Evaluating the Alignment of 106 Linearly Polarized Optical QSOs

Richard Shurtleff *

Abstract

A sample of partially linearly polarized optical quasars (QSOs) is analyzed by the Hub Test. The data originates in an online catalog of 355 such QSOs, collected and published by others. Without their efforts this article would not be possible. The 106 QSOs populate a region with a radius of about 40° centered on a point in the sky near the South Galactic Pole. We find the polarization directions to be extremely well aligned. Besides supplying convincing evidence that the alignment is not due to chance, additional quantities are calculated that describe the collective behavior of the polarization directions. The alignment function mapped onto the Celestial Sphere provides a satisfying visual representation. This article is a Mathematica notebook.

Keywords: Polarization; Alignment; Computer Program; Uncertainties; Quasars; Hub Test; Large Scale Structure

*Department of Sciences, Wentworth Institute of Technology, 550 Huntington Avenue, Boston, MA, USA, 02115, orcid.org/0000-0001-5920-759X, e-mail addresses: shurtleffr@wit.edu, momentummatrix@yahoo.com

0. Preface

UPDATES: Notes plus errata and other changes to the online pdf version may appear here.

(1) The pdf version of this notebook differs from the ready-to-run notebook which is available via the link in Ref. 1.
   (1a) Dealing with nR = 10,000 random runs and 10,000 uncertainty runs presents practical logistical problems.
   (1b) The pdf version has nR = 10,000 random runs and 10,000 uncertainty runs. The ready-to-run version has nR = 2000 runs because 2000 runs takes much less computer time than 10,000 runs.
   (1b) You can select nR to be any number of runs in Sec. 3 Settings. Setting nR = 200 runs gives results that can be compared to the 10,000 run values. And nR = 200 processes quickly.
   (1c) The pdf version has the random run generating cell and the uncertainty run generating cell inactive, as comments. The needed data has been saved in .dat files. The random runs were generated separately from the uncertainty runs and saved. Separating and saving avoided overwhelming my computer.

(2) The filename for the associated notebook: “20210205IntermediateKitFor193BestOpticalQSORegions2.nb”. This file is available for download via the link in Ref. 1.

(3) The numerical values quoted in the Concluding Remarks in Sec. 8 are associated with the random runs and uncertainty runs with the pdf version. Other sets of random runs and uncertainty runs should alter those numerical values. They are unlikely to change much.

In[1]:= Print["The date and time that this statement was evaluated: ", Now]

The date and time that this statement was evaluated: Wed 17 Feb 2021 12:45:26 GMT-5.

While this article is a Mathematica notebook, it is difficult, perhaps impossible, to run from the pdf version. A link to a ready-to-run version, Ref. 1, is provided for convenience.

The Hub Test is explained in some detail in Ref. 2, “Indirect polarization alignment with points on the sky, the Hub Test”. A template for performing calculations similar to those in this notebook, but with other data, can be found online, Ref. 3. These notebooks were created using Wolfram Mathematica, Version Number: 12.1, Ref. 4.

The formulas for creating Aitoff plots were found on Wikipedia, Ref. 5.
CONTENTS

0. Preface
1. Introduction
   1a. The Hub Test
   1b. Statistics
2. Preliminary
3. Input and Settings
   3a. Selection Process
   3b. Settings
   3c. Inputs
4. Grid
5. Significance
   5a. Random Run Generator
   5b. Analysis of the Random Run Results
   5c. Probability Distributions and Significance Formulas
6. Results and map using the Best Values $\psi_n$ of the Polarization Directions
   6a. Results Using the Best Values of $\psi_n$
   6b. Plot of the Alignment Angle Function $\eta(H)$ Using the Best Values $\psi_n$
7. Uncertainty Runs
   7a. Uncertainty Run Generator
   7b. Uncertainty in the Best Alignment Angle $\eta_{min}$
   7c. Uncertainty in the Largest Avoidance Angle $\eta_{max}$
   7d. Location ($\alpha, \delta$) of the Alignment Hubs $H_{min}$
   7e. Location of the Avoidance Hubs $H_{max}$
   7f. Map of the Hubs for the Uncertainty Runs
8. Concluding Remarks

References

1. Introduction

Given a collection of astronomical sources with linearly polarized electromagnetic emissions, one can evaluate the mutual alignment of the polarization directions. In this paper, we apply the Hub Test to judge the alignment of a set of partially polarized quasars (QSOs). Since QSOs are distant extragalactic sources, any alignment of their polarizations is remarkable, possibly providing evidence of large-scale structures or perhaps reflecting the action of the intervening medium through which the QSOs are viewed. The alignment of the sample analyzed here is matched by only about one in hundreds of thousands of randomly polarized samples.

