A re-turn to *SO*(4) algebras and the Lorentz group *O*(3,3)

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Abstract: The Lie algebra of the Lorentz group *O*(3,3) is considered. Four classes of *SO*(4) subalgebras are defined and their properties investigated.

Keywords: Lie algebra; *O*(3,3); internal symmetry; Lorentz

1. Introduction:

The Lorentz group *O*(3,1) contains transformations associated with rotations and boosts in special relativity. The related Lie algebra, *SO*(3,1), contains six group generators which can form a single *SO*(4) = *SU*(2) × *SU*(2) algebra. In quantum theory, objects with spin transform under this one algebra, with spin-½ and spin-1 objects using different representations [1,2].

The Lorentz group *O*(3,3) contains transformations that can be associated with rotations and boosts in a six dimensional mathematical space containing three time dimensions and three space dimensions [3]. The related Lie algebra, *SO*(3,3), has fifteen generators and contains a number of *SO*(4) subalgebras. One type of *SO*(4) subalgebra contains three different rotation generators and has been investigated previously [4]. A second type of *SO*(4) subalgebra contains six different rotation generators and is the subject of this article.

In section 3, four classes of *SO*(4) algebras are defined: the p, z, g, and w classes. In section 4, the p and z classes are investigated. In section 5, the g and w classes are considered.

2. Materials and methods

The article adopts the convention that roman indices can assume any value from 1 to 3.

The article uses the notation in reference [4] for generators of the Lie algebra *SO*(3,3):

$$\begin{aligned} &J_j &&: \text{three space rotation generators} \\ &T_k &&: \text{three time rotation generators} \\ &K_{kj} &&: \text{nine boost generators} \end{aligned} \qquad (1)$$

The related commutation relations for complexified *SO*(3,3) are then [4]:

$$\begin{aligned} &[T_j, T_k] = i\,\epsilon_{jkm}\,T_m && [J_j, J_k] = i\,\epsilon_{jkm}\,J_m && [T_j, J_k] = 0 \\ &[T_j, iK_{kn}] = i\,\epsilon_{jkm}\,iK_{mn} && [J_j, iK_{nk}] = i\,\epsilon_{jkm}\,iK_{nm} && \\ &[iK_{jn}, iK_{kn}] = i\,\epsilon_{jkm}\,T_m && [iK_{nj}, iK_{nk}] = i\,\epsilon_{jkm}\,J_m && \end{aligned} \qquad (2)$$

The article defines an *SO*(4) Lie algebra using the following notation [5]

$$\begin{aligned} [a_j, a_k] &= i\,\epsilon_{jkm}\,a_m \\ [b_j, b_k] &= i\,\epsilon_{jkm}\,a_m \\ [a_j, b_k] &= i\,\epsilon_{jkm}\,b_m \end{aligned} \qquad (3)$$

It also defines the direct product *SO*(3) × *SO*(2) using the notation [5]

$$
\begin{aligned}
[w_j, w_k] &= i\,\epsilon_{jkm}\, w_m \\
[w_0, w_k] &= 0
\end{aligned}
\qquad (4)
$$

We note that the groups *SU*(2) and *SO*(3) have the same Lie algebra, and that *U*(1) and *SO*(2) are isomorphic [1].

Reference [4] also defines two families of *SO*(4) subalgebras containing three rotation generators:

(5)

label	a_1	a_2	a_3	b_1	b_2	b_3
e_1	J_1	J_2	J_3	iK_{11}	iK_{12}	iK_{13}
e_2	J_1	J_2	J_3	iK_{21}	iK_{22}	iK_{23}
e_3	J_1	J_2	J_3	iK_{31}	iK_{32}	iK_{33}
m_1	T_1	T_2	T_3	iK_{11}	iK_{21}	iK_{31}
m_2	T_1	T_2	T_3	iK_{12}	iK_{22}	iK_{32}
m_3	T_1	T_2	T_3	iK_{13}	iK_{23}	iK_{33}

Finally, we define a chiral-pair of components as the sum and difference of a rotation generator and a boost generator. For example, the components $\frac{1}{2}(T_1 + iK_{11})$ and $\frac{1}{2}(T_1 - iK_{11})$ are defined to be a chiral-pair and may be written together as $\frac{1}{2}(T_1 \pm iK_{11})$.

