Abstract

We prove the Intersecting-Chords Theorem as a corollary to a relationship, derived via Geometric Algebra, about the product of the lengths of two segments of a single chord. We derive a similar theorem about the product of the lengths of a secant a chord.

“Demonstrate that \((AV)(VB) = (CV)(VD)\).”
1 Introduction

As Proposition 35 in Book III of his *Elements*, Euclid proved the Intersecting-Chords theorem, which in modern terms is stated as in Fig. 1. Here, we will prove that theorem as a corollary to the following:

The product of the lengths of the two segments into which a point \( V \) divides a chord of a circle, is equal to the square of the circle’s radius minus the square of the point’s distance from the circle’s center.

2 Proof

We’ll treat the blue chord that we showed in Fig. 1. We set up the proof as in Fig. 2. Point \( V \) is an arbitrary point within the circle, and \( AB \) is an arbitrary chord with direction \( \hat{a} \). The perpendicular from the circle’s center to the chord divides the chord in equal parts (Fig. 3). The “reject” of \( v \) from \( \hat{a} \) is \( (v \wedge \hat{a}) \hat{a} \) ([1], p. 120), the length of which is \( \|v \wedge \hat{a}\| \). Thus, by the Pythagorean Theorem, each half of the chord measures \( \sqrt{r^2 - \|v \wedge \hat{a}\|^2} \). Hence, the segments \( AV \) and \( VB \) measure \( \sqrt{r^2 - \|v \wedge \hat{a}\|^2} + v \cdot \hat{a} \) and \( \sqrt{r^2 - \|v \wedge \hat{a}\|^2} - v \cdot \hat{a} \), respectively. The product of those lengths is...
Figure 2: Setting up the proof in GA terms. Point $V$ is an arbitrary point within the circle, and $AB$ is an arbitrary chord, with direction $\hat{a}$, through $V$.

Figure 3: The “reject” of $v$ from $\hat{a}$ is $(v \wedge \hat{a})\hat{a}$ (p. 120), the length of which is $|v \wedge \hat{a}|$. Thus, by the Pythagorean theorem, each half of the chord measures $\sqrt{r^2 - |v \wedge \hat{a}|^2}$. Hence, the segments $AV$ and $VB$ measure $\sqrt{r^2 - |v \wedge \hat{a}|^2} + v \cdot \hat{a}$ and $\sqrt{r^2 - |v \wedge \hat{a}|^2} - v \cdot \hat{a}$, respectively.
Because $V$ was an arbitrary point, and $AB$ was an arbitrary chord through that point, the result we obtained is valid for every chord through every point within the circle (by the Law of Universal Generalization ([2])). Therefore, $(CV)(VD) = r^2 - v^2 = (AV)(VB)$, and $(CV)(VD) = (AV)(VB) \square$.

$$r^2 - \|v \wedge \hat{a}\|^2 - (v \cdot \hat{a})^2 = r^2 - \left[\|v \wedge \hat{a}\|^2 + (v \cdot \hat{a})^2\right] = r^2 - v^2.$$  

Because $V$ was an arbitrary point, and $AB$ was an arbitrary chord through that point, this result is valid for every chord through every point within the circle (by the Law of Universal Generalization ([2])). The Intersecting-Chords Theorem now follows as a corollary (Fig. 4): $(CV)(VD) = r^2 - v^2 = (AV)(VB)$; therefore $(CV)(VD) = (AV)(VB) \square$.

Through a similar analysis, we can establish that in Fig 5, $(AB)(AC) = v^2 - r^2$ for any point outside the circle.

References


Figure 5: Through an analysis similar to that illustrated in Fig. 3, we can establish that in the present Figure, $(AB)(AC) = v^2 - r^2$. 

\[ (AB)(AC) = v^2 - r^2. \]