In Ref. 6, the region called “A3” contains 22 of the 106 objects in the sample analyzed in this notebook. The original article, Ref. 6, which includes the QSO catalog as an Appendix, provides far ranging discussions of alignment mechanisms and other effects. However intriguing, interpretation of the results are beyond the scope of this notebook. For example contamination by interstellar polarization in our Galaxy is considered and dismissed as unlikely to be able to explain the alignments uncovered by their tests. In Ref. 8, alignments detected from the catalog data are shown to imply that “quasar spin axes are likely parallel to their host large-scale structures.” These and similar topics show how the problem of alignment is interesting and important in astronomy and
astrophysics. However, they are not considered in this notebook. One hopes that the Hub Test can help with such investigations.

One motivation for constructing this notebook is to present an application of the Hub Test. The tests in Ref. 6 differ from the Hub Test because their tests compare polarization directions directly, while the Hub Test is indirect. The Hub Test infers alignment from determining the alignment of the polarization directions with points on the sky. Essentially the test looks for the convergence of a number of great circles.

1a. The Hub Test

The Hub Test, Ref. 2, answers the question of alignment indirectly by looking at the great circles on the Celestial sphere determined by the polarization directions. At each source one great circle has a tangent parallel to the polarization direction. The collection of these great circles appear over the sphere in regions of various densities, regions of convergence and divergence.

Start with a single source depicted in Fig. 1.

![Figure 1: The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source S. The linear polarization direction \( \hat{v}_\psi \) lies in the tangent plane and determines the purple great circle on the sphere. A point \( H \) on the sphere and the location \( S \) of the source determine a second great circle, the blue circle drawn on the sphere at the left. Clearly, \( H \) and \( S \) must be distinct points on the sphere. The angle \( \eta \), with \( 0^\circ \leq \eta \leq 90^\circ \), measures the alignment of the polarization direction with the point \( H \). Perfect alignment occurs when \( \eta = 0^\circ \) and the two great circles overlap. Perpendicular great circles, \( \eta = 90^\circ \), indicates maximum avoidance of the polarization direction and the point on the sphere.

The basic concept includes “avoidance”, as well as alignment. Avoidance is high when the two directions \( \hat{v}_H \) and \( \hat{v}_H \) differ by a large angle, \( \eta \to 90^\circ \). Perpendicular great circles at \( S, \eta = 90^\circ \), would indicate the maximum avoidance of the polarization direction and the point on the sphere.

With many sources \( S_i, i = 1, ..., N \), there are \( N \) alignment angles \( \eta_{iH} \) for the point \( H \). To quantify the alignment of the \( N \) sources with the point \( H \), calculate the arithmetic average alignment angle at \( H \),

\[
\eta(H) = \frac{1}{N} \sum_{i=1}^{N} \eta_{iH}.
\]  

(1)

The alignment angle \( \eta(H) \) is a function of position \( H \) on the sphere. Since great circles that contain the point \( H \) also contain the diametrically opposite point \(-H\), the function \( \eta(H) \) is symmetric across diameters. The function \( \eta(H) \) is also a measure of convergence and divergence of the great circles determined by the polarization directions. Where the alignment function \( \eta(H) \) is small, the circles converge, where \( \eta(H) \) is large, the circles diverge.

The polarization directions are best aligned with the point \( H_{min} \) where the alignment angle is a minimum \( \eta_{min} \). The polarization directions most avoid the point \( H_{max} \) where the function \( \eta(H) \) takes its maximum value \( \eta_{max} \).

The location of their most extreme convergence is a “hub”, \( H_{min} \), called the “alignment hub”. The most extreme divergence is
another hub, $H_{\mathrm{max}}$, the “avoidance hub”. Alignment and avoidance are symmetrical concepts with the Hub Test.

