Proof of Riemann hypothesis (2)

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Abstract

This paper is a trial to prove Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of Re(s)=1/2." according to the following process.

1 We have the following (4) and (5) from the following (1) that gives $\zeta(s)$ analytic continuation to Re(s)$>$0 and the following (2) and (3) that show non-trivial zero point of $\zeta(s)$. The right side of (4) must be equal to the right side of (5).

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+ \text{-----} = (1-2^{1-s})\ \zeta(s) \qquad (1)$$

$$S_0 = 1/2+a+bi \qquad (2)$$

$$S_1 = 1-S_0 = 1/2-a-bi \qquad (3)$$

$$1 = \frac{\cos(b\log2)}{2^{1/2+a}}-\frac{\cos(b\log3)}{3^{1/2+a}}+\frac{\cos(b\log4)}{4^{1/2+a}}-\frac{\cos(b\log5)}{5^{1/2+a}}+\frac{\cos(b\log6)}{6^{1/2+a}}- \text{-----} \qquad (4)$$

$$1 = \frac{2^{2a}\cos(b\log2)}{2^{1/2+a}}-\frac{3^{2a}\cos(b\log3)}{3^{1/2+a}}+\frac{4^{2a}\cos(b\log4)}{4^{1/2+a}}-\frac{5^{2a}\cos(b\log5)}{5^{1/2+a}}+\frac{6^{2a}\cos(b\log6)}{6^{1/2+a}}- \text{---} \qquad (5)$$

2 We divide the right side of both (4) and (5) into the infinite groups after changing term order of both (4) and (5).

3 We find that the right side of (4) can be equal to the right side of (5) if only a=0 by comparing the infinite groups made from (4) with those from (5). Therefore zero point of $\zeta(s)$ must be $1/2\pm bi$ due to a=0 and other zero point does not exist.

1 Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to Re(s)$>$0. "+ -----" means infinite series in all equations in this paper.

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+ \text{-----} = (1-2^{1-s})\zeta(s) \qquad (1)$$

The following (2) shows non-trivial zero point of $\zeta(s)$. S_0 is the zero points of the left side of (1) and also zero points of $\zeta(s)$.

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$$S_0 = 1/2+a+bi \tag{2}$$

The range of a is $0 \leqq a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b>0$ and i is $\sqrt{-1}$. The following (3) also shows zero points of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1-S_0 = 1/2-a-bi \tag{3}$$

We have the following (4) by substituting S_0 for s in the left side of (1) and putting the real part of the left side of (1) at zero.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \tag{4}$$

We also have the following (5) by substituting S_1 for s in the left side of (1) and putting the real part of the left side of (1) at zero.

$$1 = \frac{2^{2a}\cos(b\log 2)}{2^{1/2+a}} - \frac{3^{2a}\cos(b\log 3)}{3^{1/2+a}} + \frac{4^{2a}\cos(b\log 4)}{4^{1/2+a}} - \frac{5^{2a}\cos(b\log 5)}{5^{1/2+a}} + \frac{6^{2a}\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \tag{5}$$

The right side of (4) must be equal to the right side of (5).

We define for easy description as follows.

$\boxed{+n}$: $+\frac{\cos(b\log n)}{n^{1/2+a}}$ when the sign of $\cos(b\log n)$ is "+" in (4) and (5).

$-\frac{\cos(b\log n)}{n^{1/2+a}}$ when the sign of $\cos(b\log n)$ is "−" in (4) and (5).

We call $\boxed{+n}$ "+term". n=2, 3, 4, 5, -----

$\boxed{-n}$: $+\frac{\cos(b\log n)}{n^{1/2+a}}$ when the sign of $\cos(b\log n)$ is "−" in (4) and (5).

$-\frac{\cos(b\log n)}{n^{1/2+a}}$ when the sign of $\cos(b\log n)$ is "+" in (4) and (5).

We call $\boxed{-n}$ "−term".

2 Changing term order of the series which starts with +term

2.1 If the values of a and b are fixed, the sign of cos(blogn) is also fixed. (n=2, 3, 4, 5, -----) And we can assume that the right sides of (4) and (5) are expressed like the following (6) and (7) as one example.

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{-6} + \boxed{+7} + \boxed{-8} + \boxed{+9} + \boxed{-10} + \boxed{-11} + \boxed{+12} + \boxed{-13} + \text{-----------} \tag{6}$$

$$1 = 2^{2a}\boxed{+2} + 3^{2a}\boxed{+3} + 4^{2a}\boxed{-4} + 5^{2a}\boxed{+5} + 6^{2a}\boxed{-6} + 7^{2a}\boxed{+7} + 8^{2a}\boxed{-8} + 9^{2a}\boxed{+9} + 10^{2a}\boxed{-10} + \text{----------} \tag{7}$$

2.2 We make (group 1a), (group 2a), (group 3a), ------ by changing term order of (6) as follows. And we have the following (8).

(1) We add $\boxed{-4}$ $\boxed{-6}$ $\boxed{-8}$ one by one after $\boxed{+2}+\boxed{+3}$ until the sum becomes negative value. If $\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6}>0$ and $\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6}+\boxed{-8}<0$, $(\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6})$ becomes (group 1a). The remaining $\boxed{-8}$ is returned to the original position of (6).

If $\boxed{+2}+\boxed{+3}+\boxed{-4}<0$, $(\boxed{+2}+\boxed{+3})$ becomes (group 1a). $\boxed{-4}$ $\boxed{-6}$ $\boxed{-8}$ are returned to the original position of (6).

(2) If $(\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6})$ is (group 1a), we add $\boxed{+5}$ which remained in making (group 1a) to the beginning of the remaining series $(\boxed{+7}+\boxed{-8}+\boxed{+9}+\boxed{-10}+\boxed{-11}+------)$. We add $\boxed{-8}$ $\boxed{-10}$ $\boxed{-11}$ one by one after $\boxed{+5}+\boxed{+7}$ until the sum becomes negative value. If $\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10}>0$ and $\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10}+\boxed{-11}<0$, $(\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10})$ becomes (group 2a). The remaining $\boxed{-11}$ is returned to the original position of (6).

If $\boxed{+5}+\boxed{+7}+\boxed{-8}<0$, $(\boxed{+5}+\boxed{+7})$ becomes (group 2a). $\boxed{-8}$ $\boxed{-10}$ $\boxed{-11}$ are returned to the original position of (6).

(3) If $(\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10})$ is (group 2a), we add $\boxed{+9}$ which remained in making (group 2a) to the beginning of the remaining series $(\boxed{-11}+\boxed{+12}+\boxed{-13}+----)$. We add $\boxed{-11}$ $\boxed{-13}$ one by one after $\boxed{+9}$ until the sum becomes negative value. If $\boxed{+9}+\boxed{-11}>0$ and $\boxed{+9}+\boxed{-11}+\boxed{-13}<0$, $(\boxed{+9}+\boxed{-11})$ becomes (group 3a). The remaining $\boxed{-13}$ is returned to the original position of (6).

If $\boxed{+9}+\boxed{-11}<0$, $(\boxed{+9})$ becomes (group 3a). $\boxed{-11}$ $\boxed{-13}$ are returned to the original position of (6).

$$1 = \boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{+5}+\boxed{-6}+\boxed{+7}+\boxed{-8}+\boxed{+9}+\boxed{-10}+\boxed{-11}+\boxed{+12}+\boxed{-13}+---- \qquad (6)$$

$$1 = \boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6}+\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10}+\boxed{+9}+\boxed{-11}+\boxed{+12}+\boxed{-13}+------ \qquad (8)$$

|←---group 1a---→|←---group 2a---→|←group 3a→|

2.3 The explanation in item 2.2 is one example. From [any infinite series starting with +term] which are assumed from (4) we can make (group na) by changing term order as follows. (n=1, 2, 3, 4, ------) (*) shows examples in n=2, the above (6) and (8).

(1) We add +terms(*1) which remained in making (group (n-1)a)(*2) one by one in descending order to the beginning of the remaining series(*3) which we sort from now.

(*1) : $\boxed{+5}$ (*2) : (group 1a) (*3) : $\boxed{+7}+\boxed{-8}+\boxed{+9}+\boxed{-10}+\boxed{-11}+------$

(2) We select -terms(*4) from the beginning of the above series(*5) which had +terms(*1) in (1) and add the -terms(*4) one by one in ascending order after the mass of +term(*6) which exist at the beginning of the

above series(*5) until the sign of the sum changes from "+" to "−".