3. *SO*(4) algebras

There are a number of *SO*(4) subalgebras in complexified *SO*(3,3) that contain six different rotation generators. Twelve algebras of interest are:

(6)

label	a_1	a_2	a_3	b_1	b_2	b_3
p_1	$(T_1 + J_1)$	$(T_2 + J_2)$	$(T_3 + J_3)$	$(T_1 - J_1)$	$(T_2 - J_2)$	$(T_3 - J_3)$
p_2	$(T_2 + J_3)$	$(T_3 + J_1)$	$(T_1 + J_2)$	$(T_2 - J_3)$	$(T_3 - J_1)$	$(T_1 - J_2)$
p_3	$(T_3 + J_2)$	$(T_1 + J_3)$	$(T_2 + J_1)$	$(T_3 - J_2)$	$(T_1 - J_3)$	$(T_2 - J_1)$
z_1	$(T_1 - J_1)$	$(T_2 - J_3)$	$(T_3 - J_2)$	$(T_1 + J_1)$	$(T_2 + J_3)$	$(T_3 + J_2)$
z_2	$(T_2 - J_2)$	$(T_3 - J_1)$	$(T_1 - J_3)$	$(T_2 + J_2)$	$(T_3 + J_1)$	$(T_1 + J_3)$
z_3	$(T_3 - J_3)$	$(T_1 - J_2)$	$(T_2 - J_1)$	$(T_3 + J_3)$	$(T_1 + J_2)$	$(T_2 + J_1)$
g_1	$(T_1 + J_1)$	$(T_2 + J_2)$	$(T_3 + J_3)$	$(iK_{23} - iK_{32})$	$(iK_{31} - iK_{13})$	$(iK_{12} - iK_{21})$
g_2	$(T_2 + J_3)$	$(T_3 + J_1)$	$(T_1 + J_2)$	$(iK_{32} - iK_{11})$	$(iK_{13} - iK_{22})$	$(iK_{21} - iK_{33})$
g_3	$(T_3 + J_2)$	$(T_1 + J_3)$	$(T_2 + J_1)$	$(iK_{11} - iK_{23})$	$(iK_{22} - iK_{31})$	$(iK_{33} - iK_{12})$
w_1	$(T_1 - J_1)$	$(T_2 - J_3)$	$(T_3 - J_2)$	$(iK_{33} - iK_{22})$	$(iK_{12} - iK_{31})$	$(iK_{21} - iK_{13})$
w_2	$(T_2 - J_2)$	$(T_3 - J_1)$	$(T_1 - J_3)$	$(iK_{11} - iK_{33})$	$(iK_{23} - iK_{12})$	$(iK_{32} - iK_{21})$
w_3	$(T_3 - J_3)$	$(T_1 - J_2)$	$(T_2 - J_1)$	$(iK_{22} - iK_{11})$	$(iK_{31} - iK_{23})$	$(iK_{13} - iK_{32})$

It is convenient to consider these as four classes, each with three members: the p, z, g, and w classes. We note that the coefficients of the time rotation generators are positive in each of these algebras. We also note that each algebra contains all three space rotation generators and all three time rotation generators.

4. The p and z classes

Three $SO(3) \times SO(2)$ algebras related to the p class are:

w_1	w_2	w_3	w_0
T_1	T_2	T_3	J_1
T_1	T_2	T_3	J_2
T_1	T_2	T_3	J_3

(7)

These algebras suggests that every time rotation generator is related to every space rotation generator by $SU(2) \times U(1)$ symmetry.

The first component of the p_1 algebra is $(T_1 + J_1)$. If we call T_1 and J_1 sub-components, then the first component of the p_1 algebra might be described as the sum of two sub-components that are related by $SU(2) \times U(1)$ symmetry.

With a change of basis to $SO(4) = SO(3) \times SO(3)$, the p class algebras become:

	$\frac{1}{2}(a_1 + b_1)$	$\frac{1}{2}(a_2 + b_2)$	$\frac{1}{2}(a_3 + b_3)$	$\frac{1}{2}(a_1 - b_1)$	$\frac{1}{2}(a_2 - b_2)$	$\frac{1}{2}(a_3 - b_3)$
p_1	T_1	T_2	T_3	J_1	J_2	J_3
p_2	T_2	T_3	T_1	J_3	J_1	J_2
p_3	T_3	T_1	T_2	J_2	J_3	J_1

(8)

Similar observations can be made with respect to the z class of algebras. In this case however, the first component of the z_1 algebra, $(T_1 - J_1)$, is the difference of two sub-components related by $SU(2) \times U(1)$ symmetry.