The Hub Test of alignment is based on the idea that the polarization directions are well-aligned with each other when they are well-aligned with some point of convergence, the hub $H_{\mathrm{min}}$.

1b. Statistics

To judge the significance of the alignment or avoidance of the observed polarization directions for a given sample of $N$ sources, one estimates how likely it would be for a sample with random polarization directions to yield the observed minimum alignment angle $\eta_{\mathrm{min}}$ and the maximum value $\eta_{\mathrm{max}}$.

By running the calculations with the same locations for the sources, but supplying those sources with random polarization directions, one can build up the statistics. The distributions of the results provide estimates of the likelihood that random data would produce certain results.

An alternative method is available. Generic formulas are presented in the Statistics Section of an earlier notebook, Ref. 3. Those formulas rely on the number of sources $N$ and a rough size of their reach on the sphere. The results are less accurate than the formulas developed here that are obtained with random runs specialized to the sources analyzed in this notebook. The specialized statistics used here are more accurate than those obtained by applying the generic formulas in Ref. 3.

2. Preliminary

Consider a sphere in 3 dimensional Euclidean space. See Fig. 1 in the Introduction. The sphere is called the “Celestial sphere” or simply the “sphere” or sometimes “the sky”. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates $(x, y, z)$. The direction of the positive $z$-axis is associated with “North”. Right ascension, RA or $\alpha$, and declination, dec or $\delta$, are measured as usual with the direction of the positive $x$-axis along $(\alpha, \delta) = (0^\circ, 0^\circ)$. Declination $\delta = 90^\circ$ indicates the North pole, the direction from the origin $(0,0,0)$ to $(0,0,1)$.

From a point-of-view located outside the sphere, as in the left-hand sketch in Fig. 1, one pictures a source $S$ plotted on the sphere and, in the 2D tangent plane at $S$, local North is upward and local East is to the right. See the right-hand sketch in Fig. 1. A “position angle” at the point $S$ on the sphere, such as the angle $\psi$ in Fig. 1, is measured in the 2D plane tangent to the sphere at $S$. The position angle $\psi$ is measured clockwise from local North with East to the right.

The rest of this section contains some useful formulas that are helpful since we often mix spherical and Cartesian coordinates.

Definitions:

- $(\alpha, \delta)$ Right Ascension RA and declination dec of a point on the sphere. Sometimes we use radians, sometimes degrees.
- $e_r(\alpha, \delta)$ radial unit vector in a Cartesian coordinate system from the Origin to the point on the sphere with $(\alpha, \delta)$, with $\alpha, \delta$ in radians
- $e_N(\alpha, \delta)$ unit vector along local North at the point $(\alpha, \delta)$ on the sphere, with $\alpha, \delta$ in radians
- $e_E(\alpha, \delta)$ unit vector along local East at the point $(\alpha, \delta)$ on the sphere, with $\alpha, \delta$ in radians
- $\alpha_{\mathrm{FROM}}(\hat{r})$ RA for the point on the sphere determined by radial unit vector $\hat{r}$, result in radians
- $\delta_{\mathrm{FROM}}(\hat{r})$ dec for the point on the sphere determined by radial unit vector $\hat{r}$, result in radians
3. Input and Settings

3a. Selection Process:

The selection of the 106 optical polarized QSOs proceeds as follows. The relevant calculations appear in a private notebook, “20210212HubTestOpticalQSOs24Deg.nb”.

The selection process employs the same grid developed below, a grid of 10,518 equally spaced points on the sphere. Each of the 10,518 grid points serves as the center of 10,518 regions each with a radius of 24°. The QSOs in the catalog make 10,518 samples with populations ranging from 0 to 73 QSOs. Seven sources are required for the generic statistics formulas to be sufficiently accurate, see Ref. 3. Of the 10,518 regions, just 4224 regions were populated with 7 or more QSOs.

