Mixed algebras and the Lorentz group *O*(3,3)

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Abstract: The Lie algebra associated with the Lorentz group *O*(3,3) is investigated. Six classes of algebras are defined. It is found that algebras in the d, s, and b classes are related to algebras in the u, c, and t classes by *SU*(2) × *U*(1) symmetry plus a rotation.

Keywords: Lie algebra; *SO*(3,3); internal symmetry; Lorentz group

1. Introduction

The Lorentz group *O*(3,3) can be associated with a six-dimensional mathematical space containing three space dimensions and three time dimensions [1]. The corresponding Lie algebra is *SO*(3,3) in which the symmetry of time and the symmetry of space are isomorphic.

This article investigates some aspects of symmetry associated with the *O*(3,3) mathematical space. We consider six *SU*(2) × *SU*(2) subalgebras and combine these in three-algebra combinations resulting in six classes of combinations, labelled d, s, b, u, c, and t. A specific type of algebra is defined, a mixed algebra, and symmetry relationships between these algebras are investigated. It is found that mixed algebras in the d, s, and b classes are related to mixed algebras in the u, c, and t classes by *SU*(2) ×*U*(1) symmetry plus a rotation . Mixed algebras in the d, s, and b classes are not related to each other by *SU*(2) ×*U*(1) symmetry, which is also true of mixed algebras in the u, c, and t classes.

In section 3, the three-algebra classes are constructed. In section 4, the symmetry relationships between algebras in these classes are investigated.

2. Materials and methods

This article adopts the convention that roman indices can assume any value from 1 to 3.

The article uses the notation in reference [2] for the generators of the Lie algebra *SO*(3,3):

J_j : three space rotation generators (1)
T_k : three time rotation generators
K_{kj} : nine boost generators

The related commutation relations for complexified *SO*(3,3) are then [2]:

$$\begin{aligned}
&[T_j, T_k] = i\,\epsilon_{jkm}\,T_m && [J_j, J_k] = i\,\epsilon_{jkm}\,J_m && [T_j, J_k] = 0 \\
&[T_j, iK_{kn}] = i\,\epsilon_{jkm}\,iK_{mn} && [J_j, iK_{nk}] = i\,\epsilon_{jkm}\,iK_{nm} && \\
&[iK_{jn}, iK_{kn}] = i\,\epsilon_{jkm}\,T_m && [iK_{nj}, iK_{nk}] = i\,\epsilon_{jkm}\,J_m &&
\end{aligned} \qquad (2)$$

The article defines an *SO*(4) Lie algebra using the following notation [3]:

$$
\begin{aligned}
[a_j, a_k] &= i\, \epsilon_{jkm}\, a_m \\
[b_j, b_k] &= i\, \epsilon_{jkm}\, a_m \\
[a_j, b_k] &= i\, \epsilon_{jkm}\, b_m
\end{aligned} \tag{3}
$$

It also defines the direct product *SO*(3) × *SO*(2) using the notation [3]:

$$
\begin{aligned}
[w_j, w_k] &= i\, \epsilon_{jkm}\, w_m \\
[w_0, w_k] &= 0
\end{aligned} \tag{4}
$$

We note that *SU*(2) and *SO*(3) have the same Lie algebra, and that *U*(1) and *SO*(2) are isomorphic [4].

Reference [2] also defines two families of *SO*(4) subalgebras:

(5)

label	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	$\mathbf{b_1}$	$\mathbf{b_2}$	$\mathbf{b_3}$
e_1	J_1	J_2	J_3	iK_{11}	iK_{12}	iK_{13}
e_2	J_1	J_2	J_3	iK_{21}	iK_{22}	iK_{23}
e_3	J_1	J_2	J_3	iK_{31}	iK_{32}	iK_{33}
m_1	T_1	T_2	T_3	iK_{11}	iK_{21}	iK_{31}
m_2	T_1	T_2	T_3	iK_{12}	iK_{22}	iK_{32}
m_3	T_1	T_2	T_3	iK_{13}	iK_{23}	iK_{33}

