Newtonian Origin of General Relativity

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This paper shows that there exists in “Classical Newtonian Mechanics” a relationship between the curvature of space and the density of energy similar to that of “General Relativity”. It is as if the General Theory of Relativity has replaced, in this relationship of Newtonian Mechanics, the scalar quantities (space curvature and Energy density) by their equivalent tensors. From this point of view, the Theory of General Relativity can be considered as an extension of Newton's classical mechanics, an extension from Euclidean flat space-time to curved space-time of Riemann.

I. INTRODUCTION

In General Theory of Relativity [1], [2], [3], the Einstein's equations relate the geometry of space-time curvature (expressed by the Einstein tensor) with the distribution of matter within it (expressed by the stress–energy tensor).

$$\boldsymbol{G}_{\mu\nu} = \frac{8\pi G}{c^4}\boldsymbol{T}_{\mu\nu}$$

where

$\boldsymbol{G}_{\mu\nu}$ is the Einstein tensor, it depends on the coordinates,

$\boldsymbol{T}_{\mu\nu}$ is the stress–energy tensor, it depends on the mass-energy distribution,

G is the Newtonian constant of gravitation and **c** is the celerity of light in vacuum.

The coefficient $\frac{\boldsymbol{8\pi G}}{\boldsymbol{c^4}}$ is called the “Einstein gravitational constant”.

The tensors in Einstein's equation are symmetric of dimension 4x4. So we have ten different equations to solve. The terms of the tensor $\mathrm{T}_{\mu\nu}$ are the entries of the problem. The goal is to find the relation between the coordinates which describes the system and which could be found in each term of the tensor $G\mu\nu$. The search for exact solutions to the equation is therefore an extremely complicated exercise,

The Solving Einstein's equation is only possible in very simple cases like:

- A spherical distribution of static mass-energy: this is the Schwarzschild metric that results in the calculation of the horizon of the black hole or Schwarzschild radius [4], [5].
- Weak field and low velocity approximation, the ten components of Einstein's equation reduce to a single equation which is the Poisson equation of Newtonian gravity [6]

Despite the resolution of Einstein's equation for very particular cases, the General Relativity remains a very difficult theory to assimilate and Newton's mechanics is considered as borderline case of this theory.

The objective of this paper is not to present a new theory, nor to find new solutions to Einstein's equations but to show that there exists in Newton's mechanics a relation between the scalar curvature in Euclidean flat space-time **and the density of energy similar to that of the General Relativity with its equivalent tensors** in the curved space-time of Riemann.

Einstein was probably inspired by this relation in Classical Mechanics to establish the equation of his Theory, this is important for the physicists to know this and then to understand that the theory of General Relativity can be considered as an extension of Newton's Classical Mechanic : an extension from Euclidean flat space-time to curved space-time of Riemann**.**

II. DESCRIPTION

Consider a point mass « $\boldsymbol{m}$ » which is not subject to any external force. According to the Newton's first law, its motion is then uniform rectilinear and its velocity is equal to « $\vec{\boldsymbol{v}}$ »

Figure 1: uniform rectilinear motion

If this point mass « $\boldsymbol{m}$ » passes near a mass « $\boldsymbol{M}$» with $\boldsymbol{M} \gg \boldsymbol{m}$, it starts to turn around «$\boldsymbol{M}$» under the effect of gravitational force (Newton's law of attraction). Its trajectory is quasi-circular with radius « $\boldsymbol{R}$ ».

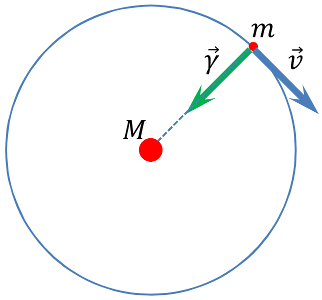

Figure 2: uniform circular motion

First Remark:

*The mass « **M**» seems to curve the linear space of the mass « **m**» but not in the meaning of space-time curvature of General Relativity.*

The force exerted by the mass « ***M*** » on « ***m*** » is, according to Newton's law of attraction,

$$F = G\frac{mM}{R^2} \qquad \textbf{(1)}$$

According to the Newton's second law of motion, we can write

$$F = m\gamma \qquad \textbf{(2)}$$

therefore

$$\gamma = G\frac{M}{R^2} \qquad \textbf{(3)}$$

which is equal at the same time to the centripetal acceleration of the uniform circular motion

$$\gamma = \frac{v^2}{R} \qquad \textbf{(4)}$$

hence

$$\frac{v^2}{R} = G\frac{M}{R^2} \qquad \textbf{(5)}$$

$$\frac{1}{R} = \frac{G}{v^2}\frac{M}{R^2} \qquad \textbf{(6)}$$

Multiplying and dividing the second term by 4π

$$\frac{\mathbf{1}}{\boldsymbol{R}} = \frac{4\pi G}{v^2}\,\frac{\boldsymbol{M}}{\boldsymbol{4\pi R^2}} \qquad \textbf{(7)}$$

$\frac{\mathbf{1}}{\mathbf{R}}$ = **curvature** of the circular trajectory of radius « ***R*** » (according to the definition of curvature).