Applying the generic statistics formulas in Ref. 3, just 418 of the 4224 regions showed very significant alignments, meaning that 1% or fewer randomly polarized 24° radius regions with the same number of sources would be better aligned. We chose to focus on the 200 most significant regions, an arbitrary choice. From Fig. 2 below it is clear that all but 7 of the 200 most significant regions are collected together, with a large gap separating the collection of 193 region centers from the group of 7. In this article, we make a sample of the QSOs in the 193 (= 200 - 7) regions in the cloud of regions depicted in Fig. 2. The 193 regions are populated with the 106 QSO sources analyzed in this notebook, completing the selection process.
Figure 2: The centers of the 200 most significantly aligned regions. The dots indicate the centers of 24° radius regions; no sources are plotted. The significances of alignment range from 0.000016 to 0.0015, i.e. 1 in 63,000 to 1 in 670 random runs. By significance, we mean the fraction of randomly directed polarized samples that were better aligned. The most significantly aligned region has 54 QSOs and is centered at (RA, dec) = (-12°, 8°). Note the island of 7 regions at (RA, dec) = (-140°, -30°). These 7 regions are thereby distinguished from the 193 others and dropped from consideration. (Figure and numerical values copied from "20210121HubTestOpticalQSOs24Deg.nb".)

Figure 3: The points $H_{\text{min}}$ where the alignment function $\eta(H)$ has the minimum value $\eta_{\text{min}}$ for the 200 most significantly aligned regions. The most significant region has its hub $H_{\text{min}}$ located at (RA, dec) = (123°, -28°) in the band of $H_{\text{min}}$ hubs stretching from RA = 100° to RA = 180°. (Figure and numerical results copied from “20210121HubTestOpticalQSOs24Deg.nb”.)

Definitions:

- **gridSpacing** separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles.
- There is no bunching at the poles.
- **d$\eta$ContourPlot** separation of successive contour lines on the map in Sec. 6b, in degrees
- **dataDirectory** folder on the computer where data files and other relevant files are stored
- **nR** number of runs generated from random polarization directions and also the number of runs generated with polarization directions consistent with the uncertainty in the observed directions
- **dataFromTheCatalog** data for the QSOs in the sample, copied from the original catalog
- **nSrc** number of sources in the sample
- **nameSrc** Object coordinate name (B1950)
- **rSrc** radial unit vectors in Cartesian coordinates from origin to sources
3b. Settings

In[14]:= gridSpacing = 2. (*, in degrees. This is a setting.*)

Print["The grid points are separated by ",
gridSpacing, "° arcs along latitude and longitude."]

The grid points are separated by 2.° arcs along latitude and longitude.

In[16]:= dηContourPlot = 4; (*, in degrees. This is a setting.*)

In[17]:= dataDirectory = "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE5OJ\\SendXXX_CJP_CEJPetc\\SendViXra\\20200715AlignmentMethod\\20200715AlignmentMMAnotebooks\\StarterKit\\20210201Optical200MostSigRegionsCombined"; (*This is a setting.*)

In[18]:= nR = 10000;
(*number of runs with random \(\psi\) for statistics and with various \(\psi\) allowed by uncertainty for determining the uncertainty in the results. The number of runs can be changed by hand at the relevant "For" statements below. *)

3c. Inputs 

Note: The angles \(\alpha\), \(\delta\), \(\psi\), \(\sigma\psi\) are expected to be input in radians.

For the 106 QSOs in the 193 of the 200 most significant regions, the relevant data from the catalog, Refs. 6 and 7, is entered as input below in this section. For details on the data, consult the “ReadMe” and other files included with the catalog online; see Ref. 7.

The description of the entries for each object in the catalog:


The catalog data is copied below as the quantity “dataFromTheCatalog”. The 106 objects analyzed in this notebook are the \(i\)th entries in the 355 object catalog, where \(i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 44, 45, 47, 50, 53, 56, 58, 62, 274, 277, 279, 280, 281, 282, 284, 286, 288, 299, 291, 292, 293, 294, 297, 300, 301, 302, 303, 304, 305, 306, 308, 310, 311, 314, 315, 317, 318, 319, 320, 321, 322, 323, 324, 325, 326, 327, 328, 329, 330, 331, 334, 335, 336, 337, 338, 339, 340, 341, 342, 344, 345, 346, 347, 348, 349, 350, 351, 353, 354, 355\}.