(*4): $\boxed{-8}$ $\boxed{-10}$ $\boxed{-11}$ (*5): $\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{+9}+\boxed{-10}+\boxed{-11}+$ ------   (*6): $\boxed{+5}+\boxed{+7}$

(3) The combination of terms like [(+terms)+(−terms)](*7) which appears immediately before the sign of the sum changes from "+" to "−" is (group na)(*8). The remaining −terms(*9) are returned to the original position of the above series(*5). +terms(*10) which remained in making (group na)(*8) are used for (group (n+1)a)(*11) as shown in (1).

(*7): $\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10}$ (*8): (group 2a)   (*9): $\boxed{-11}$ (*10): $\boxed{+9}$

(*11): (group 3a)

(4) If the sign of the sum is "−" when the first −term(*12) was added to the mass of +terms(*6), the [(+terms)](*6) is (group na)(*8). −terms(*4) are returned to the original position of the above series(*5).

(*12): $\boxed{-8}$

$$1=\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{+5}+\boxed{-6}+\boxed{+7}+\boxed{-8}+\boxed{+9}+\boxed{-10}+\boxed{-11}+\boxed{+12}+\boxed{-13}+\text{----} \qquad (6)$$

$$1=\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6}+\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10}+\boxed{+9}+\boxed{-11}+\boxed{+12}+\boxed{-13}+\text{------} \qquad (8)$$

|←---group 1a---→|←---group 2a---→|←group 3a→|

2.4 (group na) has the following features.

(1) The terms line up in ascending order.

(2) The first half and the second half consist of +terms and −terms respectively like [(+terms)+(−terms)].

(3) $-\boxed{-n}$ > the sum of (group na) >0

$\boxed{-n}$: −term next to the last −term of (group na)

When (group na) consists of only +terms

$\boxed{-n}$: −term next to the last +term of (group na)

2.5 We define as follows.

Range n : the range between the first −term of (group na) and the last +term of (group (n+1)a). (n=1, 2, 3, 4, 5, ------)

Exceptionally

Range0 : the range between the first +term of (group 1a) and the last +term of (group 1a).

Comparing (8) with (6) we can find the following.

(1) Range n is common to both (6) and (8).

(2) The terms of (6) which exist inside Range n are same as the terms of (8) which exist inside Range n. Please refer to [Appendix

1: Definition of Range n] for details.　(n=0, 1, 2, 3, 4, -----)

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{-6} + \boxed{+7} + \boxed{-8} + \boxed{+9} + \boxed{-10} + \boxed{-11} + \boxed{+12} + \text{------------} \quad (6)$$

|←-R.0-→|←-----Range1-----→|←--Range2--→|←-Range3-

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{-6} + \boxed{+5} + \boxed{+7} + \boxed{-8} + \boxed{-10} + \boxed{+9} + \boxed{-11} + \boxed{+12} + \text{------------} \quad (8)$$

|←---group 1a---→|←----group 2a---→|←--3a--→|

2.6　The sum of the right side of (6) is equal to the sum of the right side of (8) due to item 2.5 (2). The sum of the right side of (6) does not change after changing term order as shown in item 2.2. Please refer to [Appendix 2: The sum of infinite series] for details.

2.7　After enclosing the terms of (8) with ( ) for each group we have the following (9). Enclosing the terms with ( ) does not change the sum of the right side of (8) because enclosing the terms with ( ) does not change term order.

$$1 = (\boxed{+2}+\boxed{+3}+\boxed{-4}+\boxed{-6}) + (\boxed{+5}+\boxed{+7}+\boxed{-8}+\boxed{-10}) + (\boxed{+9}+\boxed{-11} + \text{------------------} \quad (9)$$

|←--group 1a--→|←---group 2a---→|←group 3a→|

2.8　By changing term order of (7) in "the same order" (*) as in (8) and enclosing the terms with ( ) for each group we have (group 1b), (group 2b), (group 3b), ------ as shown in the following (10).

$$1 = (2^{2a}\boxed{+2}+3^{2a}\boxed{+3}+4^{2a}\boxed{-4}+6^{2a}\boxed{-6}) + (5^{2a}\boxed{+5}+7^{2a}\boxed{+7}+8^{2a}\boxed{-8}+10^{2a}\boxed{-10}) + (9^{2a}\boxed{+9}+11^{2a}\boxed{-11}) + \text{--} (10)$$

|←-----group 1b------→|←------group 2b------→|←---group 3b---→|

(*): Hereafter "the same order" means that the term order of $\boxed{+n}$ and $\boxed{-n}$ is same as the term order of $n^{2a}\boxed{+n}$ and $n^{2a}\boxed{-n}$ when $\boxed{+n}$ and $\boxed{-n}$ correspond to $n^{2a}\boxed{+n}$ and $n^{2a}\boxed{-n}$ at one-to-one as shown in the following example. We define (group nb) as the group which has the same term order of (group na).

|←----group 1a----→|←------group 2a------→|←-group 3a-→|

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{-6} + \boxed{+5} + \boxed{+7} + \boxed{-8} + \boxed{-10} + \boxed{+9} + \boxed{-11} + \text{-------} \quad (9)$$

$$1 = 2^{2a}\boxed{+2}+3^{2a}\boxed{+3}+4^{2a}\boxed{-4}+6^{2a}\boxed{-6} + 5^{2a}\boxed{+5}+7^{2a}\boxed{+7}+8^{2a}\boxed{-8}+10^{2a}\boxed{-10} + 9^{2a}\boxed{+9}+11^{2a}\boxed{-11}+ \text{--------} (10)$$

|←-----group 1b------→|←------group 2b------→|←---group 3b---→|

2.9 The general formula of (group na) is as follows.

$$(\text{---} + \boxed{+N_{-3}}+\boxed{+N_{-2}}+\boxed{+N_{-1}}+\boxed{-N_0}+\boxed{-N_1}+\boxed{-N_2} + \text{---}) \qquad (11)$$

$2<\text{----}<N_{-3}<N_{-2}<N_{-1}<N_0<N_1<N_2<\text{----}$ $\boxed{-N_0}$: the first −term of (group na)

"--- +" means finite +terms and "+ ---" means finite −terms.

The general formula of (group nb) is as follows.

$$(\text{---} + N_{-3}^{2a}\boxed{+N_{-3}}+N_{-2}^{2a}\boxed{+N_{-2}}+N_{-1}^{2a}\boxed{+N_{-1}}+N_0^{2a}\boxed{-N_0}+N_1^{2a}\boxed{-N_1}+N_2^{2a}\boxed{-N_2} + \text{---}) \qquad (12)$$

(group na)−(group nb)=

$(\text{---} + \boxed{+N_{-3}}+\boxed{+N_{-2}}+\boxed{+N_{-1}}+\boxed{-N_0}+\boxed{-N_1}+\boxed{-N_2} + \text{---})$ −

$(\text{---} + N_{-3}^{2a}\boxed{+N_{-3}}+N_{-2}^{2a}\boxed{+N_{-2}}+N_{-1}^{2a}\boxed{+N_{-1}}+N_0^{2a}\boxed{-N_0}+N_1^{2a}\boxed{-N_1}+N_2^{2a}\boxed{-N_2} + \text{---})$

$\geqq (\text{---} + \boxed{+N_{-3}}+\boxed{+N_{-2}}+\boxed{+N_{-1}}+\boxed{-N_0}+\boxed{-N_1}+\boxed{-N_2} + \text{---})$ −

$(\text{---} + N_0^{2a}\boxed{+N_{-3}}+N_0^{2a}\boxed{+N_{-2}}+N_0^{2a}\boxed{+N_{-1}}+N_0^{2a}\boxed{-N_0}+N_0^{2a}\boxed{-N_1}+N_0^{2a}\boxed{-N_2} + \text{---})$

$= (1-N_0^{2a})(\text{---} + \boxed{+N_{-3}}+\boxed{+N_{-2}}+\boxed{+N_{-1}}+\boxed{-N_0}+\boxed{-N_1}+\boxed{-N_2} + \text{---})$

$= (1-N_0^{2a})$(group na) (13)

i.e. (group na)−(group nb) $\geqq (1-N_0^{2a})$(group na) (13−1)

"≧" in (13) holds due to the following reasons.

