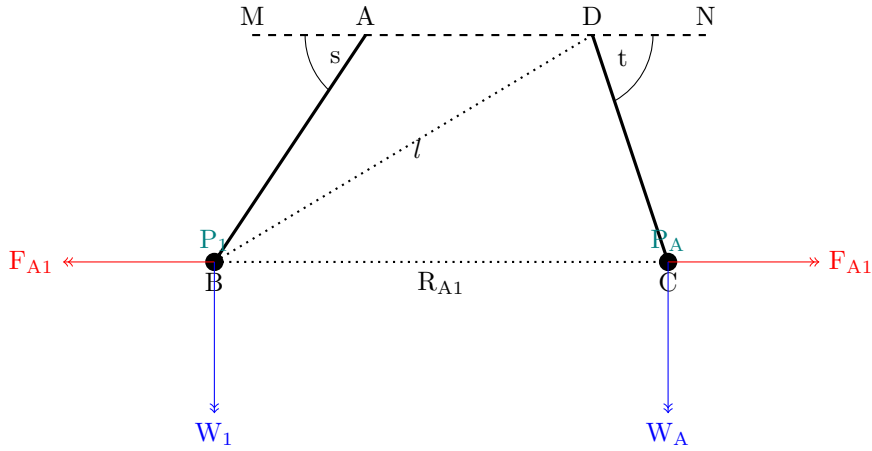
Figure 1: Static equilibrium of Charges

accuracy demanded by experimenter. *It is on experiment to find that there are situations when two identical bodies (intuitively) repel each other regardless of the orientation of the bodies.* In that case, we say that the body is charged.[1] If we find an uncharged body, we just need to know whether there is considerable force demanded by the experimenter between this and some other object which we have found charged. If there is sufficient force between them, this uncharged body can be used as the third object mentioned above.[2]

3   Definition Of Electric Charge

We need a quantitative measurement of electric charge.[3] So, we propose the following experiment: We have two charged particles P_A and P_1, both of them are used as the bob of some pendulum which we have hanged from two pivot points A and D as shown in the Figure 1.

Here, the distance between the pivot points is greater than the sum of the length of the threads. It is necessary because we don't know whether the particles attract or repel and sum of the two sides of a triangle is greater than the third side, since we don't want the particles to stick together if they attract.That is: the distance $AD > AB + DC$.[4] In addition, the particles are in static equilibrium, not necessarily stable.We assume that these can be arranged always by suitable length of the threads and the distance between the pivots.

Now, we assume that we can calculate the weight, length of the threads and also the angles s and t mentioned in the Figure. *If we have length of AD, AB, CD and the angles s, t, the quadrilateral $ABCD$ is uniquely defined and we can find*

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[1]Since two identical magnetic bodies may repel each other when two opposite poles come nearer, we have to consider in the definition whether identical bodies repel irrespective of orientation of the bodies.

[2]However, we can use *gold leaf electroscope* as a mean to know if something is charged or not.

[3]We gave a qualitative definition of charge in the previous section.

[4]Although this is not the case in Figure 1 and the particles repel each other in Figure 1, this does not effect the discussion that follows

all the other angles and sides.

Now, If we assume that the force between two particles P_A and P_1 obey the *Strong Law of Action-Reaction*[3][6], then using *Lami's theorem on statics*[2] and *Law of sines* for triangle, we arrive:

$$F_{A1} = \frac{W_A R_{A1} \cos t}{l \sin \angle BDC} \tag{1}$$

where F_{A1}, R_{A1} is respectively the mutual force, distance of P_A and P_1; W_A is the weight of particle P_A, l is the length BD. From equation(1), we find the force in all cases which can be modelled using Figure 1.

