Infinite Sum of 1 + 4 + 9 + 16 + 25 +..........+ ∞

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Abstract

This article is providing an infinite sum of $1^2 + 2^2 + \ 3^2 + 4^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots\ldots \infty = \frac{1}{70}$.

1 Introduction

The infinite sum of this kind of series can be calculated by the Riemann Zeta Function[1]. The infinite sum of this series is calculated by Srinivas Ramanujan. And Srinivas Ramanujan expressed a general formula[2] for this.

But when we solve this equation using the Algebraic and Arithmetic methods like when Srinivas Ramanujan proved his sum[3] of infinite natural numbers. And Ramanujan gets the result that is written as $1 + 2 + 3 + 4 + 5 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty = \ -\frac{1}{12}$.

Similarly, when we solve this equation $1^2 + 2^2 + \ 3^2 + 4^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$, fundamentally we get a different result.

So, this article seeks at the very delightful pedagogical approach to solving this series. And in the last we get a different result.

2 Derivation

This section is written at that level of simplicity, to emphasize how straightforward all the algebraic manipulations and concepts are. And we derive the series expressions and get our result.

2.1 Derivation of the Series –

First, we write the series with a variable named 'S'. So, S = $1^2 + 2^2 + \ 3^2 + 4^2 + 5^2 + \ldots\ldots\ldots \infty$. Now solving the power, the of this series and we get –

S = 1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + ...+ ∞. -----(1)

Now separating the number which containing the even number like $(2^2 + 4^2 + 6^2 + \ldots\ldots\ldots \infty.)$ and similarly for odd numbers like $(1^2 + \ 3^2 + 5^2 + \ldots\ldots\ldots \infty.)$ So, separating the series in this manner we get –

S = {4 + 16 + 36 + ...+ ∞} + {1 + 9 + 25 + ...+ ∞}. -----(2)

Now taking 4 as common factor from first sequence and we get –

S = 4 × {1 + 4 + 9 + ...+ ∞} + {1 + 9 + 25 + ...+ ∞}. -----(3)

And we see that the term {1 + 4 + 9 + ...+ ∞} is equal to the 'S', so put the 'S' in the place of {1 + 4 + 9 + ...+ ∞} and we get –

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S = 4S + 1 + 9 + 25 + 49 + 81 +...............................+ ∞. and then

S – 4S = 1 + 9 + 25 + 49 + 81 +...............................+ ∞. and we get –

-3S = 1 + 9 + 25 + 49 + 81 +................................+ ∞. -----(4)

So, now in equation (4) we subtract 1 from both sides from each number and we have to subtract 1 infinite time because we have infinite numbers in the right-hand side so, we get –

-3S – {1 + 1 + 1 + 1 + 1 +............+ ∞} = (1 - 1) + (9 - 1) + (25 - 1) + (49 - 1) + (81 - 1) + ...+ ∞. -----(5)

Here we have to notice that the series in the left side {1 + 1 + 1 + 1 + 1 +............+ ∞} this is called the Grandi's Series. And the Infinite sum of this Grandi's series[4] is given by ($-\frac{1}{2}$). So, we put $-\frac{1}{2}$ in the place of {1 + 1 + 1 + 1 + 1 +............+ ∞} and we get –

-3S + $\frac{1}{2}$ = 8 + 24 + 48 + 80 + 120 + ... + ∞ -----(6)

Now, we take 8 as a common factor from right side of the series and then divided the both sides of series by 8 and we get –

$-\frac{3S}{8} + \frac{1}{16} =$ 1 + 3 + 6 + 10 + 15 + ... + ∞ -----(7)

Now, we add the term in pairs of two of right-hand side of the series like –

$-\frac{3S}{8} + \frac{1}{16} =$ (1 + 3) + (6 + 10) + (15 + 21) + ... + ∞ -----(8)

On solving this equation (8), we get –

$-\frac{3S}{8} + \frac{1}{16} =$ 4 + 16 + 36 + 64 + 100 + ... + ∞ -----(9)

Now, again take 4 as a common factor from right hand side and after taking common divide the both sides of the series by 4 and we get –

$-\frac{3S}{32} + \frac{1}{64} =$ 1 + 4 + 9 + 16 + 25 + 36 + 49 + ... + ∞ -----(10)

And we see that the right-hand side of the equation is equal to the 'S' as we take in the beginning of the series now put the 'S' in the place of 1 + 4 + 9 + 16 + 25 + 36 + 49 + ... + ∞ and solve we get –

$-\frac{3S}{32} + \frac{1}{64} = S \Rightarrow \frac{1}{64} = S + \frac{3S}{32} = \frac{35S}{32}$ So, $\frac{1}{64} = \frac{35S}{32}$ -----(11)

On solving the Equation (11) we get the value of 'S'. **S = $\mathbf{\frac{1}{70}}$.** So, finally the result is written as –

1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 + ...+ ∞ = $\mathbf{\frac{1}{70}}$.

3 References –

[1] Riemann Zeta Function, https://en.wikipedia.org/wiki/Riemann_zeta_function

[2] Ramanujan Summation, https://en.wikipedia.org/wiki/Ramanujan_summation

[3] Ramanujan Infinite sum of Natural Numbers,

https://en.wikipedia.org/wiki/1_%2B_2_%2B_3_%2B_4_%2B_%E2%8B%AF

[4] Grandi's Series Sum of (1+1+1+1+1+1+1+..................+∞)

https://en.wikipedia.org/wiki/1_%2B_1_%2B_1_%2B_1_%2B_%E2%8B%AF