$N_0^{2a}\boxed{+N_{-3}} \geqq N_{-3}^{2a}\boxed{+N_{-3}} > 0$ due to $N_0 > N_{-3}$

$0 > N_0^{2a}\boxed{-N_2} \geqq N_2^{2a}\boxed{-N_2}$ due to $N_2 > N_0$

$0 \geqq 1-N_0^{2a}$ holds from $N_0 > N_0^{2a} \geqq 1$ due to $0 \leqq a < 1/2$.

From (13−1)

If (group na)<0 i.e. (group na)−(group nb) $\geqq (1-N_0^{2a})$(group na)≧0 ,

(group na)≧(group nb) is true. (Equal sign "=" holds if only a=0.)

If (group na)≦(group nb)

i.e. 0≧(group na)−(group nb) $\geqq (1-N_0^{2a})$(group na), (group na)>0 is true.

(Equal sign "=" holds if only a=0.)

As shown in item 2.3 we selected terms for (group na) in order to make the situation of (group na)>0. So (group nb)≧(group na) holds. (Equal sign "=" holds if only a=0.)

2.10 When (group na) consists of just +terms like $(\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2})$

(group nb)−(group na)=

$(N_0^{2a}\boxed{+N_0}+N_1^{2a}\boxed{+N_1}+N_2^{2a}\boxed{+N_2}) - (\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2})$

$\geqq (N_0^{2a}\boxed{+N_0}+N_0^{2a}\boxed{+N_1}+N_0^{2a}\boxed{+N_2}) - (\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2})$

$= (N_0^{2a}-1)(\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2}) \geqq 0$ (14)

Here $2 < N_0 < N_1 < N_2$

$N_0^{2a}-1 \geqq 0$ holds from $N_0 > N_0^{2a} \geqq 1$ due to $0 \leqq a < 1/2$.

The second "≧" in (14) holds due to $(N_0^{2a}-1) \geqq 0$ and $(\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2}) > 0$.

So (group nb)≧(group na) holds. (Equal sign "=" holds if only a=0.)

2.11 [Changing term order of the series which starts with +term] will change to [Changing term order of the series which starts with -term] at the following situation. [Changing term order of the series which starts with -term] is explained in item 3.

(1) When we made (group 2a) +term does not remain and the term next to (group 2a), $\boxed{-9}$ is -term as follows.

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{-6} + \boxed{+7} + \boxed{-8} + \boxed{-9} + \boxed{+10} + \boxed{-11} + \boxed{+12} + \boxed{-13} + -----$$

|←-R.0-→|←-----Range1-----→|←R. 2→|

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{-6} + \boxed{+5} + \boxed{+7} + \boxed{-8} + \boxed{-9} + \boxed{+10} + \boxed{-11} + \boxed{+12} + \boxed{-13} + ----$$

|←---group 1a----→|←--group 2a-→|←--- done by item 3 -------

(2) When we made (group 3a), (group 3a) consists of only +term. The term next to (group 3a), $\boxed{-11}$ is -term as follows.

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{-6} + \boxed{+7} + \boxed{-8} + \boxed{-9} + \boxed{+10} + \boxed{-11} + \boxed{+12} + \boxed{-13} + -----$$

|←-R.0-→|←-----Range1------→|←---Range2---→|

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{-6} + \boxed{+5} + \boxed{+7} + \boxed{-8} + \boxed{-9} + \boxed{+10} + \boxed{-11} + \boxed{+12} + \boxed{-13} + ----$$

|←----group 1a----→|←----group 2a----→|←3a→|←done by item 3--

2.12 Conclusion

(1) From [any infinite series starting with +term] which are assumed from (4) by fixing the value of a and b, we can make (group na) by changing term order as shown in item 2.3. (n=1, 2, 3, 4, ----)

(2) From [any infinite series starting with +term] which are assumed from (5) by fixing a and b to the same value as in (1), we can make (group nb) by changing term order in the same order as in making (group na).

(3) (group nb)≧(group na) holds. (Equal sign "=" holds if only a=0.)

3 Changing term order of the series which starts with -term

3.1 If the values of a and b are fixed, the sign of cos(blogn) is also fixed. (n=2, 3, 4, 5, -----) And we can assume that the right sides of (4) and (5) are expressed like (15) and (16) as one example.

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{-5} + \boxed{+6} + \boxed{-7} + \boxed{+8} + \boxed{-9} + \boxed{+10} + \boxed{+11} + \boxed{-12} + \boxed{+13} + \boxed{-14} + ----- \quad (15)$$

$$1 = 2^{2a}\boxed{-2} + 3^{2a}\boxed{-3} + 4^{2a}\boxed{+4} + 5^{2a}\boxed{-5} + 6^{2a}\boxed{+6} + 7^{2a}\boxed{-7} + 8^{2a}\boxed{+8} + 9^{2a}\boxed{-9} + 10^{2a}\boxed{+10} + --------- \quad (16)$$

3.2 We make (group 1a), (group 2a), (group 3a), ------ as follows by changing term order of (15). And we have the following (17).

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{-5} + \boxed{+6} + \boxed{-7} + \boxed{+8} + \boxed{-9} + \boxed{+10} + \boxed{+11} + \boxed{-12} + \boxed{+13} + --------- \tag{15}$$

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{+6} + \boxed{-5} + \boxed{-7} + \boxed{+8} + \boxed{+10} + \boxed{-9} + \boxed{+11} + \boxed{+13} + \boxed{-12} + --------- \tag{17}$$

|←---group 1a---→|←------group 2a-----→|←--group 3a --→|

(1) We add $\boxed{+4}$ $\boxed{+6}$ $\boxed{+8}$ one by one after $\boxed{-2}+\boxed{-3}$ until the sum becomes positive value. If $\boxed{-2}+\boxed{-3}+\boxed{+4}<0$ and $\boxed{-2}+\boxed{-3}+\boxed{+4}+\boxed{+6}>0$, ($\boxed{-2}+\boxed{-3}+\boxed{+4}+\boxed{+6}$) becomes (group 1a). The remaining $\boxed{+8}$ is returned to the original position of (15).

(2) We add $\boxed{-5}$ which remained in making (group 1a) to the beginning of the remaining series($\boxed{-7}+\boxed{+8}+\boxed{-9}+\boxed{+10}+\boxed{+11}$ + -----). We add $\boxed{+8}$ $\boxed{+10}$ $\boxed{+11}$ one by one after $\boxed{-5}+\boxed{-7}$ until the sum becomes positive value. If $\boxed{-5}+\boxed{-7}+\boxed{+8}<0$ and $\boxed{-5}+\boxed{-7}+\boxed{+8}+\boxed{+10}>0$, ($\boxed{-5}+\boxed{-7}+\boxed{+8}+\boxed{+10}$) becomes (group 2a). The remaining $\boxed{+11}$ is returned to the original position of (15).

(3) We add $\boxed{-9}$ which remained in making (group 2a) to the beginning of the remaining series($\boxed{+11}+\boxed{-12}+\boxed{+13}$ + ----). We add $\boxed{+11}$ $\boxed{+13}$ one by one after $\boxed{-9}$ until the sum becomes positive value. If $\boxed{-9}+\boxed{+11}<0$ and $\boxed{-9}+\boxed{+11}+\boxed{+13}>0$, ($\boxed{-9}+\boxed{+11}+\boxed{+13}$) becomes (group 3a).

3.3 The explanation in item 3.2 is one example. From [any infinite series starting with -term] which are assumed from (4) we can make (group na) by changing term order as follows. (n=1,2,3,4,------) (*) shows examples in n=2, the above (15) and (17).

(1) We add -terms(*1) which remained in making (group (n-1)a)(*2) one by one in descending order to the beginning of the remaining series(*3) which we sort from now.

(*1): $\boxed{-5}$ (*2): (group 1a) (*3): $\boxed{-7}+\boxed{+8}+\boxed{-9}+\boxed{+10}+\boxed{+11}$ + -----

(2) We select +terms(*4) from the beginning of the above series(*5) which had -terms(*1) in (1) and add the +terms(*4) one by one in ascending order after the mass of -terms(*6) which exist at the beginning of the above series(*5) until the sign of the sum changes from "-" to "+".

(*4): $\boxed{+8}$ $\boxed{+10}$ $\boxed{+11}$ (*5) : $\boxed{-5}+\boxed{-7}+\boxed{+8}+\boxed{-9}+\boxed{+10}+\boxed{+11}$ + ----- (*6): $\boxed{-5}+\boxed{-7}$

(3) The combination of terms like [(-terms)+(+terms)](*7) which appears immediately after the sign of the sum changes from "-" to "+" is (group na)(*8). The remaining +terms(*9) are returned to the original position of the above series(*5). -terms(*10) which remained in making (group na)(*8) are used for (group (n+1)a)(*11) as shown in (1).