$\frac{\mathbf{M}}{\mathbf{4\pi R^2}}$ = **fictitious areal density of M**. It is as if this mass **M** is distributed (by its effect or its gravitational field) over the surface $\mathbf{4\pi R^2}$ of the sphere of radius « ***R*** » where the mass « ***m*** » moves.

Equation **(7)** is a good example to understand this relationship of proportionality between curvature of space and density of mass (or density of energy, see below equation **18**).

Indeed, when the radius « ***R*** » decreases (*) both the curvature $\frac{\mathbf{1}}{\mathbf{R}}$ of a smaller circle and the fictitious areal density of mass $\frac{\mathbf{M}}{\mathbf{4\pi R^2}}$ on a smaller sphere increase **and in the same ratio**,

* *According to* **(5)**, *the radius « **R** »*

$$R = \frac{GM}{v^2} \qquad \textbf{(8)}$$

*decreases when the velocity of rotation « **v**» of the mass « **m**» increases*

$$v = \sqrt{\frac{GM}{R}} \qquad \textbf{(9)}$$

We will demonstrate now that the relation **(7)** between the curvature of space and the density of the mass is independent of the variation of the velocity« ***v*** » because the radius « ***R*** » depends on « ***v*** »,

<u>Demonstration</u>

Multiply the velocity « ***v*** » of « ***m*** » around the same mass « ***M*** » by a factor « ***a*** »

$$v \to a.v$$

Which corresponds, according to equation **(8)**, to a radius « ***R*** », divided by « $\boldsymbol{a^2}$»

$$R \to \frac{R}{a^2}$$

The equation **(7)**

$$\left(\frac{\mathbf{1}}{\boldsymbol{R}}\right) = \frac{4\pi G}{v^2}\left(\frac{\boldsymbol{M}}{\boldsymbol{4\pi R^2}}\right)$$

becomes

$$\frac{\mathbf{1}}{\frac{\boldsymbol{R}}{\boldsymbol{a^2}}} = \frac{4\pi G}{(\boldsymbol{av})^2}\,\frac{\boldsymbol{M}}{\boldsymbol{4\pi}(\frac{\boldsymbol{R}}{\boldsymbol{a^2}})^2}$$

$$\frac{\boldsymbol{a^2}}{\boldsymbol{R}} = \frac{4\pi G}{\boldsymbol{a^2}v^2}\,\frac{\boldsymbol{a^4 M}}{\boldsymbol{4\pi R^2}}$$

$$\boldsymbol{a^2}\left(\frac{\mathbf{1}}{\boldsymbol{R}}\right) = \frac{\boldsymbol{4\pi G}}{\boldsymbol{v^2}}\;\boldsymbol{a^2}\left(\frac{\boldsymbol{M}}{\boldsymbol{4\pi R^2}}\right)$$

The curvature $\boldsymbol{a^2}\left(\frac{\mathbf{1}}{\boldsymbol{R}}\right)$ and the density of mass $\boldsymbol{a^2}\left(\frac{\boldsymbol{M}}{\boldsymbol{4\pi R^2}}\right)$ have been multiplied by $\boldsymbol{a^2}$, the proportionality factor $\frac{4\pi G}{v^2}$ remained the same as before multiplying the velocity by « ***a*** ».

Second Remark:

*When the velocity « **v** » is multiplied by a factor « **a** », the curvature and the mass density will be multiplied by « $\boldsymbol{a}^2$ », the proportionality factor $\frac{4\pi G}{v^2}$ between these two quantities remains the same; it is independent of the variation « **a** » of the velocity «**v**». We can therefore choose a reference velocity «v_0».*

Which value should we choose for v_0 ?

It makes more sense to choose as a reference velocity «v_0» for a mass in uniform circular motion the maximum velocity it can have on its orbit.

The reference velocity is given by the equation **(9)**

$$v_0 = \sqrt{\frac{GM}{R_0}} \qquad \textbf{(10)}$$

Its maximum value corresponds to the maximum escape velocity « v_{esc} » [7]. « v_{esc} » is calculated according to the principle of energy conservation:

kinetic energy + potential energy = 0

$$\frac{1}{2} m\, v_{esc}^2 - \frac{GmM}{R_0} = 0$$

$$v_{esc} = \sqrt{\frac{2GM}{R_0}} \qquad \textbf{(11)}$$

Taking into account equation **(10)**

$$v_{esc} = \sqrt{2}\, v_0 \qquad \textbf{(12)}$$

therfore

$$(v_0)_{max} = \frac{(v_{esc})max}{\sqrt{2}} \qquad \textbf{(13)}$$

We replace «v_0» by its value in **(7)** we obtain:

$$\frac{\boldsymbol{1}}{\boldsymbol{R}} = \frac{8\pi G}{(v_{esc})_{max}^2} \frac{\boldsymbol{M}}{\boldsymbol{4\pi R^2}} \qquad \textbf{(14)}$$

What becomes of this equation (14) with the Restreinte Relativity ?

