Schrödinger Equation and Free Particle Wave Function

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Abstract

Using the wave function of a free particle we obtain a solution of the Schrödinger equation for a class of potentials.

1 Time dependent accelerating frame of reference

Consider an accelerating frame of reference $\mathcal{F}'$ with coordinates x', t' and an inertial frame of reference $\mathcal{F}$ with coordinates x, t. The coordinates of the frames being related by

$$x' = x - f(t) \qquad t' = t \tag{1}$$

Since $dx' = dx$ and position probabilites are the same for $\mathcal{F}'$ and $\mathcal{F}$ we have for the wave function $\psi(x,t)$ with respect to $\mathcal{F}$ and corresponding wave function $\psi'(x',t')$ with respect to $\mathcal{F}'$ that [1]

$$|\psi'(x',t')|^2 = |\psi(x,t)|^2 \tag{2}$$

Consequently there is a real valued function $\beta(x,t)$ such that

$$\psi'(x',t') = e^{-\frac{i}{\hbar}\beta(x,t)}\psi(x,t) \tag{3}$$

With respect to $\mathcal{F}$ let the wave function $\psi(x,t)$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}(x,t) = i\hbar\frac{\partial\psi}{\partial t}(x,t) \tag{4}$$

With respect to $\mathcal{F}'$ we have an additional force $m\ddot{f}(t)$ and hence an additional potential $m\ddot{f}(t)x' + V_0(t')$. The wave function $\psi'(x',t')$ then satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi'}{\partial x'^2}(x',t') + \Big(m\ddot{f}(t')x' + V_0(t')\Big)\psi'(x',t') = i\hbar\frac{\partial\psi'}{\partial t'}(x',t') \tag{5}$$

Now

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial t'} = \dot{f}\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \tag{6}$$

and on substituting (3) in (5) and using (4) and (6) gives

$$\begin{aligned}\left[\frac{i\hbar}{2m}\frac{\partial^2\beta}{\partial x'^2}\right. &+ \left.\frac{1}{2m}\left(\frac{\partial\beta}{\partial x'}\right)^2 + m\ddot{f}(x-f) + V_0 - \dot{f}\frac{\partial\beta}{\partial x} - \frac{\partial\beta}{\partial t}\right]\psi \\ &+ \frac{i\hbar}{m}\left[\frac{\partial\beta}{\partial x} - m\dot{f}\right]\frac{\partial\psi}{\partial x} = 0\end{aligned} \tag{7}$$

We have

$$\beta(x,t) = m\dot{f}(t)x + \int_0^t [V_0(s) - mf(s)\ddot{f}((s) - \frac{1}{2}m\dot{f}(s)^2]ds + C \tag{8}$$

is the unique solution of (7) satisfying the initial condition [2]

$$\beta(x,0) = m\dot{f}(0)x + C \tag{9}$$

2 Space and time dependent velocity

Let $v_\epsilon(x,t)$ be a smooth function in variables ϵ, x, t. Require $v_\epsilon(x,0) = 0$. Define $X_\epsilon(u;t)$ to be the curve $x(t)$ such that

$$\frac{dx}{dt} = v_\epsilon(x,t) \tag{10}$$

and $x(0) = u$. Require that the curves are defined for all t and the curves do not intersect. We then have a frame of reference $\mathcal{F}_\epsilon$ with coordinates x_ϵ, t_ϵ such that

$$x_\epsilon = X_\epsilon(x;t) \qquad t_\epsilon = t \tag{11}$$

Let $\psi(x,t)$ satisfy (4). Let $V_\epsilon(x_\epsilon, t_\epsilon)$ be the potential in these coordinates. We have

$$\frac{1}{m}\frac{\partial V_\epsilon}{\partial x_\epsilon}(x_\epsilon, t_\epsilon) = v(x,t)\frac{\partial v}{\partial x}(x,t) + \frac{\partial v}{\partial t}(x,t) \tag{12}$$

Let $\psi_\epsilon(x_\epsilon, t_\epsilon)$ be the wave function satisfying the Schrödinger equation in x_ϵ, t_ϵ coordinates and $\psi_\epsilon(x,0) = \psi(x,0)$. Let $B(x_0;\epsilon)$ be the set of points $x_0 - \epsilon < x < x_0 + \epsilon$. Choose $v_\epsilon(x,t)$ so that for $u \in B(x_0;\epsilon)$

$$X_\epsilon(u;t) = X_0(x_0;t) + u - x_0 \tag{13}$$

Let $\widehat{\mathcal{F}}$ be a frame of reference with coordinates $\hat{x}, \hat{t}$ related to coordinates x, t of $\mathcal{F}$ by

$$\hat{x} = x - X_0(x_0;t) \qquad \hat{t} = t \tag{14}$$

The potential in these coordinates is $m\ddot{X}_0(\hat{x}_0 : \hat{t})\hat{x} + V_0(\hat{t})$. Let $\widehat{\psi}(\hat{x}, \hat{t})$ be the wave function satisfying the Schrödinger equation with this potential and $\widehat{\psi}(x,0) = \psi(x,0)$. We have by (8) a $\widehat{\beta}(x,t)$ such that

$$\frac{\widehat{\psi}(\hat{x},\hat{t})}{\psi(x,t)} = e^{-\frac{i}{\hbar}\widehat{\beta}(x,t)} \qquad \frac{\partial \widehat{\beta}}{\partial x}(x,t) = m\dot{X}_0(x_0;t) \tag{15}$$

hence for points $(X_\epsilon(u;t), t)$ where $u \in B(x_0;\epsilon)$ we have

$$\frac{\psi_\epsilon(x_\epsilon,t_\epsilon)}{\psi(x,t)} = e^{-\frac{i}{\hbar}\widehat{\beta}(x,t)} \qquad \frac{\partial \widehat{\beta}}{\partial x}(x,t) = m\dot{X}_0(x_0;t) \tag{16}$$

Define coordinates $x' = x_0, t' = t_0$. Let $\psi'(x',t') = \psi_0(x',t')$. Now x_0 is arbitrary and let $\beta(x,t)$ be the limit of $\widehat{\beta}(x,t)$ as $\epsilon \to 0$ so we get

$$\frac{\partial \beta}{\partial x}(x,t) = mv_0(x,t) \tag{17}$$

Require $v(x,t) \to 0$ as $v \to -\infty$. We then have $\beta(x,t) \to 0$ as $x \to -\infty$ hence by (17)

$$\beta(x,t) = \int_{-\infty}^{x} v_0(u,t)du \tag{18}$$

Consequently

$$\psi'(x',t') = e^{-\frac{i}{\hbar}\int_{-\infty}^{x} v_0(u,t)du}\psi(x,t) \tag{19}$$

References

[1] K. De Paepe, Physics Essays, September 2008

[2] A. Colcelli, G. Mussardo, G. Sierra, A. Trombettoni, Phys. Rev. Lett. **123**, 130401 (2019)