(*7)： $\boxed{-5}+\boxed{-7}+\boxed{+8}+\boxed{+10}$　　(*8)： (group 2a)　　(*9)： $\boxed{+11}$　　(*10)： $\boxed{-9}$

(*11)： (group 3a)

3.4　(group na) has the following features.

(1)　The terms line up in ascending order.

(2)　The first half and the second half consist of -terms and +terms respectively like [(-terms)+(+terms)].

(3)　$\boxed{+n}$ ＞ the sum of (group na) ＞ 0

$\boxed{+n}$ ： the last +term of (group na)

3.5　We define as follows.

Range n ： the range between the first +term of (group na) and the last -term of (group (n+1)a).　(n=1, 2, 3, 4, 5, -----)

Exceptionally

Range0 ： the range between the first -term of (group 1a) and the last -term of (group 1a).

Comparing (15) with (17) we can find the following.

(1)　Range n is common to both (15) and (17).

(2)　The terms of (15) which exist inside Range n are same as the terms of (17) which exist inside Range n. Please refer to [Appendix 1: Definition of Range n] for details.　n=0, 1, 2, 3, 4, -----

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{-5} + \boxed{+6} + \boxed{-7} + \boxed{+8} + \boxed{-9} + \boxed{+10} + \boxed{+11} + \boxed{-12} + \boxed{+13} + \boxed{-14} + ---- \quad (15)$$

|←--R.0--→|←------Range1------→|←----Range2----→|←-------Range3---------

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{+6} + \boxed{-5} + \boxed{-7} + \boxed{+8} + \boxed{+10} + \boxed{-9} + \boxed{+11} + \boxed{+13} + \boxed{-12} + \boxed{-14} + ---- \quad (17)$$

|←----group 1a----→|←-----group 2a-----→|←---group 3a ---→|←--group 4a ---

3.6　The sum of the right side of (15) is equal to the sum of the right side of (17) due to item 3.5 (2). The sum of the right side of (15) does not change after changing term order as shown in item 3.2. Please refer to [Appendix 2: The sum of infinite series] for details.

3.7　After enclosing the terms of (17) with ( ) for each group we have the following (18). Enclosing the terms with ( ) does not change the sum of the right side of (17) because enclosing the terms with ( ) does not change term order.

$$1 = (\boxed{-2}+\boxed{-3}+\boxed{+4}+\boxed{+6}) + (\boxed{-5}+\boxed{-7}+\boxed{+8}+\boxed{+10}) + (\boxed{-9}+\boxed{+11}+\boxed{+13}) + (\boxed{-12}+\boxed{-14}+\boxed{+15}) + --------- \quad (18)$$

|←---group 1a---→|←---group 2a---→|←--group 3a --→|←--group 4a ---

3.8 By changing terms order of (16) in the same order as in (17) and enclosing the terms with ( ) for each group we have (group 1b), (group 2b), (group 3b), ------ as shown in the following (19).

$$1 = (2^{2a}\boxed{-2}+3^{2a}\boxed{-3}+4^{2a}\boxed{+4}+6^{2a}\boxed{+6})+(5^{2a}\boxed{-5}+7^{2a}\boxed{-7}+8^{2a}\boxed{+8}+10^{2a}\boxed{+10})+(9^{2a}\boxed{-9}+11^{2a}\boxed{+11}+13^{2a}\boxed{+13})+\text{--------} \quad (19)$$

|←------group 1b------→|←--------group 2b-------→|←------group 3b------→|

3.9 The general formula of (group na) is as follows.

$$(\text{---} + \boxed{-N_{-3}}+\boxed{-N_{-2}}+\boxed{-N_{-1}}+\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2} + \text{---}) \quad (20)$$

$2<\text{----}<N_{-3}<N_{-2}<N_{-1}<N_0<N_1<N_2<\text{-----}$ $\boxed{+N_0}$: the first +term of (group na)

"--- +" means finite −terms and "+ ---" means finite +terms.

The general formula of (group nb) is as follows.

$$(\text{---} + N_{-3}^{2a}\boxed{-N_{-3}}+N_{-2}^{2a}\boxed{-N_{-2}}+N_{-1}^{2a}\boxed{-N_{-1}}+N_0^{2a}\boxed{+N_0}+N_1^{2a}\boxed{+N_1}+N_2^{2a}\boxed{+N_2} + \text{---}) \quad (21)$$

(group nb) − (group na) =

$$(\text{---} + N_{-3}^{2a}\boxed{-N_{-3}}+N_{-2}^{2a}\boxed{-N_{-2}}+N_{-1}^{2a}\boxed{-N_{-1}}+N_0^{2a}\boxed{+N_0}+N_1^{2a}\boxed{+N_1}+N_2^{2a}\boxed{+N_2} + \text{---})$$

$$-(\text{---} + \boxed{-N_{-3}}+\boxed{-N_{-2}}+\boxed{-N_{-1}}+\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2} + \text{---})$$

$$\geqq (\text{---} + N_0^{2a}\boxed{-N_{-3}}+N_0^{2a}\boxed{-N_{-2}}+N_0^{2a}\boxed{-N_{-1}}+N_0^{2a}\boxed{+N_0}+N_0^{2a}\boxed{+N_1}+N_0^{2a}\boxed{+N_2} + \text{---})$$

$$-(\text{---} + \boxed{-N_{-3}}+\boxed{-N_{-2}}+\boxed{-N_{-1}}+\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2} + \text{---})$$

$$= (N_0^{2a}-1)(\text{---} + \boxed{-N_{-3}}+\boxed{-N_{-2}}+\boxed{-N_{-1}}+\boxed{+N_0}+\boxed{+N_1}+\boxed{+N_2} + \text{---})$$

$$= (N_0^{2a}-1)(\text{group na}) \quad (22)$$

i.e. (group nb) − (group na) $\geqq (N_0^{2a}-1)$(group na) (22−1)

"$\geqq$" in (22) holds due to the following reasons.

$N_0^{2a}\boxed{-N_{-3}} \leqq N_{-3}^{2a}\boxed{-N_{-3}} < 0$ due to $N_0 > N_{-3}$

$0 < N_0^{2a}\boxed{+N_2} \leqq N_2^{2a}\boxed{+N_2}$ due to $N_0 < N_2$

$N_0^{2a}-1 \geqq 0$ holds from $N_0 > N_0^{2a} \geqq 1$ due to $0 \leqq a < 1/2$.

From (22−1)

If (group na) > 0 i.e. (group nb) − (group na) $\geqq (N_0^{2a}-1)$(group na) $\geqq 0$,

(group nb) $\geqq$ (group na) is true. (Equal sign "=" holds if only a=0.)

If (group nb) $\leqq$ (group na) i.e. $0 \geqq$ (group nb) − (group na) $\geqq (N_0^{2a}-1)$(group na),

(group na) < 0 is true. (Equal sign "=" holds if only a=0.)

As shown in item 3.3 we selected terms for (group na) in order to make the situation of (group na) > 0. So (group nb) $\geqq$ (group na) holds. (Equal sign "=" holds if only a=0.)

3.10 [Changing term order of the series which starts with −term] will change to [Changing term order of the series which starts with +term] at the following situation.

(1) When we made (group 2a) −term does not remain. The term next to (group 2a), $\boxed{+6}$ is +terms as follows.

$$1 = \boxed{-2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{+6} + \boxed{+7} + \boxed{-8} + \boxed{-9} + \boxed{+10} + \boxed{-11} + \boxed{+12} + \text{-----}$$

|←R.0→|←Range1→|←R. 2→|

$$1 = \boxed{-2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{+6} + \boxed{+7} + \boxed{-8} + \boxed{-9} + \boxed{+10} + \boxed{-11} + \boxed{+12} + \text{-----}$$

|←group 1a→|←group 2a→|←--- done by item 2 ---

3.11 Conclusion

(1) From [any infinite series starting with −term] which are assumed from (4) by fixing the value of a and b, we can make (group na) by changing term order as shown in item 3.3. (n=1, 2, 3, 4, ----)

(2) From [any infinite series starting with −term] which are assumed from (5) by fixing a and b to the same value as in (1), we can make (group nb) by changing term order in the same order as in making (group na).

(3) (group nb) ≧ (group na) holds. (Equal sign "=" holds if only a=0.)