In the *SO*(4) = *SU*(2) × *SU*(2) basis these become:

(6)

label	$\frac{1}{2}(\mathbf{a_1+b_1})$	$\frac{1}{2}(\mathbf{a_2+b_2})$	$\frac{1}{2}(\mathbf{a_3+b_3})$	$\frac{1}{2}(\mathbf{a_1-b_1})$	$\frac{1}{2}(\mathbf{a_2-b_2})$	$\frac{1}{2}(\mathbf{a_3-b_3})$
e_1	$\frac{1}{2}(J_1+iK_{11})$	$\frac{1}{2}(J_2+iK_{12})$	$\frac{1}{2}(J_3+iK_{13})$	$\frac{1}{2}(J_1-iK_{11})$	$\frac{1}{2}(J_2-iK_{12})$	$\frac{1}{2}(J_3-iK_{13})$
e_2	$\frac{1}{2}(J_1+iK_{21})$	$\frac{1}{2}(J_2+iK_{22})$	$\frac{1}{2}(J_3+iK_{23})$	$\frac{1}{2}(J_1-iK_{21})$	$\frac{1}{2}(J_2-iK_{22})$	$\frac{1}{2}(J_3-iK_{23})$
e_3	$\frac{1}{2}(J_1+iK_{31})$	$\frac{1}{2}(J_2+iK_{32})$	$\frac{1}{2}(J_3+iK_{33})$	$\frac{1}{2}(J_1-iK_{31})$	$\frac{1}{2}(J_2-iK_{32})$	$\frac{1}{2}(J_3-iK_{33})$
m_1	$\frac{1}{2}(T_1+iK_{11})$	$\frac{1}{2}(T_2+iK_{21})$	$\frac{1}{2}(T_3+iK_{31})$	$\frac{1}{2}(T_1-iK_{11})$	$\frac{1}{2}(T_2-iK_{21})$	$\frac{1}{2}(T_3-iK_{31})$
m_2	$\frac{1}{2}(T_1+iK_{12})$	$\frac{1}{2}(T_2+iK_{22})$	$\frac{1}{2}(T_3+iK_{32})$	$\frac{1}{2}(T_1-iK_{12})$	$\frac{1}{2}(T_2-iK_{22})$	$\frac{1}{2}(T_3-iK_{32})$
m_3	$\frac{1}{2}(T_1+iK_{13})$	$\frac{1}{2}(T_2+iK_{23})$	$\frac{1}{2}(T_3+iK_{33})$	$\frac{1}{2}(T_1-iK_{13})$	$\frac{1}{2}(T_2-iK_{23})$	$\frac{1}{2}(T_3-iK_{33})$

We define a *chiral-pair* of algebra components as the sum and difference of a rotation generator and a boost generator. For example, the components $\frac{1}{2}(J_1+iK_{11})$ and $\frac{1}{2}(J_1-iK_{11})$ are defined to be a chiral-pair and may be written together as $\frac{1}{2}(J_1 \pm iK_{11})$.

Finally, we define a *mixed algebra* as a six component algebra containing three chiral-pairs, where each chiral-pair is associated with a different *SU*(2) × *SU*(2) subalgebra. For example, in the following array of components, the components in each column form an *SU*(2) × *SU*(2) algebra and the components in each row form a mixed algebra:

e_1	m_1	m_2
$\frac{1}{2}(J_1 \pm iK_{11})$	$\frac{1}{2}(T_1 \pm iK_{11})$	$\frac{1}{2}(T_1 \pm iK_{12})$
$\frac{1}{2}(J_2 \pm iK_{12})$	$\frac{1}{2}(T_2 \pm iK_{21})$	$\frac{1}{2}(T_2 \pm iK_{22})$
$\frac{1}{2}(J_3 \pm iK_{13})$	$\frac{1}{2}(T_3 \pm iK_{31})$	$\frac{1}{2}(T_3 \pm iK_{32})$

(7)

