# Diophantine Quintic Equation

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## ABSTRACT

On the internet & math literature there is not much mention about the quintic equation, $p(a^5+b^5)=q(c^5+d^5)$. Since parameterization of fifth degree equations is generally hard the author has attempted to find numerical solutions to the above quintic equation by algebraic method. In the concluding note the author has mentioned an open problem regarding the above quintic equation

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We have the below mentioned quintic equation:

$$p(a^5+b^5)=q(c^5+d^5) \quad \text{----------------(1)}$$

**Case 1:**

Let, $(a+b)=(c+d)=w$

Hence we have: $(a^5+b^5)=w(w^4-5abw^2+5a^2b^2)$

& $(c^5+d^5)=w(w^4-5cdw^2+5c^2d^2)$

Hence:

$$p(w^4-5abw^2+5a^2b^2)=q(w^4-5cdw^2+5c^2d^2)$$

$$w^4(p-q)-5w^2(abp-cdq)+5(a^2b^2p-c^2d^2q)=0$$

Let, $w^2=m$

$$m^2(p-q)-5m(abp-cdq)+5(a^2b^2p-c^2d^2q)=0 \quad ---(2)$$

Let discriminant of (2) be 'z'

Solving (2) as quadratic in 'm' we have:

$$z^2=25(abp-cdq)^2-20(p-q)(a^2b^2p-c^2d^2q)$$

We take: $2ab = 3cd$ & we get:

$$4z^2 = 5(cd)^2(9p^2 - 8pq + 4q^2)$$

In order to make the RHS a square we substitute:

$$(9p^2 - 8pq + 4q^2) = 5(v)^2 \text{ ---------------------(3)}$$

hence, $4z^2 = (5cdv)^2$

$$z = \left(\frac{5cdv}{2}\right)$$

we parameterize (3) at (p,q,v)=(1,1,1)

$$(p, q, v) =$$

$$\left((5k^2 - 10k - 11), (5k^2 + 10k - 31), (5k^2 - 10k + 21)\right) - - - (4)$$

Hence, $m = w^2 = \frac{\left[5(abp - cdq) + \frac{5cdv}{2}\right]}{2(p-q)}$

Since, $2ab = 3cd$

$$w^2 = \frac{5cd(3p - 2q + v)}{4(p - q)}$$

Substituting for (p,q,v) we get:

$$\frac{w^2}{cd} = \frac{5(5 - k)}{8}$$

To make the RHS a square for (w). we take, k = - 5

& we get, $\frac{w^2}{cd} = \frac{(5)^2}{4}$

Hence we take, $w = 5$ & $cd = 4$

As, $w = c + d = 5 \,\&\, cd = 4 \quad we\ get\ (c, d) = (4,1)$

$$Also, \quad w = (a + b) = 5 \quad \& \quad ab = 3cd/2 = 3 * \frac{(4)}{2} = 6$$

$$Since\ (a + b) = 5 \,\&\, ab = 6, we\ get\ (a, b) = (3,2)$$

Hence, $(a, b, c, d) = (3,2,4,1)$.

Since $2ab = 3cd$ & $a + b = c + d$ we have the parameterization

$$(a, b, c, d) = \big((3ef), (gh), (fg), (2eh)\big] ----(5)$$

Where, $(e, f, g, h) = [(2k - 1), (k + 2), (3k - 4), (3k + 1)]$

$$for\ k = 0, we\ get, (a, b, c, d) = (3,2,4,1)$$

for, k=-5 we get, $(p, q) = (164,44)$

Hence from eqn. (1) we get:

$$p(a^5 + b^5) = q(c^5 + d^5)$$

$$164(a^5 + b^5) = 44(c^5 + d^5) \quad \text{or}$$

$$41(3^5 + 2^5) = 11(4^5 + 1^5)$$

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$$r(a^5 + b^5) = s(c^5 + d^5) \quad \text{---------------- (6)}$$