In the $SO(4) = SO(3) \times SO(3)$ basis the z class algebras become:

	$\frac{1}{2}(a_1 + b_1)$	$\frac{1}{2}(a_2 + b_2)$	$\frac{1}{2}(a_3 + b_3)$	$\frac{1}{2}(a_1 - b_1)$	$\frac{1}{2}(a_2 - b_2)$	$\frac{1}{2}(a_3 - b_3)$
z_1	T_1	T_2	T_3	$-J_1$	$-J_3$	$-J_2$
z_2	T_2	T_3	T_1	$-J_2$	$-J_1$	$-J_3$
z_3	T_3	T_1	T_2	$-J_3$	$-J_2$	$-J_1$

(9)

5. The g and w classes:

Nine $SO(3) \times SO(2)$ algebras of interest with respect to the g class are:

w_1	w_2	w_3	w_0
J_1	iK_{12}	iK_{13}	T_1
J_1	iK_{22}	iK_{23}	T_2
J_1	iK_{32}	iK_{33}	T_3
J_2	iK_{13}	iK_{11}	T_1
J_2	iK_{23}	iK_{21}	T_2
J_2	iK_{33}	iK_{31}	T_3

(10)

J_3	iK_{11}	iK_{12}	T_1
J_3	iK_{21}	iK_{22}	T_2
J_3	iK_{31}	iK_{32}	T_3

If we consider the first algebra,

w_1	w_2	w_3	w_0
J_1	iK_{12}	iK_{13}	T_1

(11)

then with a change of basis we may obtain:

$\frac{1}{2}(w_1 \pm w_2)$	$\frac{1}{2}(w_0 \pm w_3)$
$\frac{1}{2}(J_1 \pm iK_{12})$	$\frac{1}{2}(T_1 \pm iK_{13})$

(12)

We note that the generators T_1 and iK_{13} are associated with the m_3 algebra (of Section 2). Rotating $\frac{1}{2}(w_0 \pm w_3)$ in the associated m_3 vector space gives:

$\frac{1}{2}(w_1 \pm w_2)$	$\frac{1}{2}(w_0 \pm w_3)'$
$\frac{1}{2}(J_1 \pm iK_{12})$	$\frac{1}{2}(T_2 \pm iK_{33})$

(13)

We conclude that the two chiral-pairs $\frac{1}{2}(J_1 \pm iK_{12})$ and $\frac{1}{2}(T_2 \pm iK_{33})$ are related by $SO(3) \times SO(2)$ symmetry plus a rotation.

In this way, the following $SU(2) \times U(1)$ partnered chiral-pairs can be constructed:

$\frac{1}{2}(J_1 \pm iK_{12}) \leftrightarrow \frac{1}{2}(T_2 \pm iK_{33})$ $\quad \frac{1}{2}(J_2 \pm iK_{13}) \leftrightarrow \frac{1}{2}(T_2 \pm iK_{31})$ $\quad \frac{1}{2}(J_3 \pm iK_{11}) \leftrightarrow \frac{1}{2}(T_2 \pm iK_{32})$ (14)
$\frac{1}{2}(J_1 \pm iK_{22}) \leftrightarrow \frac{1}{2}(T_3 \pm iK_{13})$ $\quad \frac{1}{2}(J_2 \pm iK_{23}) \leftrightarrow \frac{1}{2}(T_3 \pm iK_{11})$ $\quad \frac{1}{2}(J_3 \pm iK_{21}) \leftrightarrow \frac{1}{2}(T_3 \pm iK_{12})$
$\frac{1}{2}(J_1 \pm iK_{32}) \leftrightarrow \frac{1}{2}(T_1 \pm iK_{23})$ $\quad \frac{1}{2}(J_2 \pm iK_{33}) \leftrightarrow \frac{1}{2}(T_1 \pm iK_{21})$ $\quad \frac{1}{2}(J_3 \pm iK_{31}) \leftrightarrow \frac{1}{2}(T_1 \pm iK_{22})$

Changing the g class algebras into the $SO(4) = SO(3) \times SO(3)$ basis gives:

	$\frac{1}{2}(a_1 \pm b_1)$	$\frac{1}{2}(a_2 \pm b_2)$	$\frac{1}{2}(a_3 \pm b_3)$
g_1	$\frac{1}{2}(T_1 + J_1) \pm \frac{1}{2}(iK_{23} - iK_{32})$	$\frac{1}{2}(T_2 + J_2) \pm \frac{1}{2}(iK_{31} - iK_{13})$	$\frac{1}{2}(T_3 + J_3) \pm \frac{1}{2}(iK_{12} - iK_{21})$
g_2	$\frac{1}{2}(T_2 + J_3) \pm \frac{1}{2}(iK_{32} - iK_{11})$	$\frac{1}{2}(T_3 + J_1) \pm \frac{1}{2}(iK_{13} - iK_{22})$	$\frac{1}{2}(T_1 + J_2) \pm \frac{1}{2}(iK_{21} - iK_{33})$
g_3	$\frac{1}{2}(T_3 + J_2) \pm \frac{1}{2}(iK_{11} - iK_{23})$	$\frac{1}{2}(T_1 + J_3) \pm \frac{1}{2}(iK_{22} - iK_{31})$	$\frac{1}{2}(T_2 + J_1) \pm \frac{1}{2}(iK_{33} - iK_{12})$

(15)

Rearranging generators gives:

	$\frac{1}{2}(a_1 \pm b_1)$	$\frac{1}{2}(a_2 \pm b_2)$	$\frac{1}{2}(a_3 \pm b_3)$
g_1	$\frac{1}{2}(T_1 \pm iK_{23}) + \frac{1}{2}(J_1 \mp iK_{32})$	$\frac{1}{2}(T_2 \pm iK_{31}) + \frac{1}{2}(J_2 \mp iK_{13})$	$\frac{1}{2}(T_3 \pm iK_{12}) + \frac{1}{2}(J_3 \mp iK_{21})$
g_2	$\frac{1}{2}(T_2 \pm iK_{32}) + \frac{1}{2}(J_3 \mp iK_{11})$	$\frac{1}{2}(T_3 \pm iK_{13}) + \frac{1}{2}(J_1 \mp iK_{22})$	$\frac{1}{2}(T_1 \pm iK_{21}) + \frac{1}{2}(J_2 \mp iK_{33})$
g_3	$\frac{1}{2}(T_3 \pm iK_{11}) + \frac{1}{2}(J_2 \mp iK_{23})$	$\frac{1}{2}(T_1 \pm iK_{22}) + \frac{1}{2}(J_3 \mp iK_{31})$	$\frac{1}{2}(T_2 \pm iK_{33}) + \frac{1}{2}(J_1 \mp iK_{12})$

(16)

We note that the first paired component of the g_1 algebra, $\frac{1}{2}(T_1 \pm iK_{23}) + \frac{1}{2}(J_1 \mp iK_{32})$, is the sum of two sub-components, $\frac{1}{2}(T_1 \pm iK_{23})$ and $\frac{1}{2}(J_1 \mp iK_{32})$, that are related by $SU(2) \times U(1)$ symmetry plus a rotation.

Using the same nine $SO(3) \times SO(2)$ algebras as above but rotating in the opposite direction, leads to the following $SU(2) \times U(1)$ partnered chiral-pairs:

$$\begin{array}{lll} \tfrac{1}{2}(J_1 \pm iK_{12}) \leftrightarrow \tfrac{1}{2}(T_3 \pm iK_{23}) & \tfrac{1}{2}(J_2 \pm iK_{13}) \leftrightarrow \tfrac{1}{2}(T_3 \pm iK_{21}) & \tfrac{1}{2}(J_3 \pm iK_{11}) \leftrightarrow \tfrac{1}{2}(T_3 \pm iK_{22}) \\ \tfrac{1}{2}(J_1 \pm iK_{22}) \leftrightarrow \tfrac{1}{2}(T_1 \pm iK_{33}) & \tfrac{1}{2}(J_2 \pm iK_{23}) \leftrightarrow \tfrac{1}{2}(T_1 \pm iK_{31}) & \tfrac{1}{2}(J_3 \pm iK_{21}) \leftrightarrow \tfrac{1}{2}(T_1 \pm iK_{32}) \\ \tfrac{1}{2}(J_1 \pm iK_{32}) \leftrightarrow \tfrac{1}{2}(T_2 \pm iK_{13}) & \tfrac{1}{2}(J_2 \pm iK_{33}) \leftrightarrow \tfrac{1}{2}(T_2 \pm iK_{11}) & \tfrac{1}{2}(J_3 \pm iK_{31}) \leftrightarrow \tfrac{1}{2}(T_2 \pm iK_{12}) \end{array} \quad (17)$$