3. Classes and sub-classes of mixed algebras

If we consider the six $SU(2) \times SU(2)$ algebras of the e-family and m-family as a set, then there are twenty three-algebra combinations. The combinations $\{m_1, m_2, m_3\}$ and $\{e_1, e_2, e_3\}$ have been addressed previously [2]. The remaining eighteen combinations can be organized into classes and sub-classes:

class	sub-class	combination	class	sub-class	combination
d	d_1	$\{e_1, m_1, m_2\}$	u	u_1	$\{e_1, e_2, m_1\}$
	d_2	$\{e_1, m_1, m_3\}$		u_2	$\{e_1, e_2, m_2\}$
	d_3	$\{e_1, m_2, m_3\}$		u_3	$\{e_1, e_2, m_3\}$
s	s_1	$\{e_2, m_1, m_2\}$	c	c_1	$\{e_1, e_3, m_1\}$
	s_2	$\{e_2, m_1, m_3\}$		c_2	$\{e_1, e_3, m_2\}$
	s_3	$\{e_2, m_2, m_3\}$		c_3	$\{e_1, e_3, m_3\}$
b	b_1	$\{e_3, m_1, m_2\}$	t	t_1	$\{e_2, e_3, m_1\}$
	b_2	$\{e_3, m_1, m_3\}$		t_2	$\{e_2, e_3, m_2\}$
	b_3	$\{e_3, m_2, m_3\}$		t_3	$\{e_2, e_3, m_3\}$

(8)

Here, each combination of three $SU(2) \times SU(2)$ algebras, gives rise to a sub-class containing twenty-seven unique mixed algebras. For example, the mixed algebras in the d_1 sub-class are:

	d_1 mixed algebra		d_1 mixed algebra
1	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{12})\}$	**16**	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_2 \pm iK_{22})\}$
2	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_2 \pm iK_{22})\}$	**17**	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{32})\}$
3	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_3 \pm iK_{32})\}$	**18**	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_1 \pm iK_{12})\}$
4	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{32})\}$	**19**	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_1 \pm iK_{12})\}$
5	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_1 \pm iK_{12})\}$	**20**	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_2 \pm iK_{22})\}$
6	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_2 \pm iK_{22})\}$	**21**	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{32})\}$
7	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{22})\}$	**22**	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_3 \pm iK_{32})\}$
8	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_3 \pm iK_{32})\}$	**23**	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_1 \pm iK_{12})\}$
9	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_1 \pm iK_{12})\}$	**24**	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{22})\}$
10	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_1 \pm iK_{12})\}$	**25**	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_2 \pm iK_{22})\}$
11	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{22})\}$	**26**	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_3 \pm iK_{32})\}$
12	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_3 \pm iK_{32})\}$	**27**	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{12})\}$
13	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_3 \pm iK_{32})\}$		

(9)

14	{½(J_2±iK_{12}), ½(T_1±iK_{11}), ½(T_1±iK_{12})}			
15	{½(J_3±iK_{13}), ½(T_2±iK_{21}), ½(T_2±iK_{22})}			

These are all the possible six component algebras that contain one chiral-pair from e_1, one chiral-pair from m_1 and one chiral-pair from m_2.

4. *SU*(2) × *U*(1) symmetry

Nine *SO*(3) × *SO*(2) algebras of interest are:

(10)

w_1	w_2	w_3	w_0
J_1	iK_{12}	iK_{13}	T_1
J_1	iK_{22}	iK_{23}	T_2
J_1	iK_{32}	iK_{33}	T_3
J_2	iK_{13}	iK_{11}	T_1
J_2	iK_{23}	iK_{21}	T_2
J_2	iK_{33}	iK_{31}	T_3
J_3	iK_{11}	iK_{12}	T_1
J_3	iK_{21}	iK_{22}	T_2
J_3	iK_{31}	iK_{32}	T_3

If we consider the first algebra

(11)

w_1	w_2	w_3	w_0
J_1	iK_{12}	iK_{13}	T_1

then with a change of basis we may obtain the algebra

(12)

½(w_1 ± w_2)	½(w_0 ± w_3)
½(J_1 ± iK_{12})	½(T_1 ± iK_{13})

We note that the generators J_1 and iK_{12} are associated with the *SU*(2) × *SU*(2) algebra e_1. Rotating ½(w_1 ± w_2) in the associated e_1 vector space gives:

(13)

½(w_1 ± w_2)′	½(w_0 ± w_3)
½(J_3 ± iK_{13})	½(T_1 ± iK_{13})

We conclude that the two chiral-pairs ½(J_3 ± iK_{13}) and ½(T_1 ± iK_{13}) are related by *SU*(2) × *U*(1) symmetry plus a rotation.