(*)Input*)

dataFromTheCatalog = {{"B+0003-066", 0.347°, 3.5°, 1.6°, 160, 12, 4}, {{"B+0003+158", 0.45°, 0.62°, 0.16°, 114, 7, 1}, {"B+0004+017", 1.71°, 1.29°, 0.28°, 122, 6, 8}, {"B+0010-002", 2.145°, 1.7°, 0.77°, 116, 13, 8}, {"B+0013-004", 2.084°, 1.03°, 0.33°, 115, 10, 0}, {"B+0017+154", 2.012°, 1.14°, 0.52°, 137, 13, 3}, . . .}
In[20]:= nSrc = Length[dataFromTheCatalog]; (*calculated from Input.*)
Print["There are ", nSrc, " sources."]

There are 106 sources.

Coordinates:

The coordinate name for each object contains the Right Ascension and Declination. For example “B*2320-035” has (RA,dec) = (α,δ) = (23hr 20min, -3.5°).

RA: The 23hr 20min is converted to radians: \(23 + \frac{20}{60} = 6.1087\) radians, which is close to \(2\pi\). We want \(\alpha\) to be between \(-\pi\) and \(+\pi\), i.e. \(-\pi \leq \alpha \leq +\pi\), so we subtract \(2\pi\) from the RA and get 6.1087 - 2\(\pi\) = -0.1745, which can be found in the values for \(\alpha_{\text{Src}}\) below.

Dec: The -3.5° is converted to radians: \((-3.5^\circ) \left(\frac{\pi}{180}\right)\) = -0.0611 radians, and that value can be found with \(\delta_{\text{Src}}\) below.

(*names of objects from the catalog. Determined by Input.*)
nameSrc = Table[dataFromTheCatalog[[ii, 1]], {ii, nSrc}];
nameSrc[[86]] (*The object discussed in regard to the above Coordinate calculation.*)

Out[23]= B*2320-035
(*The Right Ascensions of the sources $S_i$. Determined by Input.*)

$$a_{\text{Src}} = (0.0131, 0.0131, 0.0175, 0.0436, 0.0567, 0.0742, 0.0829, 0.0916, 0.1047, 0.1091, 0.1265, 0.1658, 0.2051, 0.2094, 0.2182, 0.2225, 0.24, 0.2574, 0.2618, 0.2749, 0.2836, 0.288, 0.3011, 0.3054, 0.336, 0.336, 0.3447, 0.3622, 0.3927, 0.4058, 0.4232, 0.4232, 0.4276, 0.4581, 0.4625, 0.4712, 0.4974, 0.5411, 0.5454, 0.5847, 0.6589, 0.6938, 0.829, 0.8814, 0.9861, -0.7636, -0.6938, -0.6632, -0.6589, -0.6501, -0.6458, -0.6152, -0.6065, -0.589, -0.5454, -0.5192, -0.5105, -0.4887, -0.4538, -0.4538, -0.4407, -0.4232, -0.4232, -0.4145, -0.4058, -0.3927, -0.3927, -0.3911, -0.3185, -0.3185, -0.3011, -0.3075, -0.3011, -0.2923, -0.288, -0.2836, -0.2618, -0.2574, -0.2531, -0.2269, -0.1876, -0.1745, -0.1222, -0.1178, -0.1091, -0.0873, -0.0829, -0.0785, -0.0698, -0.0654, -0.0567, -0.048, -0.0436, -0.0393, -0.0305, -0.0305, -0.0262, -0.0262, -0.0175, -0.0131, -0.0087);$$

(*The Declinations of the sources $S_i$. Determined by Input.*)