Changing the w class algebras into the $SO(4) = SO(3) \times SO(3)$ basis and rearranging generators gives:

(18)

	$\tfrac{1}{2}(a_1 \pm b_1)$	$\tfrac{1}{2}(a_2 \pm b_2)$	$\tfrac{1}{2}(a_3 \pm b_3)$
w_1	$\tfrac{1}{2}(T_1 \pm iK_{33}) - \tfrac{1}{2}(J_1 \pm iK_{22})$	$\tfrac{1}{2}(T_2 \pm iK_{12}) - \tfrac{1}{2}(J_3 \pm iK_{31})$	$\tfrac{1}{2}(T_3 \pm iK_{21}) - \tfrac{1}{2}(J_2 \pm iK_{13})$
w_2	$\tfrac{1}{2}(T_2 \pm iK_{11}) - \tfrac{1}{2}(J_2 \pm iK_{33})$	$\tfrac{1}{2}(T_3 \pm iK_{23}) - \tfrac{1}{2}(J_1 \pm iK_{12})$	$\tfrac{1}{2}(T_1 \pm iK_{32}) - \tfrac{1}{2}(J_3 \pm iK_{21})$
w_3	$\tfrac{1}{2}(T_3 \pm iK_{22}) - \tfrac{1}{2}(J_3 \pm iK_{11})$	$\tfrac{1}{2}(T_1 \pm iK_{31}) - \tfrac{1}{2}(J_2 \pm iK_{23})$	$\tfrac{1}{2}(T_2 \pm iK_{13}) - \tfrac{1}{2}(J_1 \pm iK_{32})$

We note that the first paired component of the w_1 algebra, $\tfrac{1}{2}(T_1 \pm iK_{33}) - \tfrac{1}{2}(J_1 \pm iK_{22})$, is the difference of two sub-components related by $SU(2) \times U(1)$ symmetry plus a rotation.

Conclusion

The article has investigated four classes of $SO(4)$ subalgebras in complexified $SO(3,3)$. Components of these algebras appear to be the sum or difference of sub-components related by $SU(2) \times U(1)$ symmetry.

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Appendix:

Complexified *SO*(3,3) commutation table:

	T_1	T_2	T_3	J_1	J_2	J_3	iK_{11}	iK_{12}	iK_{13}	iK_{21}	iK_{22}	iK_{23}	iK_{31}	iK_{32}	iK_{33}
T_1		iT_3	$-iT_2$							i^2K_{31}	i^2K_{32}	i^2K_{33}	$-i^2K_{21}$	$-i^2K_{22}$	$-i^2K_{23}$
T_2	$-iT_3$		iT_1				$-i^2K_{31}$	$-i^2K_{32}$	$-i^2K_{33}$				i^2K_{11}	i^2K_{12}	i^2K_{13}
T_3	iT_2	$-iT_1$					i^2K_{21}	i^2K_{22}	i^2K_{23}	$-i^2K_{11}$	$-i^2K_{12}$	$-i^2K_{13}$			
J_1					iJ_3	$-iJ_2$		i^2K_{13}	$-i^2K_{12}$		i^2K_{23}	$-i^2K_{22}$		i^2K_{33}	$-i^2K_{32}$
J_2				$-iJ_3$		iJ_1	$-i^2K_{13}$		i^2K_{11}	$-i^2K_{23}$		i^2K_{21}	$-i^2K_{33}$		i^2K_{31}
J_3				iJ_2	$-iJ_1$		i^2K_{12}	$-i^2K_{11}$		i^2K_{22}	$-i^2K_{21}$		i^2K_{32}	$-i^2K_{31}$	
iK_{11}		i^2K_{31}	$-i^2K_{21}$		i^2K_{13}	$-i^2K_{12}$		iJ_3	$-iJ_2$	iT_3			$-iT_2$		
iK_{12}		i^2K_{32}	$-i^2K_{22}$	$-i^2K_{13}$		i^2K_{11}	$-iJ_3$		iJ_1		iT_3			$-iT_2$	
iK_{13}		i^2K_{33}	$-i^2K_{23}$	i^2K_{12}	$-i^2K_{11}$		iJ_2	$-iJ_1$				iT_3			$-iT_2$
iK_{21}	$-i^2K_{31}$		i^2K_{11}		i^2K_{23}	$-i^2K_{22}$	$-iT_3$				iJ_3	$-iJ_2$	iT_1		
iK_{22}	$-i^2K_{32}$		i^2K_{12}	$-i^2K_{23}$		i^2K_{21}		$-iT_3$		$-iJ_3$		iJ_1		iT_1	
iK_{23}	$-i^2K_{33}$		i^2K_{13}	i^2K_{22}	$-i^2K_{21}$				$-iT_3$	iJ_2	$-iJ_1$				iT_1
iK_{31}	i^2K_{21}	$-i^2K_{11}$			i^2K_{33}	$-i^2K_{32}$	iT_2			$-iT_1$				iJ_3	$-iJ_2$
iK_{32}	i^2K_{22}	$-i^2K_{12}$		$-i^2K_{33}$		i^2K_{31}		iT_2			$-iT_1$		$-iJ_3$		iJ_1
iK_{33}	i^2K_{23}	$-i^2K_{13}$		i^2K_{32}	$-i^2K_{31}$				iT_2			$-iT_1$	iJ_2	$-iJ_1$	