Applying this procedure to all nine *SO*(3) × *SO*(2) algebras of interest, gives the following *SU*(2) × *U*(1) partnered chiral-pairs:

$$\tfrac{1}{2}(J_1 \pm iK_{11}) \leftrightarrow \tfrac{1}{2}(T_1 \pm iK_{11}) \qquad \tfrac{1}{2}(J_1 \pm iK_{21}) \leftrightarrow \tfrac{1}{2}(T_2 \pm iK_{21}) \qquad \tfrac{1}{2}(J_1 \pm iK_{31}) \leftrightarrow \tfrac{1}{2}(T_3 \pm iK_{31}) \tag{14}$$
$$\tfrac{1}{2}(J_2 \pm iK_{12}) \leftrightarrow \tfrac{1}{2}(T_1 \pm iK_{12}) \qquad \tfrac{1}{2}(J_2 \pm iK_{22}) \leftrightarrow \tfrac{1}{2}(T_2 \pm iK_{22}) \qquad \tfrac{1}{2}(J_2 \pm iK_{32}) \leftrightarrow \tfrac{1}{2}(T_3 \pm iK_{32})$$
$$\tfrac{1}{2}(J_3 \pm iK_{13}) \leftrightarrow \tfrac{1}{2}(T_1 \pm iK_{13}) \qquad \tfrac{1}{2}(J_3 \pm iK_{23}) \leftrightarrow \tfrac{1}{2}(T_2 \pm iK_{23}) \qquad \tfrac{1}{2}(J_3 \pm iK_{33}) \leftrightarrow \tfrac{1}{2}(T_3 \pm iK_{33})$$

Using these relationships, we may find the $SU(2) \times U(1)$ partners of a sub-class of mixed algebras. For the d_1 family this gives:

	d_1 mixed algebra		***SU*(2) x *U*(1) partner algebra**	**sub-class**
1	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{12})\}$	
2	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{22})\}$	
3	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{32})\}$	
4	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{32})\}$	c_1
5	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{12})\}$	u_2
6	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{22})\}$	t_3
7	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{22})\}$	u_1
8	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{32})\}$	t_2
9	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{12})\}$	c_3
10	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{12})\}$	c_1
11	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{22})\}$	u_2
12	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{32})\}$	t_3
13	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{32})\}$	
14	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{12})\}$	
15	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{22})\}$	
16	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{22})\}$	t_1
17	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{32})\}$	c_2
18	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{12})\}$	u_3
19	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{12})\}$	u_1
20	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{22})\}$	t_2
21	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{32})\}$	c_3
22	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{32})\}$	t_1
23	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{12})\}$	c_2
24	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{22})\}$	u_3
25	$\{\frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(T_2 \pm iK_{21}), \frac{1}{2}(T_2 \pm iK_{22})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(J_1 \pm iK_{21}), \frac{1}{2}(J_2 \pm iK_{22})\}$	
26	$\{\frac{1}{2}(J_2 \pm iK_{12}), \frac{1}{2}(T_3 \pm iK_{31}), \frac{1}{2}(T_3 \pm iK_{32})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{12}), \frac{1}{2}(J_1 \pm iK_{31}), \frac{1}{2}(J_2 \pm iK_{32})\}$	
27	$\{\frac{1}{2}(J_3 \pm iK_{13}), \frac{1}{2}(T_1 \pm iK_{11}), \frac{1}{2}(T_1 \pm iK_{12})\}$	$\leftrightarrow$	$\{\frac{1}{2}(T_1 \pm iK_{13}), \frac{1}{2}(J_1 \pm iK_{11}), \frac{1}{2}(J_2 \pm iK_{12})\}$	