Now, we replace the charge P_A with another charge P_B in Figure 1 and we may even change the length of the threads and the distance between the pivots. We assume that we can arrange static equilibrium again. Then we have $F_{A1}, F_{B1}, R_{A1}, R_{B1}$ using (1) and we take the ratio $\frac{F_{A1} R_{A1}^2}{F_{B1} R_{B1}^2}$ which we denote by $q1$. Again, we replace P_1 by $P_2, P_3, P_4 \ldots$[5] and measure the corresponding $F_{A2}, F_{A3}, F_{A4} \ldots F_{B2}, F_{B3}, F_{B4} \ldots, R_{A2}, R_{A3}, R_{A4} \ldots, R_{B2}, R_{B3}, R_{B4} \ldots$ using (1) and using these results, we deduce the corresponding ratios $q_2, q_3, q_4 \ldots$. *It is on experiment to conclude that $q_1 = q_2 = q_3 = q_4 = \cdots = q$.* Then we conclude that the ratios depends on P_B and P_A and not on $P_1, P_2, P_3, P_4, \ldots$. Therefore, the ratio is a relative property. That is:

$$\frac{F_{Ai} R_{Ai}^2}{F_{Bi} R_{Bi}^2} = q \tag{2}$$

equation (2) is valid independent of i. Before proceeding, we need some discussions about the meaning of (2). The equation here means that *in Figure 1 if we replace only one charge with another charge (we may need to change the threads and pivots for equilibrium or we can change them at will.), the ratio in the left side of the equation (2) is independent of the unchanged particle.*

We, therefore finds a way to define charge relative to P_B. We define the charge(relative to P_B) in accordance with the preceding sentence, of P_A to be q unit. Thus if P_A and P_B are the same, in that case the substitution of P_B for P_A is in the Figure 1 is unnecessary, and also if we do not change the threads and pivots we find that P_B has an unit charge. We, therefore, have almost completed our definition of electric charge.

4   Derivation Of Coulomb's Law

We take any charge P_A and P_1 in Figure 1 and then replace unit charge P_B for P_A with proper length of threads and pivot for equilibrium, we then have from (2):

$$\frac{F_{A1} R_{A1}^2}{F_{B1} R_{B1}^2} = q \tag{3}$$

The discussion after (2) implies that in (3) q is the charge of P_A. Now, using charge P_C instead of P_A and P_1 in Figure 1 and then replacing P_C with P_B

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[5]In each case we may need to change the length of the threads and pivots for static equilibrium

with appropriate threads and pivots, and using (2) again, we have:

$$\frac{F_{C1}R_{C1}^2}{F_{B1}R_{B1}^2} = q* \tag{4}$$

Here again $q*$ is the charge of P_C. Dividing (3) by (4), we have:

$$\frac{F_{A1}R_{A1}^2}{F_{C1}R_{C1}^2} = \frac{q}{q*} \tag{5}$$

The meaning of (5) is dependent on the meaning given for (2). That is:(5) *means that the ratio, independent of the unchanged particle, threads and pivots, in* (2) *is equal to the ratio of the charge that is changed (P_A in (2)) and the charge which is used after replacement (P_B in (2) and in (5) P_C).*
Now, in (5) if we take all the particles (P_A, P_1, P_C) to be of unit charge and if we change the threads and pivots, allowed by the discussion after (2) and also in the discussion above on (5), we get in the right side of (5) 1 (one). This suggests the following: *Mutual electric force times the square of the distance of two unit charge is a constant of nature. This constant we denote by K_0.*
We are now ready to get back to the derivation as the title of the section suggests. We take, now, two charged particles S and T with charge q_S and q_T whose mutual force we want to calculate to establish Coulomb's law.
First, we take S and a unit charge u in Figure 1 so that equilibrium is established in accordance with our assumption. Then we replace S with v, we may or may need to change the pivots and threads, in Figure 1. If the force and distance between S and u, we denote by F_{Su} and R_{Su} respectively.[6] We get, by (2) and using the discussion about the constant K_0 above in Italic.

