Klein-Gordon Equation and Wave Function in Robertson-Walker and Schwarzschild space-time

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ABSTRACT
In the general relativity theory, we find Klein-Gordon wave functions in Robertson-Walker and Schwarzschild space-time. Specially, this article is that Klein-Gordon wave equations is treated by gauge fixing equations in Robertson-Walker space-time and Schwarzschild space-time.

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1. Introduction

In the general relativity theory, our article’s aim is that we find Klein-Gordon wave equations and functions by gauge fixing equations in Robertson-Walker space-time and Schwarzschild space-time.

The gauge fixing equation in general relativity theory

\[ A^\mu_{\;;\mu} = \frac{\partial A^\mu}{\partial x^\mu} + \Gamma^\mu_{\;\mu\rho} A^\rho \]

\[ \rightarrow \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^\mu_{\;\mu\rho} (A^\rho + \partial_\rho \Lambda) \]

\[ = \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^\mu_{\;\mu\rho} (A^\rho + g^{\rho\nu} \partial_\nu \Lambda) \]

(1)

2. Klein-Gordon wave equation in Robertson-Walker space-time

Because the gauge fixing equation is the electro-magnetic wave equation, Klein-Gordon wave equation is in Robertson-Walker space-time.

The Robertson-Walker solution is

\[ d\tau^2 = dt^2 - \frac{1}{c^2} \Omega^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \]

(2)

In this time, 2-dimensional solution is

\[ d\Omega = 0 \]

\[ d\tau^2 = dt^2 - \frac{1}{c^2} \Omega^2(t) \frac{dr^2}{1 - kr^2} \]

(3)

The gauge fixing equation is in 2-dimensional solution[3]

\[ \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^\mu_{\;\mu\rho} (A^\rho + g^{\rho\nu} \partial_\nu \Lambda) \]

\[ = \partial_\mu A^\mu + \Gamma^1_{\;10} A^0 + \Gamma^1_{\;11} A^1 + \partial_\mu g^{\mu\nu} \partial_\nu \Lambda + g^{\mu\nu} \partial_\mu \partial_\nu \Lambda + \Gamma^1_{\;10} g^{00} \frac{1}{c} \partial \frac{\partial \Lambda}{\partial t} + \Gamma^1_{\;11} g^{11} \frac{\partial \Lambda}{\partial r} \]

(4)

Hence, we can find Klein-Gordon wave equation in 2-dimentional Robertson-Walker space-time.

\[ \partial_\mu g^{\mu\nu} \partial_\nu \phi + g^{\mu\nu} \partial_\mu \partial_\nu \phi + \Gamma^1_{\;10} g^{00} \frac{1}{c} \frac{\partial \phi}{\partial t} + \Gamma^1_{\;11} g^{11} \frac{\partial \phi}{\partial r} \]

\[ = \left[ \frac{-2kr}{\Omega^2(t)} \frac{\partial}{\partial r} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{1 - kr^2}{\Omega^2(t)} \frac{\partial^2}{\partial r^2} - \frac{\Omega}{c\Omega} \frac{1}{c} \frac{\partial}{\partial t} + \frac{kr}{\Omega^2(t)} \frac{\partial}{\partial r} \right] \phi = \frac{m^2 c^4}{\hbar^2} \phi \]

\[ \Gamma^1_{\;10} = \frac{\Omega}{c\Omega}, \quad \Gamma^1_{\;11} = \frac{kr}{1 - kr^2} \]

(5)

In this time, we can think the shape of Klein-Gordon wave function from 2-dimensional Robertson-Walker space-time. In this case, light is
\[ d\tau^2 = dt^2 - \frac{1}{c^2} \Omega^2(t) \frac{dr^2}{1-kr^2} = 0 \]

\[ \int \frac{dt}{\Omega(t)} = \frac{1}{c} \int \frac{dr}{\sqrt{1-kr^2}} \quad (6) \]

Hence, matter wave function is in 2-dimensional Robertson-Walker space-time.

\[ \phi = A_0 \exp j\Phi, \quad A_0 \text{ is amplitude} \]

\[ \Phi = \omega_0 \int \frac{dt}{\Omega(t)} - k_0 \int \frac{dr}{\sqrt{1-kr^2}}, \quad \omega_0 \text{ is angular frequency}, \quad k_0 = \left| \vec{k}_0 \right| \text{ is wave number} \]

i) \( k = 1, \Phi = \omega_0 \int \frac{dt}{\Omega(t)} - k_0 \sin^{-1} r \)

ii) \( k = 0, \Phi = \omega_0 \int \frac{dt}{\Omega(t)} - k_0 r \)

iii) \( k = -1, \Phi = \omega_0 \int \frac{dt}{\Omega(t)} - k_0 \sinh^{-1} r \) \quad (7)

If the definition of energy and momentum is

\[ E = \frac{\hbar \omega_0}{\Omega(t)} \bar{p} = \frac{\hbar \vec{k}_0}{\Omega^2(t)} \quad (8) \]

Energy-Momentum relation is in Robertson-Walker space-time,

\[ m^2 c^4 = E^2 - \frac{\Omega^2(t)}{1-kr^2} \rho^2 c^2, E = mc^2 \frac{d\tau}{d\tau}, \bar{p} = m \frac{d\vec{r}}{d\tau} \quad (9) \]

Finally, angular frequency-wave number relation is in Robertson-Walker space-time,

\[ \frac{\hbar^2 \omega_0^2}{\Omega^2(t)} - \frac{\hbar^2 k_0^2 c^2}{\Omega^2(t)} \frac{1}{1-kr^2} = m^2 c^4 \quad (10) \]

Hence, Klein-Gordon wave equation-Eq(5) is satisfied by matter wave function-Eq(7) in Robertson-Walker space-time.