(15)

We conclude that mixed algebras in the d_1 sub-class are related by $SU(2) \times U(1)$ symmetry plus a rotation, to mixed algebras in the u_1, u_2, u_3, c_1, c_2, c_3, t_1, t_2, and t_3 sub-classes. Investigation of the other sub-classes finds the following relationships:

class	sub-class	$SU(2) \times U(1)$ partners
d	d_1	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
	d_2	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
	d_3	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
s	s_1	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
	s_2	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
	s_3	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
b	b_1	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
	b_2	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$
	b_3	$u_1,u_2,u_3,c_1,c_2,c_3,t_1,t_2,t_3$

class	sub-class	$SU(2) \times U(1)$ partners
u	u_1	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
	u_2	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
	u_3	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
c	c_1	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
	c_2	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
	c_3	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
t	t_1	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
	t_2	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$
	t_3	$d_1,d_2,d_3,s_1,s_2,s_3,b_1,b_2,b_3$

(16)

On the level of classes this gives:

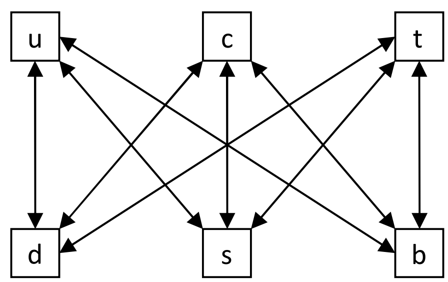

Figure 1 : *SU*(2) x *U*(1) class relationships

Conclusion

The article has investigated some aspects of symmetry in the mathematical space $O(3,3)$. In particular we find:

1) The d, s and b classes of mixed algebras are related to the u, c and t classes by $SU(2) \times U(1)$ symmetry plus a rotation.
2) The d, s and b classes of mixed algebras are not related to each other by $SU(2) \times U(1)$ symmetry. This is also true of the u, c and t classes.

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Appendix:

Complexified *SO*(3,3) commutation table:

	T_1	T_2	T_3	J_1	J_2	J_3	iK_{11}	iK_{12}	iK_{13}	iK_{21}	iK_{22}	iK_{23}	iK_{31}	iK_{32}	iK_{33}
T_1		iT_3	$-iT_2$							i^2K_{31}	i^2K_{32}	i^2K_{33}	$-i^2K_{21}$	$-i^2K_{22}$	$-i^2K_{23}$
T_2	$-iT_3$		iT_1				$-i^2K_{31}$	$-i^2K_{32}$	$-i^2K_{33}$				i^2K_{11}	i^2K_{12}	i^2K_{13}
T_3	iT_2	$-iT_1$					i^2K_{21}	i^2K_{22}	i^2K_{23}	$-i^2K_{11}$	$-i^2K_{12}$	$-i^2K_{13}$			
J_1					iJ_3	$-iJ_2$		i^2K_{13}	$-i^2K_{12}$		i^2K_{23}	$-i^2K_{22}$		i^2K_{33}	$-i^2K_{32}$
J_2				$-iJ_3$		iJ_1	$-i^2K_{13}$		i^2K_{11}	$-i^2K_{23}$		i^2K_{21}	$-i^2K_{33}$		i^2K_{31}
J_3				iJ_2	$-iJ_1$		i^2K_{12}	$-i^2K_{11}$		i^2K_{22}	$-i^2K_{21}$		i^2K_{32}	$-i^2K_{31}$	
iK_{11}		i^2K_{31}	$-i^2K_{21}$		i^2K_{13}	$-i^2K_{12}$		iJ_3	$-iJ_2$	iT_3			$-iT_2$		
iK_{12}		i^2K_{32}	$-i^2K_{22}$	$-i^2K_{13}$		i^2K_{11}	$-iJ_3$		iJ_1		iT_3			$-iT_2$	
iK_{13}		i^2K_{33}	$-i^2K_{23}$	i^2K_{12}	$-i^2K_{11}$		iJ_2	$-iJ_1$				iT_3			$-iT_2$
iK_{21}	$-i^2K_{31}$		i^2K_{11}		i^2K_{23}	$-i^2K_{22}$	$-iT_3$				iJ_3	$-iJ_2$	iT_1		
iK_{22}	$-i^2K_{32}$		i^2K_{12}	$-i^2K_{23}$		i^2K_{21}		$-iT_3$		$-iJ_3$		iJ_1		iT_1	
iK_{23}	$-i^2K_{33}$		i^2K_{13}	i^2K_{22}	$-i^2K_{21}$				$-iT_3$	iJ_2	$-iJ_1$				iT_1
iK_{31}	i^2K_{21}	$-i^2K_{11}$			i^2K_{33}	$-i^2K_{32}$	iT_2			$-iT_1$				iJ_3	$-iJ_2$
iK_{32}	i^2K_{22}	$-i^2K_{12}$		$-i^2K_{33}$		i^2K_{31}		iT_2			$-iT_1$		$-iJ_3$		iJ_1
iK_{33}	i^2K_{23}	$-i^2K_{13}$		i^2K_{32}	$-i^2K_{31}$				iT_2			$-iT_1$	iJ_2	$-iJ_1$	