According to Restreint Relativity [8] and [9]

$$(v_{esc})_{max} = c \qquad \textbf{(15)}$$

$$E_M = Mc^2 \qquad \textbf{(16)}$$

Equation (16) allows us to write:

$$M = \frac{E_M}{c^2} \qquad \textbf{(17)}$$

where «c» represents the celerity of light, and «E_M» the energy associated with the mass *M.*

By replacing $(v_{esc})_{max}$ and M by their value in the equation **(14)** we obtain:

$$\frac{\boldsymbol{1}}{\boldsymbol{R}} = \frac{8\pi G}{c^4} \frac{\boldsymbol{E_M}}{\boldsymbol{4\pi R^2}} \qquad \textbf{(18)}$$

where :

$\frac{\mathbf{1}}{\mathbf{R}}$ = **curvature** of the circular trajectory of radius « ***R*** »

$\frac{\mathbf{E_M}}{\mathbf{4\pi R^2}}$ = **areal energy density of M**. This energy E_M is distributed over the surface $\mathbf{4\pi R^2}$ of the sphere of radius « ***R*** » where the mass « ***m*** » moves.

Third Remark:

*Equation **(18)** shows a relationship between curvature and energy density similar to the equation of General Relativity **(19)***

$$\boldsymbol{G_{\mu\nu}} = \frac{8\pi G}{c^4} \boldsymbol{T_{\mu\nu}} \qquad \textbf{(19)}$$

where

$\boldsymbol{G_{\mu\nu}}$ = *Einstein tensor, it represents the **curvature** of space-time*

$\boldsymbol{T_{\mu\nu}}$ = *Energy-impulse tensor it represents the total **energy density** (mass and energy).*

Let's continue our analysis to find the meaning of « $\boldsymbol{R_0}$ ». From **(11)** and **(15)**, we deduce

$$\boldsymbol{R_0} = \frac{\boldsymbol{2GM}}{\boldsymbol{v_{esc}^2}} = \frac{\boldsymbol{2GM}}{\boldsymbol{c^2}} \qquad \textbf{(20)}$$

« $\boldsymbol{R_0}$ » is the minimum radius of the orbit that corresponds to the maximum escape velocity « ***c*** ». Below this value the escape velocity becomes greater than « ***c*** », and since it is impossible, the mass « ***m*** » can never escape and the mass « ***M*** » becomes a black hole. « $\boldsymbol{R_0}$ » is the Schwarzschild radius in General Relativity [10]. For the sun, R_0 = 2.948 km. This means that if we concentrate the mass of the sun in a sphere with a radius less than « $\boldsymbol{R_0}$ », it becomes a black hole for everything that passes at a distance less than « $\boldsymbol{R_0}$ » from it.

Fourth Remark:

The value of the reference velocity $v_0 = \frac{c}{\sqrt{2}}$ chosen in the equation of General Relativity corresponds to the *maximum escape velocity or to the orbit whose radius is equal to Schwarzschild radius.*

Fifth Remark:

*If the inertial mass "**m**" in "**m**γ" of equation (2) were not equal to the gravitational mass "**m**" in "$G\frac{\boldsymbol{mM}}{R^2}$" of equation (1), we would not have had equation (3) to compare it to equation (4) and then establish equation (7) or (18). This may explain why Einstein said that Newton's principle of equivalence between inertial mass and gravitational mass was the starting point of the famous theory of General Relativity.*

Sixth Remark:

The principle of equivalence cannot be applied to electric force, because we have a mass on one side and a charge on the other side:

$$F = m\gamma = \frac{1}{4\pi\varepsilon}\frac{qQ}{R^2} \qquad (21)$$

*where "**q**" is the charge of masse "**m**" and "**Q**" is the charge of masse "**M**", therefore, we cannot simplify to have an equation similar to equation (3), i.e.*

$$\gamma = \frac{1}{4\pi\varepsilon}\frac{Q}{R^2} \qquad (22)$$

to then compare it to the equation (4) and establish an equation for the electric charge equivalent to equation (18), that is to say:

$$\frac{\boldsymbol{1}}{\boldsymbol{R}} \neq \frac{1}{\varepsilon\, v^2}\frac{\boldsymbol{Q}}{\boldsymbol{4\pi R^2}} \qquad (23)$$

*The conclusion is that **we cannot have a relation between** a "**charge density tensor**" and the "space **curvature tensor**" equivalent to relation of Einstein's General Relativity.*

III. CONCLUSION

This paper is a new approach in the understanding of the General Theory of Relativity. It shows the origin of its equations in Newton's mechanics and it enables a better understanding of this relationship between the curvature of space and the density of energy without going through the non-intuitive tensorial equation of this theory. This paper also shows the importance of the principle of equivalence "gravitational mass and inertial mass" in the establishment of the equation of this theory. It demonstrates the impossibility to having an equivalent theory for the other fundamental forces like the electric force.

IV. REFERENCES

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