$$F_{Su}R_{Su}^2 = K_0 q_S \tag{6}$$

In (6), we have used the ratio q in (2) to be q_S, the charge of S, by the discussion after (2).
Because of the meaning given here about (5), if we replace u with T instead of replacing S with v , we get from (5),

$$\frac{F_{Su}R_{Su}^2}{F_{ST}R_{ST}^2} = \frac{1}{q_T} \tag{7}$$

[u is unit charge] which implies $K_0 q_S q_T = F_{ST}R_{ST}^2$ (using (6)) from which we get,

$$F_{ST} = \frac{K_0 q_S q_T}{R_{ST}^2} \tag{8}$$

(8) is the Coulomb's law by its very form which is easily recognized. However, one sees immediately after seeing the Coulomb Law in (8), that there is some problem in our definition of charge. This is the very reason we have said earlier that our definition is almost complete. The problem is: It is known that there are two types of charge in Nature[4][5][9][12] but our definition is not able to distinguish between them and as a consequence, we do not see modulus of any quantity in the right side of (8), although we are focusing on the magnitude of the force. For doing this, we will use a *algebraic trick*, so to speak.

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[6]Once again, we are using *Strong Law of Action-Reaction*[3][6]

5   Completing the Definition

Suppose we have a charge X. We define two collections A and B where we assume that there are at least two elements in each of the collection other than X. We further assume that any member of A, other than X which also belongs to A, repels X and any member of B attracts X and this completes our definition of the collections. Now, we ask ourselves what will happen if we take any two members from the sets. It is on experiment to find that *if the two members belong to the same set, they repel whereas if they belong to different sets they attract.*[4][5][9][12] For formulating this mathematically, take the X to be the unit charge P_B. For any member of the collection B, we designate their charge by putting a negative sign in front of the magnitude of the charge we defined earlier. It should be noted that we could put the minus in front of the charge of the members of A instead of B and this is merely a matter of convention. After taking such convention granted, we are already done distinguishing between two types of charge in nature. But we have not yet outlined how to find, in a simple way, whether two particles will repel or not. For doing this, we mention that following is true how remarkable this may seem : For any two particles S and T, if we designate the unit vector from S to T as $\hat{i}$, charge of S and T as q_S and q_T respectively, electric force on T by S as $\overrightarrow{F}_{ST}$, then form of (8) will be

$$\overrightarrow{F}_{ST} = \frac{K_0 q_S q_T}{R_{ST}^2}\hat{i} \tag{9}$$

(9) is a vector equation and in this equation we have to take into consideration the sign of the charge. One can check by considering different cases, the consistency of the direction of force in (9) with the requirement of the sentence in Italic above and *the Strong Law of Action-Reaction*[3][6]. This last equation, however, is the Coulomb's law in its general form.[4][9]

6   Conservation of Charge

For the conservation of charge[4][9], we don't see any other way but to postulate it[7] but we warn the reader that we have to take into account the sign of the charge here[4][5][9][12] and this is the very portion of this law which makes it look artificial. If the law seems obvious to some reader, we should mention that in the case we did not use the minus sign for the collection B, the law can be stated thus: *In a closed system, summation of initial total charges that belong to A and final total charges that belong to B is equal to the summation of final total charges that belong to A and the initial total charges that belong to B.* And it can safely be said that this form of the law does not seem obvious. Moreover, in the Italic sentence above we can interchange A and B and that also becomes a statement of the conservation of charge. This property of interchanging represents that the collection A and B are dual to each other.

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[7]Although we have derived Coulomb's Law, there seems to be no way to derive the conservation of charge. It would be interesting if we found some operational definition which enables us to derive both the Coulomb's Law and Conservation of charge.

7 Conclusion

We have tried here to give an elementary definition of electric charge and derive Coulomb's law with some other hypothesis at least for the systems that can be modeled using Figure 1. The definition used here is operational and so is not just a mathematical truism, rather it has some physical content which uses result from experiment. Although we have implicitly used Coulomb's law for predicting the result of the experiments, it is not a circularity in argument since we could proceed by performing the experiments.

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