Proof of the Riemann Hypothesis

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December 24, 2020

Abstract
In this article we will prove the well-known Riemann hypothesis.

1 Introduction
The Riemann Hypothesis is a famous conjecture made by Bernhard Riemann in his article on prime numbers. Riemann, as indicated by the title of his article [1], wanted to know the number of prime numbers in a given interval of the real line, for this he extended an observation by Euler and defined a function called Zeta. Riemann obtained an explicit formula, which depends on the non-trivial zeros of the Zeta function, for the quantity he sought. Throughout the process, Riemann mentions that probably all the non-trivial zeros of the Zeta function are, in the now called critical line, that is, when the complex argument $s = \sigma + iT$ of the Zeta function has a real part equal to half - $\sigma = \frac{1}{2}$. The proof of the Riemann hypothesis has direct implications for the distribution of prime numbers. In the following sections, we will demonstrate that the hypothesis is true.

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2 Proof

We started the proof by writing the Taylor series of Riemann's $\xi(s)$ function, which has the same zeros as Riemann's zeta function, around $s = \sigma_1 + iT_0$ where $T_0$ is the imaginary part in which there is an assumed zero of the $\xi(s)$ function outside the critical line. We have

$$\xi(s) = \sum_{i=0}^{\infty} a_i (s - \sigma_1 - iT_0)^i.$$ (1)

Suppose we have a zero of the function $\xi(s)$, $s = \sigma_1 + C + iT_0$, outside the critical line

$$\xi (\sigma_1 + IT_0 + C) = \sum_{i=0}^{\infty} a_i C^i = 0$$ (2)

or

$$\xi (\sigma_1 + IT_0 + C) = \sum_{i=0; i \neq k}^{\infty} a_i C^i + a_k C^k = 0$$ (3)

where $C$ is a positive real value less than the unit and $k$ is one of the sum factors (2). Therefore we would have

$$\Re a_k C^k = - \sum_{i=0; i \neq k}^{\infty} \Re a_i C^i$$ (4)

or choosing $k = 0$ we get

$$\Re \xi (\sigma_1 + IT_0) = - \sum_{i=1}^{\infty} \Re a_i C^i.$$ (5)

We can write the equation (5) in the form

$$\frac{\Re \xi (\sigma_1 + IT_0)}{\Re \xi (1 + \sigma_1 + IT_0) - \Re \xi (\sigma_1 + IT_0)} = \frac{\sum_{i=1}^{\infty} \Re a_i C^i}{\Re \xi (1 + \sigma_1 + IT_0) - \Re \xi (\sigma_1 + IT_0)} = - \sum_{i=1}^{\infty} \Re a_i C^i$$ (6)

where we initially assume that $\Re \xi (1 + \sigma_1 + IT_0) - \Re \xi (\sigma_1 + IT_0) \neq 0$. Taking the module of equation (6) we are left with

$$\left| \frac{\Re \xi (\sigma_1 + IT_0)}{\Re \xi (1 + \sigma_1 + IT_0) - \Re \xi (\sigma_1 + IT_0)} \right| = \frac{\sum_{i=1}^{\infty} \Re a_i C^i}{\sum_{i=1}^{\infty} \Re a_i} < 1$$ (7)

because if all the coefficients $a_i$ were equal to the equation (3) it would imply that the function $\xi(s)$ would always be null. Due to the equation (7) we should have

$$|\Re \xi (\sigma_1 + IT_0)| < |\Re \xi (1 + \sigma_1 + IT_0) - \Re \xi (\sigma_1 + IT_0)|.$$ (8)
Let $\sigma_0 = 1 - \sigma_1 - C$ be the real part of the simetric zero of the $\xi(s)$ function—see equation (2). Let’s do

$$
1 + \sigma_1 = \sigma_0 \\
\sigma_1 = \sigma_0 - 1
$$

(9)

therefore

$$
|\Re \xi (\sigma_0 - 1 + IT_0)| < |\Re \xi (\sigma_0 + IT_0) - \Re \xi (\sigma_0 - 1 + IT_0)|
$$

(10)

satisfying $|\Re \xi (\sigma_0 + IT_0) - \Re \xi (\sigma_0 - 1 + IT_0)| \neq 0$ because $\Re \xi (\sigma_0 + IT_0) = 0$ and $\Re \xi (\sigma_0 - 1 + IT_0) \neq 0$, where the last inequality occurs because the argument is outside the critical strip. Consequently we will have the contradiction

$$
|\Re \xi (\sigma_0 - 1 + IT_0)| < |\Re \xi (\sigma_0 - 1 + IT_0)|.
$$

(11)

Notice that

$$
1 + \sigma_1 = 1 - \sigma_1 - C \\
\sigma_1 = -\frac{C}{2} \\
\sigma_1 + C = \frac{C}{2}
$$

(12)

and as we must have that $C < 1$ we show, see the equation (2), that there can be no zeros on the left side of the critical line and by symmetry across the critical strip and therefore there can be no zero outside the critical line and the Riemann hypothesis is true.

### 3 Conclusion

After the efforts of several mathematicians and scientific disseminators, the problem has reached its maturity and can be solved.

### 4 Acknowledgment

Posthumously, I thank my dear mother Edna Vieira Rocha de Rezende who always motivated me in life.

I also thank my father Rodolpho Antônio de Rezende who always encouraged me in my personal life and in the habit of reading.
References