4 Comparing the right side of (4) with the right side of (5)

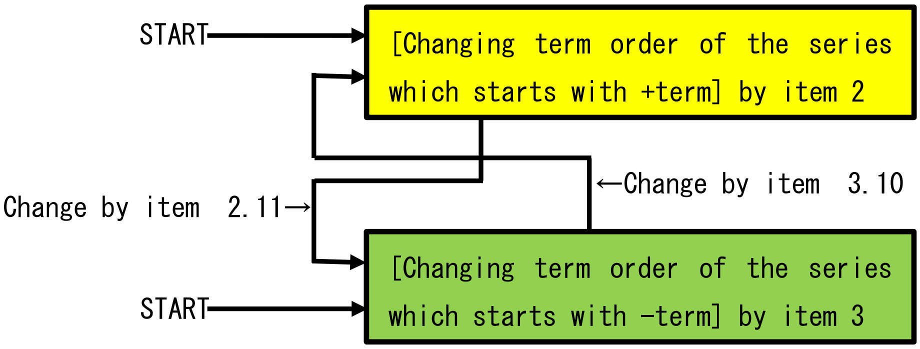
4.1 As shown in item 2 and item 3 we can divide the right side of the following (4) into the infinite groups like (group 1a), (group 2a), (group 3a), ----- by changing term order of (4). The sum of the right side of (4) does not change after changing term order. Please refer to [Appendix 2: The sum of infinite series] for details.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \qquad (4)$$

$$1 = \frac{2^{2a}\cos(b\log 2)}{2^{1/2+a}} - \frac{3^{2a}\cos(b\log 3)}{3^{1/2+a}} + \frac{4^{2a}\cos(b\log 4)}{4^{1/2+a}} - \frac{5^{2a}\cos(b\log 5)}{5^{1/2+a}} + \frac{6^{2a}\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \qquad (5)$$

The workflow for changing term order of (4) is shown in the following (Figure 1). We can continue changing term order infinitely because in any infinite series which are assumed from (4) the sum of +terms and the sum of −terms are +∞ and −∞ respectively. Please refer to [Appendix 3: The sum of +terms] for details.

Figure 1: The workflow for changing term order



4.2 The following (Table 1) shows the type of (group na).

Table 1: The type of (group na)

Type	The sum of (group na) : S	Proof of (group nb) ≧(group na)
(+terms)+(−terms)	$\boxed{-n} > S > 0$ Refer to item 2.4 $\boxed{-n}$: −term next to the last −term of (group na)	Refer to item 2.9
(−terms)+(+terms)	$\boxed{+n} > S > 0$ Refer to item 3.4 $\boxed{+n}$: the last +term of (group na)	Refer to item 3.9
(+terms)	$\boxed{-n} > S > 0$ Refer to item 2.4 $\boxed{-n}$: −term next to the last +term of (group na)	Refer to item 2.10

4.3 As shown in item 2 and item 3 we can divide the right side of (5) into the infinite groups like (group 1b), (group 2b), (group 3b), ----- by changing term order of (5) just in the same order as in changing term order of the right side of (4). The sum of the right side of (5) does not change after changing term order. Please refer to [Appendix 2: The sum of infinite series] for details.

4.4 We can have the following (24) and (25) after changing term order of (4) and (5).

1 = the right side of (4)

= (group 1a)+(group 2a)+(group 3a)+(group 4a)+(group 5a)+ ---------- (24)

1 = the right side of (5)

= (group 1b)+(group 2b)+(group 3b)+(group 4b)+(group 5b)+ ---------- (25)

4.5 In the above (24) and (25) $\boxed{+N}$ and $\boxed{-N}$ in (group na) correspond to $N^{2a}\boxed{+N}$ and $N^{2a}\boxed{-N}$ in (group nb) at one-to-one respectively as shown in the following example.

|←-----group 1a------→|←-------group 2a-------→|←----group 3a ----→|

$$1 = (\boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{+6}) + (\boxed{-5} + \boxed{-7} + \boxed{+8} + \boxed{+10}) + (\boxed{-9} + \boxed{+11} + \boxed{+13}) + \text{------} \quad (18)$$

$$1 = (2^{2a}\boxed{-2}+3^{2a}\boxed{-3}+4^{2a}\boxed{+4}+6^{2a}\boxed{+6}) + (5^{2a}\boxed{-5}+7^{2a}\boxed{-7}+8^{2a}\boxed{+8}+10^{2a}\boxed{+10}) + (9^{2a}\boxed{-9}+11^{2a}\boxed{+11}+13^{2a}\boxed{+13}) + \text{--------} \quad (19)$$

|←-----group 1b------→|←--------group 2b-------→|←------group 3b------→|

4.6 As shown in item 2.9, item 2.10 and item 3.9 (group nb) ≧ (group na) holds. (Equal sign "=" holds if only a=0. n=1,2,3,4,-----) From (24) and (25) [the right side of (5)] ≧ [the right side of (4)] holds as follows. (Equal sign "=" holds if only a=0.)

(group 1b)	≧	(group 1a)
(group 2b)	≧	(group 2a)
(group 3b)	≧	(group 3a)
(group 4b)	≧	(group 4a)
(group 5b)	≧	(group 5a)
¦	¦	¦
(group nb)	≧	(group na)
¦	¦	¦
+ ¦	¦	+ ¦
the right side of (5)	≧	the right side of (4)

4 Conclusion

5.1 If the following (2) and (3) show the non-trivial zero points of Riemann zeta function ξ(s), the sum of the right side of (4) must be equal to the sum of the right side of (5).

$$S_0 = 1/2+a+bi \quad (2)$$

$$S_1 = 1-S_0 = 1/2-a-bi \quad (3)$$

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \quad (4)$$

$$1 = \frac{2^{2a}\cos(b\log 2)}{2^{1/2+a}} - \frac{3^{2a}\cos(b\log 3)}{3^{1/2+a}} + \frac{4^{2a}\cos(b\log 4)}{4^{1/2+a}} - \frac{5^{2a}\cos(b\log 5)}{5^{1/2+a}} + \frac{6^{2a}\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \quad (5)$$

5.2 [the right side of (5)] ≧ [the right side of (4)] is true and equal sign "=" holds if only a=0. a has the range of $0 \leqq a < 1/2$ by the critical strip of $\zeta(s)$. But a cannot have any value but zero. Because the sum of the right side of (4) must be equal to the sum of the right side of (5). Therefore zero point of $\zeta(s)$ must be $1/2 \pm bi$ from (2) and (3) and other zero point does not exist. Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s)=1/2$." is true.

Appendix 1: Definition of Range n

1 Introduction

The following (26) and (27) are one example in [Changing term order of the series which starts with −term]. (26) is one example for (4) and (27) is the series changed from (26). Red numbered terms are “fixed term” which does not change its location after changing term order. Blue numbered terms are the last term of (group na).

$$1 = \boxed{-2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{-6} + \boxed{+7} + \boxed{-8} + \boxed{+9} + \boxed{-10} + \boxed{+11} + \boxed{-12} + \boxed{+13} + \boxed{-14} + \boxed{+15} + \text{----------} \quad (26)$$

|←R.0→|←---Range1---→|←----------Range2----------→|←-------Range3----------

$$1 = \boxed{-2} + \boxed{+3} + \boxed{+5} + \boxed{-4} + \boxed{-6} + \boxed{+7} + \boxed{+9} + \boxed{+11} + \boxed{-8} + \boxed{-10} + \boxed{-12} + \boxed{+13} + \boxed{+15} + \boxed{+17} + \boxed{+19} + \text{-----} \quad (27)$$

|←-group 1a-→|←-----group 2a-------→|←------------group 3a ------------→|

2 The first +term of (group na)

We assume as follows.

(1) The first +term of (group 2a), $\boxed{+7}$ is a fixed term.

(2) $\boxed{+11}$ is the last term of (group 2a).

(3) Arranging terms of $\boxed{-4}+\boxed{-6}+\boxed{+7}$ for (group 2a) in (27) is already finished.

To complete (group 2a) we must change term order of $\boxed{-8}+\boxed{+9}+\boxed{-10}+\boxed{+11}$ in (26) to term order of $\boxed{+9}+\boxed{+11}+\boxed{-8}+\boxed{-10}$ in (27). That is moving +terms backward and moving −terms forward between the first +term of (group 2a), $\boxed{+7}$ and the last +term of (group 2a), $\boxed{+11}$ in (26) due to (1) and (2). Sorting like this completes (group 2a) and the first half of (group 3a). $\boxed{-12}$ does not have to change its location due to (2). We find that the term next to the last term of (group 2a) in (26), $\boxed{-12}$ is a fixed term.