SO(3,3) commutation table:

	T_1	T_2	T_3	J_1	J_2	J_3	K_{11}	K_{12}	K_{13}	K_{21}	K_{22}	K_{23}	K_{31}	K_{32}	K_{33}
T_1		iT_3	$-iT_2$							iK_{31}	iK_{32}	iK_{33}	$-iK_{21}$	$-iK_{22}$	$-iK_{23}$
T_2	$-iT_3$		iT_1				$-iK_{31}$	$-iK_{32}$	$-iK_{33}$				iK_{11}	iK_{12}	iK_{13}
T_3	iT_2	$-iT_1$					iK_{21}	iK_{22}	iK_{23}	$-iK_{11}$	$-iK_{12}$	$-iK_{13}$			
J_1					iJ_3	$-iJ_2$		iK_{13}	$-iK_{12}$		iK_{23}	$-iK_{22}$		iK_{33}	$-iK_{32}$
J_2				$-iJ_3$		iJ_1	$-iK_{13}$		iK_{11}	$-iK_{23}$		iK_{21}	$-iK_{33}$		iK_{31}
J_3				iJ_2	$-iJ_1$		iK_{12}	$-iK_{11}$		iK_{22}	$-iK_{21}$		iK_{32}	$-iK_{31}$	
K_{11}		iK_{31}	$-iK_{21}$		iK_{13}	$-iK_{12}$		$-iJ_3$	iJ_2	$-iT_3$			iT_2		
K_{12}		iK_{32}	$-iK_{22}$	$-iK_{13}$		iK_{11}	iJ_3		$-iJ_1$		$-iT_3$			iT_2	
K_{13}		iK_{33}	$-iK_{23}$	iK_{12}	$-iK_{11}$		$-iJ_2$	iJ_1				$-iT_3$			iT_2
K_{21}	$-iK_{31}$		iK_{11}		iK_{23}	$-iK_{22}$	iT_3				$-iJ_3$	iJ_2	$-iT_1$		
K_{22}	$-iK_{32}$		iK_{12}	$-iK_{23}$		iK_{21}		iT_3		iJ_3		$-iJ_1$		$-iT_1$	
K_{23}	$-iK_{33}$		iK_{13}	iK_{22}	$-iK_{21}$				iT_3	$-iJ_2$	iJ_1				$-iT_1$
K_{31}	iK_{21}	$-iK_{11}$			iK_{33}	$-iK_{32}$	$-iT_2$			iT_1				$-iJ_3$	iJ_2
K_{32}	iK_{22}	$-iK_{12}$		$-iK_{33}$		iK_{31}		$-iT_2$			iT_1		iJ_3		$-iJ_1$
K_{33}	iK_{23}	$-iK_{13}$		iK_{32}	$-iK_{31}$				$-iT_2$			iT_1	$-iJ_2$	iJ_1	