$$\delta_{\text{Src}} = (-0.1152, 0.2758, 0.0297, -0.0035, -0.007, 0.2688, 0.0192, -0.0384, 0.391, -0.0314, 0.0035, 0.4887, 0.4852, 0.5096, 0.2164, 0.5079, 0.274, 0.4555, 0.2269, 0.4485, 0.3752, 0.0227, -0.0244, 0.5184, 0.3718, 0.3438, 0.0716, 0.4485, 0.4224, 0.3613, -0.0314, -0.0175, -0.1693, 0.0733, 0.0297, 0.1571, 0.295, 0.5096, 0.0419, 0.1885, 0.4259, 0.0105, 0.3648, 0.3072, 0.2217, -0.1134, 0.0873, -0.1536, -0.1257, -0.0367, -0.0192, -0.1484, 0.0698, 0.1169, -0.2653, -0.3229, -0.3281, -0.3752, -0.3019, -0.0663, -0.1588, 0.3421, 0.3438, -0.0908, 0.3438, -0.096, -0.1536, 0.0436, 0.199, -0.4538, -0.2147, 0.2443, 0.0262, 0.1972, 0.2758, 0.4259, 0.0105, -0.2007, 0.0419, -0.4922, 0.4433, 0.1047, -0.4869, 0.171, -0.0105, -0.0611, -0.0297, -0.1763, -0.0471, -0.0628, -0.4102, 0.2094, 0.3211, -0.2915, 0.0035, -0.1833, -0.0175, 0.014, -0.2688, 0.4939, -0.014, -0.2042, 0.0035, -0.0105, -0.2251, 0.0384);$$

$r_{\text{Src}} = \text{Table}[\text{er}(a_{\text{Src}}[[i]], \delta_{\text{Src}}[[i]]), \{i, n_{\text{Src}}\}];$ (*calculated from Input.*)

$e_{\text{NSrc}} = \text{Table}[\text{en}(a_{\text{Src}}[[i]], \delta_{\text{Src}}[[i]]) ,\{i, n_{\text{Src}}\}];$ (*calculated from Input.*)

$e_{\text{ESrc}} = \text{Table}[\text{eE}(a_{\text{Src}}[[i]], \delta_{\text{Src}}[[i]]), \{i, n_{\text{Src}}\}];$ (*calculated from Input.*)

The polarization position angles (PPA) $\psi$ and their uncertainties $\sigma\psi$ are given in degrees in the catalog. They are converted to radians below.

(*The polarization position angles in radians for the EM radiation from the sources. Determined by Input.*)

$$\psi_n = \{2.7925, 1.9897, 2.1293, 2.0246, 2.0071, 2.3911, 0.4538, 2.9671, 1.5708, 1.9024, 2.7576, 1.7977, 0.8552, 1.7104, 0.1396, 2.0769, 0.2618, 2.0944, 1.9548, 1.9897, 2.0769, 2.4958, 1.3265, 1.0996, 1.7802, 2.234, 1.0297, 2.4435, 1.9199, 0.8552, 1.0647, 2.6878, 2.9322, 2.2864, 2.4609, 2.426, 1.1519, 2.042, 0.384, 2.1118, 1.7279, 2.9147, 2.5656, 2.0769, 1.2043, 2.5656, 1.1868, 2.9845, 0.7679, 1.6232, 1.9722, 2.7925, 1.9373, 2.4086, 0.1222, 0.5411, 0.8203, 2.5831, 2.426, 0.0175, 1.9824, 2.4086, 2.3213, 1.0123, 2.8274, 3.0194, 2.0769, 2.0595, 2.2864, 2.7227, 1.309, 1.4312, 0.8552, 2.5133, 1.9722, 2.5255, 2.2689, 0.0349, 1.9548, 2.4435, 2.8449, 0.1571, 1.8326, 2.8623, 1.5708, 1.6057, 2.7925, 1.9199, 2.2689, 2.1293, 2.2166, 1.5359, 1.2217, 2.3387, 1.85, 2.4958, 0.4712, 0.2269, 1.3265, 0.2793, 1.8326, 1.2915, 2.7576, 2.6354, 0.7854);
(The uncertainties in the polarization position angles in radians. Determined by Input. *)