If $\boxed{-12}$ is a fixed term, we find the following.

(4) $\boxed{+13}$ is also a fixed term because $\boxed{-12}+\boxed{+13}$ can be used for (group 3a) without changing their location. And $\boxed{+13}$ becomes the first +term of (group 3a).

(5) If we assume $\boxed{-12}+\boxed{-13}+\boxed{+14}$ in (26) instead of $\boxed{-12}+\boxed{+13}+\boxed{-14}$, $\boxed{-12}$ $\boxed{-13}$ $\boxed{+14}$ are fixed terms because $\boxed{-12}+\boxed{-13}+\boxed{+14}$ can be used for (group 3a) without changing their location. And $\boxed{+14}$ becomes the first +term of (group 3a).

(6) If we assume $\boxed{-12}+\boxed{-13}+\boxed{-14}+\boxed{+15}$ in (26) instead of $\boxed{-12}+\boxed{+13}+\boxed{-14}+\boxed{+15}$, $\boxed{-12}$ $\boxed{-13}$ $\boxed{-14}$ $\boxed{+15}$ are fixed terms because $\boxed{-12}+\boxed{-13}+\boxed{-14}+\boxed{+15}$ can be used for (group 3a) without changing their location. And $\boxed{+15}$ becomes the first +term of (group 3a).

(7) If we assume $\boxed{+12}$ in (26) instead of $\boxed{-12}$, $\boxed{+12}$ becomes the first +term of (group 3a).

As shown above we can say "If the first +term of (group 2a) is a fixed term, the first +term of (group 3a) also becomes a fixed term." Through the same discussion as above in the following generalized series we can also say "If the first +term of (group na), $\boxed{+N_4}$ is a fixed term, the first +term of (group (n+1)a), $\boxed{+N_8}$ also becomes a fixed term." in [any infinite series starting with −term] which are assumed from (4).

$$1 = \boxed{-2} + \text{-----------------------} + \boxed{+N_4} + \boxed{-N_5} + \text{--------} + \boxed{+N_6} + \boxed{-N_7} + \boxed{+N_8} + \boxed{-N_9} + \boxed{+N_{10}} + \text{-----}$$

-----Range(n−1)---→|←-----------Range n-----------→|←------Range(n+1)------

$$1 = \boxed{-2} + \text{---} + \boxed{+N_2} + \boxed{-N_1} + \text{---} + \boxed{-N_3} + \boxed{+N_4} + \text{---} + \boxed{+N_6} + \boxed{-N_5} + \text{---} + \boxed{-N_7} + \boxed{+N_8} + \boxed{+N_{10}} + \boxed{+N_{11}} + \text{----}$$

g. (n−1)a→|←-----------group na-----------→|←-------------group (n+1)a ------------

$2 < N_1 < N_2 < N_3 < N_4 < N_5 < N_6 < N_7 < N_8 < N_9 < N_{10} < N_{11}$

The first +term of (group 1a) is always a fixed term. Therefore "the first +term of (group na) is a fixed term." is true by mathematical induction. (n=1, 2, 3, 4, -----)

3 Definition of Range n

If we define Range n as the range between the first +term of (group na) and the last −term of (group (n+1)a) in [Changing term order of the series which starts with −term], the terms which exist inside Range n do not change after changing term order. Because the first +term of (group na) is a fixed term and changing term order is just moving +terms backward and moving −terms forward within Range n. (n=1, 2, 3, 4, -----)

Exceptionally Range0 is the range between the first −term of (group 1a) and the last −term of (group 1a). All the terms which exist inside Range0 are fixed terms.

Similarly in [Changing term order of the series which starts with +term] "the first −term of (group na) is a fixed term." is true. If we define Range n as the range between the first −term of (group na) and the last +term of (group (n+1)a), the terms which exist inside Range n do not change after changing term order. Because the first −term of (group na) is fixed term and changing term order is just moving −terms backward and moving +terms forward within Range n. (n=1, 2, 3, 4, -----)

Exceptionally Range0 is the range between the first +term of (group 1a) and the last +term of (group 1a). All the terms which exist inside Range0 are fixed terms.

Appendix 2: The sum of infinite series

1 Introduction

Figure 2 : Changing term order within Range

A =	$a_1+a_2+a_3$	+	$a_4+a_5+a_6+a_7$	+	$a_8+a_9+a_{10}+a_{11}+a_{12}$	+	$a_{13}+a_{14}+a_{15}+a_{16}+a_{17}+a_{18}$	+	-------	(28)
	Range1		Range2		Range3		Range4		Range5	
B =	$a_3+a_2+a_1$	+	$a_7+a_6+a_5+a_4$	+	$a_{12}+a_{11}+a_{10}+a_9+a_8$	+	$a_{18}+a_{17}+a_{16}+a_{15}+a_{14}+a_{13}$	+	-------	(29)

As shown in (Figure 2) the right side of (28) converges. We divide the right side of (28) into infinite number of Range like Range1, Range2, Range3,-----. We change term order of the right side of (28) within each Range and make another infinite series of (29).

We define as follows.

The inside sum of Range1: R1 The inside sum of Range2: R2 The inside sum of Range3: R3 The inside sum of Range4: R4 ---------------------

(28) is expressed as (28-1).

$$\begin{aligned} A &= a_1+a_2+a_3 + a_4+a_5+a_6+a_7 + a_8+a_9+a_{10}+a_{11}+a_{12} + a_{13}+a_{14}+a_{15}+a_{16}+a_{17}+a_{18} + \text{-------} \\ &= (a_1+a_2+a_3) + (a_4+a_5+a_6+a_7) + (a_8+a_9+a_{10}+a_{11}+a_{12}) + (a_{13}+a_{14}+a_{15}+a_{16}+a_{17}+a_{18}) + \text{-------} \\ &= R1 + R2 + R3 + R4 + R5 + \text{---------} \qquad (28\text{-}1) \end{aligned}$$

The inside sum of each Range does not change after changing term order within Range. Therefore (29) is expressed as (29-1).

$$\begin{aligned} B &= a_3+a_2+a_1 + a_7+a_6+a_5+a_4 + a_{12}+a_{11}+a_{10}+a_9+a_8 + a_{18}+a_{17}+a_{16}+a_{15}+a_{14}+a_{13} + \text{-------} \\ &= (a_3+a_2+a_1) + (a_7+a_6+a_5+a_4) + (a_{12}+a_{11}+a_{10}+a_9+a_8) + (a_{18}+a_{17}+a_{16}+a_{15}+a_{14}+a_{13}) + \text{-------} \\ &= R1 + R2 + R3 + R4 + R5 + \text{---------} \qquad (29\text{-}1) \end{aligned}$$

A=B is true from (28-1) and (29-1). Therefore the sum of infinite series which converges does not change after dividing the infinite series into infinite number of Range and changing term order within Range.

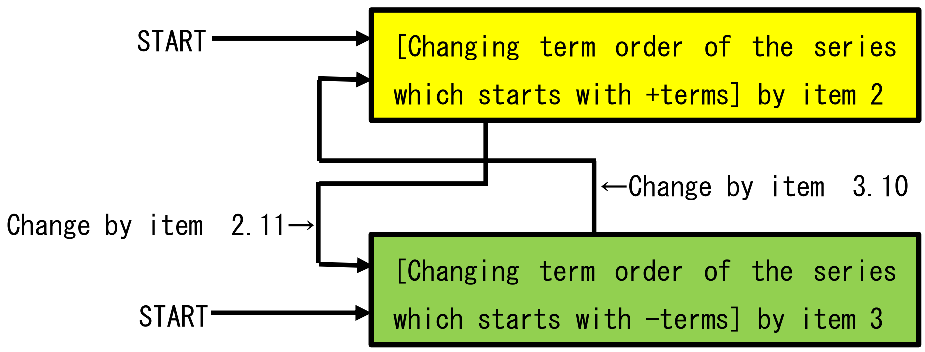
2　Changing term order of (4)

As shown in item 2 and item 3 we can divide the right side of the following (4) into the infinite groups like (group 1a), (group 2a), (group 3a),----- by repeating [Changing term order of the series which starts with +term] by item 2 and [Changing term order of the series which starts with -term] by item 3 alternately as shown in the following (Figure 1).