SO(3,3) commutation table:

	T_1	T_2	T_3	J_1	J_2	J_3	K_{11}	K_{12}	K_{13}	K_{21}	K_{22}	K_{23}	K_{31}	K_{32}	K_{33}
T_1		iT_3	$-iT_2$							iK_{31}	iK_{32}	iK_{33}	$-iK_{21}$	$-iK_{22}$	$-iK_{23}$
T_2	$-iT_3$		iT_1				$-iK_{31}$	$-iK_{32}$	$-iK_{33}$				iK_{11}	iK_{12}	iK_{13}
T_3	iT_2	$-iT_1$					iK_{21}	iK_{22}	iK_{23}	$-iK_{11}$	$-iK_{12}$	$-iK_{13}$			
J_1					iJ_3	$-iJ_2$		iK_{13}	$-iK_{12}$		iK_{23}	$-iK_{22}$		iK_{33}	$-iK_{32}$
J_2				$-iJ_3$		iJ_1	$-iK_{13}$		iK_{11}	$-iK_{23}$		iK_{21}	$-iK_{33}$		iK_{31}
J_3				iJ_2	$-iJ_1$		iK_{12}	$-iK_{11}$		iK_{22}	$-iK_{21}$		iK_{32}	$-iK_{31}$	
K_{11}		iK_{31}	$-iK_{21}$		iK_{13}	$-iK_{12}$		$-iJ_3$	iJ_2	$-iT_3$			iT_2		
K_{12}		iK_{32}	$-iK_{22}$	$-iK_{13}$		iK_{11}	iJ_3		$-iJ_1$		$-iT_3$			iT_2	
K_{13}		iK_{33}	$-iK_{23}$	iK_{12}	$-iK_{11}$		$-iJ_2$	iJ_1				$-iT_3$			iT_2
K_{21}	$-iK_{31}$		iK_{11}		iK_{23}	$-iK_{22}$	iT_3				$-iJ_3$	iJ_2	$-iT_1$		
K_{22}	$-iK_{32}$		iK_{12}	$-iK_{23}$		iK_{21}		iT_3		iJ_3		$-iJ_1$		$-iT_1$	
K_{23}	$-iK_{33}$		iK_{13}	iK_{22}	$-iK_{21}$				iT_3	$-iJ_2$	iJ_1				$-iT_1$
K_{31}	iK_{21}	$-iK_{11}$			iK_{33}	$-iK_{32}$	$-iT_2$			iT_1				$-iJ_3$	iJ_2
K_{32}	iK_{22}	$-iK_{12}$		$-iK_{33}$		iK_{31}		$-iT_2$			iT_1		iJ_3		$-iJ_1$
K_{33}	iK_{23}	$-iK_{13}$		iK_{32}	$-iK_{31}$				$-iT_2$			iT_1	$-iJ_2$	iJ_1	