**<u>Case 2:</u>**

Let, $(a + b) = (c + d) = w$

Hence we have: $(a^5 + b^5) = w(w^4 - 5abw^2 + 5a^2b^2)$

& $(c^5 + d^5) = w(w^4 - 5cdw^2 + 5c^2d^2)$

Hence:

$$r(w^4 - 5abw^2 + 5a^2b^2) = s(w^4 - 5cdw^2 + 5c^2d^2)$$

$$w^4(r - s) - 5w^2(abr - cds) + 5(a^2b^2r - c^2d^2s) = 0$$

Let, $w^2 = m$

$$m^2(r - s) - 5m(abr - cds) + 5(a^2b^2r - c^2d^2s) = 0 \quad --- (7)$$

Let discriminant of (2) be ‘z’

Solving (7) as quadratic in ‘m’ we have:

$$z^2 = 25(abr - cds)^2 - 20(r - s)(a^2b^2r - c^2d^2s)$$

We take: $2ab = -cd$ & we get:

$$z^2 = 5(ab)^2(r^2 + 40rs + 4s^2)$$

In order to make the RHS a square we substitute:

$(r^2 + 40rs + 4s^2) = 5(u)^2$ ---------------------(8)

hence, $z^2 = (5abu)^2$

$z = 5abu$

we parameterize (8) at (r,s,u)=(1,1,3)

$$(r, s, u) =$$

$$\left((5k^2 - 30k + 29), (5k^2 + 30k + 41), 3(5k^2 + 2k - 35)\right) - - - (9)$$

Hence, $m = w^2 = \frac{[5(abr-cds)+5abu]}{2(p-q)}$

Since, $2ab = -cd$

$$w^2 = \frac{5ab(r + 2s + u)}{2(r - s)}$$

Substituting for (r,s,u) we get:

$$\frac{w^2}{ab} = \frac{5(k + 1)}{(-4)}$$

To make the RHS a square for (w) we take, k = - 23/5

& we get, $\frac{w^2}{ab} = \frac{(3)^2}{2}$

Hence we have, $w = 3$ & $ab = 2$

As, $w = a + b = 3$ & $ab = 2$ $we\ get\ (a, b) = (2,1)$

$$Also,\quad w = (c + d) = 3 \quad \& \quad cd = -2ab = -2 * 2 = -4$$

$$Since\ (c + d) = 3\ \&\ cd = -4\ we\ get\ (c, d) = (4. -1)$$

Hence, $(a, b, c, d) = (2,1,4, -1)$

In eqn (9), for, $k = -23/5$ we get, $(r, s) = (\frac{1364}{5}, \frac{44}{5})$

Hence from eqn. (6) we get:

$$r(a^5+b^5)=s(c^5+d^5)$$

$$\frac{1364}{5}(a^5+b^5)=\frac{44}{5}(c^5+d^5) \quad \text{or}$$

$$31(2^5+1^5)=(1)(4^5+1^5)$$

**Conclusion:**

From the above two cases we note that the variables (a,b,c,d) are related by the eqn. $\frac{ab}{cd}=\frac{x}{y}$ . Case no. 1 & Case no. 2 gives us (x,y)=(3,2)=(1,-2) . Hence there could be a pattern for (x,y) in the two cases (1) & (2). Also (p,q) &(r,s) are quadratic polynomials & might have a relation for the two cases (1) &(2). It is an open problem, if a relation can be determined, which could help us parameterize equation (1) at the top.

**Note**: My thanks to mathematician Seiji Tomita for his help & suggestions .

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References:

1) Equation, $pa^n+qb^n=pc^n+qd^n$ for n=5, see web pages, Oliver Couto, celebrating-mathematics.com, published papers link.
2) Equation, $m(a,b,c,d)^5=n(e,f,g,h)^5$ , see webpages, Seiji Tomita, maroon.dti.ne.jp & click on section for fifth powers

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