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{----------------} \quad (4)$$

$$1 = \frac{2^{2a}\cos(b\log 2)}{2^{1/2+a}} - \frac{3^{2a}\cos(b\log 3)}{3^{1/2+a}} + \frac{4^{2a}\cos(b\log 4)}{4^{1/2+a}} - \frac{5^{2a}\cos(b\log 5)}{5^{1/2+a}} + \frac{6^{2a}\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \quad (5)$$

Figure 1: The workflow for changing term order



We can make Range like Range0, Range1, Range2, Range3, ----- in both (4) and the infinite series changed from (4) according to item 2.5 and item 3.5. [The terms of the right side of (4)] which exist inside Range are same as [the terms of the series changed from (4)] which exist inside the same Range due to item 3 of [Appendix 1: Definition of Range n]. Please refer to the following examples.

(1) [Changing term order of the series which starts with +term] by item 2

(6) is an example for (4) and (8) is the series changed from (6). Red numbered terms are "fixed term" which does not change its location after changing term order. Changing term order is just moving -terms backward and moving +terms forward within Range. So the inside sum of Range does not change after changing term order.

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{+5} + \boxed{-6} + \boxed{+7} + \boxed{-8} + \boxed{+9} + \boxed{-10} + \boxed{-11} + \boxed{+12} + \boxed{-13} + \text{-----} \qquad (6)$$

|←R.0→|←----Range1-----→|←--Range2--→|←-Range3-

$$1 = \boxed{+2} + \boxed{+3} + \boxed{-4} + \boxed{-6} + \boxed{+5} + \boxed{+7} + \boxed{-8} + \boxed{-10} + \boxed{+9} + \boxed{-11} + \text{-----} \qquad (8)$$

|←---group 1a--→|←---group 2a----→|←group 3a→|

(2) [Changing term order of the series which starts with -term] by item 3

(15) is an example for (4) and (17) is the series changed from (15). Changing term order is just moving +terms backward and moving -terms forward within Range. So the inside sum of Range does not change after changing term order.

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{-5} + \boxed{+6} + \boxed{-7} + \boxed{+8} + \boxed{-9} + \boxed{+10} + \boxed{+11} + \boxed{-12} + \boxed{+13} + \boxed{-14} + \boxed{+15} + \text{-------} \qquad (15)$$

|←-R.0-→|←---Range1---→|←--Range2-→|←------Range3------→|

$$1 = \boxed{-2} + \boxed{-3} + \boxed{+4} + \boxed{+6} + \boxed{-5} + \boxed{-7} + \boxed{+8} + \boxed{+10} + \boxed{-9} + \boxed{+11} + \boxed{+13} + \boxed{-12} + \boxed{-14} + \boxed{+15} + \text{-------} \qquad (17)$$

|←--group 1a--→|←---group 2a----→|←--group 3a --→|←-group 4a ---

Therefore changing term order of (4) for making the infinite groups like (group 1a), (group 2a), (group 3a),------ does not change the sum of the right side of (4). Because changing term order is done within Range.

3 Changing term order of (5)

Similarly changing term order of (5) for making the infinite groups like (group 1b), (group 2b), (group 3b),------ does not change the sum of the right side of (5). Because changing term order of (5) is done in the same term order as in changing term order of (4). The same Range as in changing term order of (4) exists as shown in the following example. And changing term order is done within Range.

$$1 = 2^{2a}\boxed{-2}+3^{2a}\boxed{-3}+4^{2a}\boxed{+4}+5^{2a}\boxed{-5}+6^{2a}\boxed{+6}+7^{2a}\boxed{-7} + 8^{2a}\boxed{+8}+9^{2a}\boxed{-9}+10^{2a}\boxed{+10} + 11^{2a}\boxed{+11} + \text{-------------} \qquad (16)$$

|←--R.0--→|←------Range1--------→|←------Range2----→|←-----Range3-----

$$1 = 2^{2a}\boxed{-2}+3^{2a}\boxed{-3}+4^{2a}\boxed{+4}+6^{2a}\boxed{+6} + 5^{2a}\boxed{-5}+7^{2a}\boxed{-7}+8^{2a}\boxed{+8}+10^{2a}\boxed{+10} + 9^{2a}\boxed{-9}+11^{2a}\boxed{+11}+13^{2a}\boxed{+13} + \text{------} \qquad (19\text{-}1)$$

|←-----group 1b------→|←-------group 2b------→|←-----group 3b-----→|

Appendix 3: The sum of +terms

1 In the following (4) if N is a large number, the value of blogN increases very slowly with the increase of N and +term and -term appear alternately. We define f(n) as following (31).

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \text{--------} + \frac{\cos(b\log N)}{N^{1/2+a}} - \text{------------------} \quad (4)$$

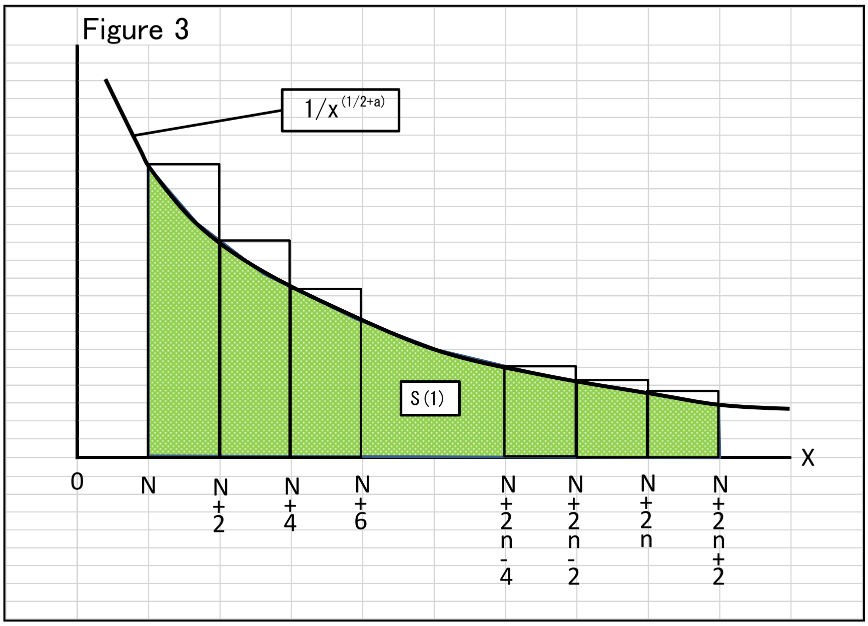
$$\boxed{+N} + \boxed{+(N+2)} + \boxed{+(N+4)} + \boxed{+(N+6)} + \text{----} + \boxed{+(N+2n-4)} + \boxed{+(N+2n-2)} + \boxed{+(N+2n)}$$
$$< 1/N^{1/2+a} + 1/(N+2)^{1/2+a} + 1/(N+4)^{1/2+a} + 1/(N+6)^{1/2+a} + \text{----}$$
$$+ 1/(N+2n-4)^{1/2+a} + 1/(N+2n-2)^{1/2+a} + 1/(N+2n)^{1/2+a} = f(n) \quad (31)$$

$$f(n) > 1/2 \int_{N}^{N+2n+2} 1/x^{1/2+a}\, dx = \{1/(1-2a)\}\left[x^{1/2-a}\right]_{N}^{N+2n+2}$$

$$= \{1/(1-2a)\}\{(N+2n+2)^{1/2-a} - N^{1/2-a}\} = 1/2\ S(1) \quad (32)$$

S(1) is the green area in (Figure 3).

If we do $n\to\infty$ in (32), $(N+2n+2)^{1/2-a}\to\infty$ due to $0\leqq a<1/2$, $S(1)\to\infty$ and $f(n)\to\infty$ hold.



2　In page 2 $\boxed{+n}$ is defined as follows.

$\boxed{+n}$: $+\frac{\cos(b\log n)}{n^{1/2+a}}$ when the sign of $\cos(b\log n)$ is "+".

$-\frac{\cos(b\log n)}{n^{1/2+a}}$ when the sign of $\cos(b\log n)$ is "−".