3. Klein-Gordon wave equation in Schwarzschild space-time

Because the gauge fixing equation is the electro-magnetic wave equation, Klein-Gordon wave equation is in Schwarzschild space-time.

The Schwarzschild solution is

\[ d\tau^2 = (1-\frac{2GM}{rc^2})dt^2 - \frac{1}{c^2} \left[ \frac{dr^2}{1-\frac{2GM}{rc^2}} + r^2 d\Omega^2 \right] \]

\[ a \Omega = 0 \]
\[ \text{d}t^2 = (1 - \frac{2GM}{r c^2}) \text{d}t^2 - \frac{1}{c^2} \frac{\text{d}r^2}{1 - \frac{2GM}{rc^2}} \] (12)

The gauge fixing equation is in 2-dimensional solution[3]

\[ \partial_\mu (A^\mu + g^{\mu\nu} \partial_\nu \Lambda) + \Gamma^{\mu}_{\rho\rho}(A^\rho + g^{\rho\nu} \partial_\nu \Lambda) \]

\[ = \partial_\mu A^\mu + \Gamma^{01}_0 A^1 + \Gamma^{11}_1 A^1 + \partial_\mu g^{\mu\nu} \partial_\nu \Lambda + g^{\mu\nu} \partial_\mu \partial_\nu \Lambda + \Gamma^{01}_0 \frac{\partial \Lambda}{\partial r} + \Gamma^{11}_1 \frac{\partial \Lambda}{\partial r} \]

\[ = \partial_\mu A^\mu + \partial_\mu g^{\mu\nu} \partial_\nu \Lambda + g^{\mu\nu} \partial_\mu \partial_\nu \Lambda \]

\[ \Gamma^{01}_0 = \frac{GM}{r^2 c^2} \frac{1}{1 - \frac{2GM}{rc^2}}, \quad \Gamma^{11}_1 = - \frac{GM}{r^2 c^2} \frac{1}{1 - \frac{2GM}{rc^2}} \]

(13)

Hence, we can find Klein-Gordon wave equation in 2-dimentional Schwarzschild space-time.

\[ \partial_\mu g^{\mu\nu} \partial_\nu \phi + g^{\mu\nu} \partial_\mu \partial_\nu \phi \]

\[ = \left[ \frac{2GM}{r^2 c^2} \frac{\partial}{\partial r} - \frac{1}{1 - \frac{2GM}{rc^2}} \frac{\partial^2}{\partial t^2} + (1 - \frac{2GM}{rc^2}) \frac{\partial^2}{\partial r^2} \right] \phi = \frac{m^2 c^4}{\hbar^2} \phi \] (14)

In this time, we can think the shape of Klein-Gordon wave function from 2-dimentional Schwarzschild space-time. In this case, light is

\[ \text{d}t^2 = (1 - \frac{2GM}{r c^2}) \text{d}t^2 - \frac{1}{c^2} \frac{\text{d}r^2}{1 - \frac{2GM}{rc^2}} = 0 \]

\[ t = \frac{1}{c} \int \frac{\text{d}r}{1 - \frac{2GM}{rc^2}} = \frac{r}{c} + \frac{2GM}{c^3 \ln | r - \frac{2GM}{c^2} |} \] (15)

Hence, Klein-Gordon wave function is in 2-dimentional Schwarzschild space-time-

\[ \phi = A_0 \exp (i \Phi), A_0 \text{ is amplitude} \]

\[ \Phi = \omega_0 t - k_0 r - k_0 \frac{2GM}{c^2} \ln | r - \frac{2GM}{c^2} | \]

\[ \omega_0 \text{ is angular frequency, } k_0 = \left| \vec{k}_0 \right| \text{ is wave number} \] (16)

If the definition of energy and momentum is
\[ E = \frac{\hbar \omega_0}{(1 - \frac{2GM}{rc^2})}, \bar{\rho} = \hbar \bar{k}_0^2 (1 - \frac{2GM}{rc^2}) \] (17)

Energy-Momentum relation is in Schwarzschild space-time,

\[ m^2 c^4 = (1 - \frac{2GM}{rc^2})E^2 - \frac{\rho^2 c^2}{(1 - \frac{2GM}{rc^2})}, E = mc^2 \frac{d\tau}{dt}, \bar{\rho} = m \frac{df}{d\tau} \] (18)

Finally, angular frequency-wave number relation is in Schwarzschild space-time,

\[ \frac{\hbar^2 \omega_0^2}{(1 - \frac{2GM}{rc^2})} - \hbar^2 k_0^2 (1 - \frac{2GM}{rc^2}) = m^2 c^4 \] (19)

Hence, Klein-Gordon wave equation-Eq(14) is satisfied by matter wave function-Eq(16) in Schwarzschild space-time.

4. Conclusion

We find Klein-Gordon wave equation and function in Robertson-Walker space-time. We find Klein-Gordon wave equation and function in Schwarzschild space-time.

References