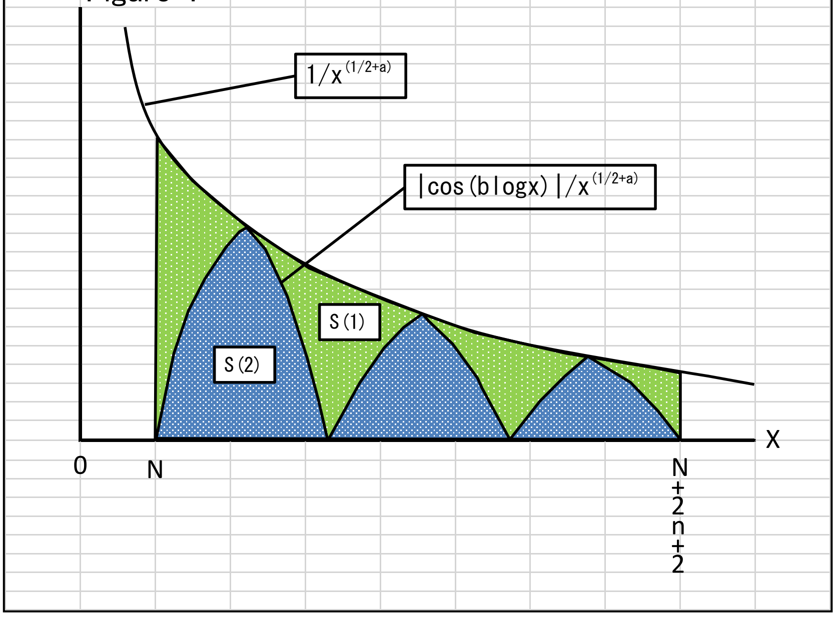
Therefore $\boxed{+N} = |\cos(b\log N)|/N^{1/2+a}$

$\boxed{+N} + \boxed{+(N+2)} + \boxed{+(N+4)} + \boxed{+(N+6)} + \text{----} + \boxed{+(N+2n-4)} + \boxed{+(N+2n-2)} + \boxed{+(N+2n)}$
$= |\cos(b\log N)|/N^{1/2+a} + |\cos(b\log(N+2))|/(N+2)^{1/2+a} + |\cos(b\log(N+4))|/(N+4)^{1/2+a} +$
$|\cos(b\log(N+6))|/(N+6)^{1/2+a} + \text{----} + |\cos(b\log(N+2n-4))|/(N+2n-4)^{1/2+a} +$
$|\cos(b\log(N+2n-2))|/(N+2n-2)^{1/2+a} + |\cos(b\log(N+2n))|/(N+2n)^{1/2+a}$

We define S(2) as $\int_N^{N+2n+2} |\cos(b\log x)|/x^{1/2+a}\,dx$.

S(2) is the blue area in (Figure 4). Due to $[0 \leqq |\cos(b\log x)|/x^{1/2+a} \leqq 1/x^{1/2+a}]$ S(2) is inscribed in area S(1) as shown in (Figure 4). S(1) diverges to ∞ with n →∞. Therefore S(2) also diverges to ∞ with n→∞.

Figure 4



2 To make the situation of

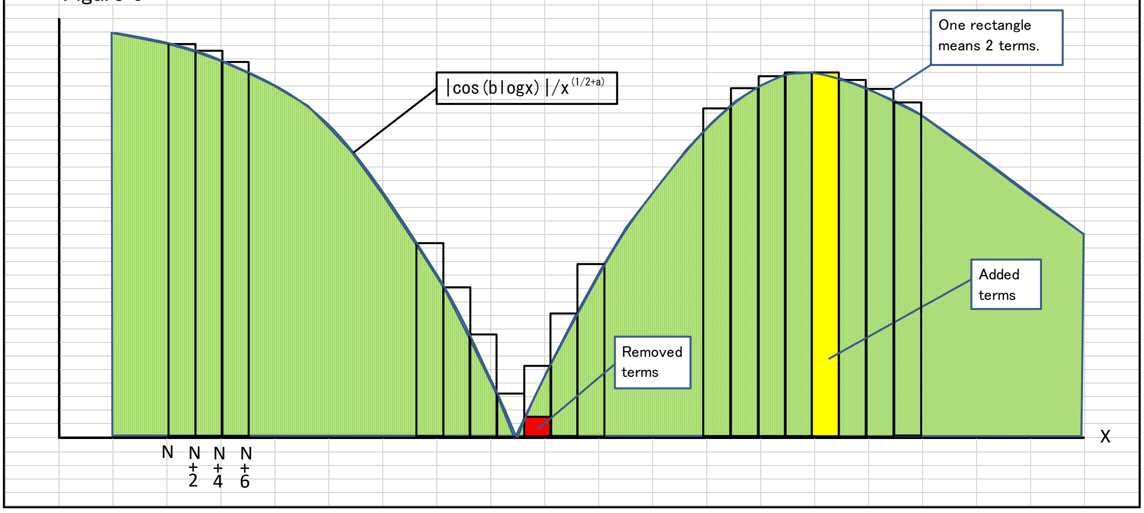
$$2x\,(+N++(N+2)++(N+4)++(N+6)\ + \text{--------}) > \int_N^{\infty} |\cos(b\log x)|/x^{1/2+a}\,dx$$

we need the following operation as shown in (Figure 5).

(1) We remove 2 terms of the minimum value. By doing that way the terms after the removed terms shift backward by 2 terms.

(2) We add 2 terms of the maximum value after 2 terms of the maximum value. By doing that way the terms after the added terms shift forward by 2 terms.

Figure 5



Now we have the following inequality.

$2x[(+N++(N+2)++(N+4)++(N+6)\ + \text{-------})$ - (the sum of infinite number of the removed terms) + (the sum of infinite number of the added terms)]

$$> \int_N^{\infty} |\cos(b\log x)|/x^{1/2+a}\,dx = \infty \qquad (33)$$

In the added term $|\cos(b\log N)|=1$ holds and $N=e^{k/b\pi}$ also holds due to $|\cos(b\log N)|=1$. ($k=L, L+1, L+2, L+3$, ---- L: natural number)

$g(k)$ = [The value of the added term] $= 1/N^{1/2+a} = e^{-k/b(1/2+a)\pi} = e^{-Mk}$

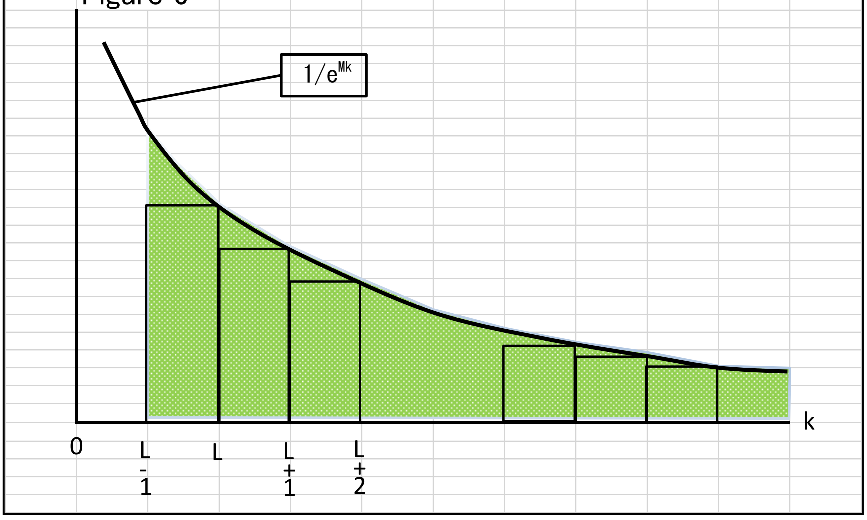
($(1/2+a)\pi/b = M>0$)

(the sum of infinite number of the added terms) has upper bound of A as follows.

(the sum of infinite number of the added terms)

$$= g(L)+g(L+1)+g(L+2)+g(L+3)+ \text{--------} < \int_{L-1}^{\infty} e^{-Mk}dk = -1/M[e^{-Mk}]_{L-1}^{\infty} = e^{-(L-1)M}/M = A > 0$$

Figure 6



(the sum of infinite number of the removed terms) has also upper bound of B (>0) due to the following reasons.

(1) [The value of (the added term) which exists next to (the removed term)] > the value of (the removed term)

(2) (the removed term) appears at the same frequency as (the added term).

(3) (the sum of infinite number of the added terms) has upper bound.

From (33) we can find that (the sum of infinite number of +terms) is $+\infty$ as follows.

$$2x[(+N++(N+2)++(N+4)++(N+6) + \text{--------}) + A] > [\text{the left side of (33)}]$$

$$> \int_{N}^{\infty} |\cos(b\log x)|/x^{1/2+a}\,dx = \infty$$

We can also find that (the sum of infinite number of -terms) is $-\infty$ through the same discussion as above.