\[ \sigma_\psi = \{0.2094^\circ, 0.1222^\circ, 0.1047^\circ, 0.2269^\circ, 0.1745^\circ, 0.2269^\circ, 0.1222^\circ, 0.2443^\circ, 0.2443^\circ, 0.2269^\circ, 0.1745^\circ, 0.0349^\circ, 0.0873^\circ, 0.0524^\circ, 0.2443^\circ, 0.2269^\circ, 0.1571^\circ, 0.192^\circ, 0.1047^\circ, 0.1407^\circ, 0.0873^\circ, 0.0175^\circ, 0.1222^\circ, 0.1571^\circ, 0.1222^\circ, 0.192^\circ, 0.2443^\circ, 0.0698^\circ, 0.0175^\circ, 0.2269^\circ, 0.0349^\circ, 0.0524^\circ, 0.2443^\circ, 0.0175^\circ, 0.0175^\circ, 0.1047^\circ, 0.1396^\circ, 0.1222^\circ, 0.0698^\circ, 0.2443^\circ, 0.1571^\circ, 0.1396^\circ, 0.2269^\circ, 0.1047^\circ, 0.1222^\circ, 0.0698^\circ, 0.0873^\circ, 0.0175^\circ, 0.0175^\circ, 0.1745^\circ, 0.0349^\circ, 0.1396^\circ, 0.1571^\circ, 0.1745^\circ, 0.1396^\circ, 0.1222^\circ, 0.0873^\circ, 0.192^\circ, 0.0873^\circ, 0.1047^\circ, 0.2443^\circ, 0.1222^\circ, 0.0175^\circ, 0.1222^\circ \} ;

4. Grid

We avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle \( \theta \).

We grid one hemisphere at a time, then the hemispheres are combined.

Definitions:

- gridSpacing: separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles. Set by the user in Sec. 2.
- \( d\theta \): grid spacing in radians
- \( \alpha_{\text{pointH}}, \delta_{\text{pointH}} \): RA and dec of the grid points \( H_j \)
- grid, gridN, gridS: tables of points on the sphere, "grid points", including associated information. See listing below for "grid" table entries
- nGrid: number of grid points
- rGrid: radial unit vectors from origin to grid points, in 3D Cartesian coordinates
- \( \alpha_{\text{Grid}}, \delta_{\text{Grid}} \): RAs and decs for the grid points

Tables:

**grid, gridN and gridS**

1. sequential point #  2. RA index  3. dec index  4. RA (rad)  5. dec (rad)  6. Cartesian coordinates of the grid point

(When gridSpacing = 2°, we get a 2° x 2° grid.*)

Print["The grid spacing is a setting that was chosen in Sec. 3 to be gridSpacing = ", gridSpacing, ", "];

d\theta = ((2. \pi) / 360.) gridSpacing; (*Convert gridSpacing to radians*)
The grid spacing is a setting that was chosen in Sec. 3 to be gridSpacing = 2.°.

The grid spacing is a setting that was chosen in Sec. 3 to be gridSpacing = 2.°.

In[33]=
(*The Northern Grid "gridN".*)
gridN = {}; idN = 1;
For[j = 0., j < π / (2. °), j++, δpointH = δj °;
  For[ai = 0., ai < Ceiling[(2. π) / δθ] (Cos [δpointH] + 0.01)],
    ai++, apointH = ai δθ / (Cos [δpointH] + 0.01);
    AppendTo[gridN, {idN, ai, δj, apointH, δpointH, er[apointH, δpointH]}];
  idN = idN + 1]
]

In[35]=
(*The Southern Grid "gridS".*)
gridS = {}; idS = 1;
For[j = 0., j < π / (2. °), j++, δpointH = -δj °;
  (*Print["{δj, δpointH} = ", {δj, δpointH}];*) For[ai = 0.,
    ai < Ceiling[(2. π) / δθ] (Cos [δpointH] + 0.01)], ai++, apointH = ai δθ / (Cos [δpointH] + 0.01);
    (*Print["{ai, apointH} = ", {ai, apointH}];*)
    AppendTo[gridS, {idS, ai, δj, apointH, δpointH, er[apointH, δpointH]}];
  idS = idS + 1]
]

In[37]=
grid = {}; j = 1;
For[j = 1, jN ≤ Length[gridN], jN++, AppendTo[grid,
  {j, gridN[[jN, 2]], gridN[[jN, 3]], gridN[[jN, 4]], gridN[[jN, 5]], gridN[[jN, 6]]}];
  j = j + 1]
For[j = 1, jS ≤ Length[gridS], jS++, AppendTo[grid,
  {j, gridS[[jS, 2]], gridS[[jS, 3]], gridS[[jS, 4]], gridS[[jS, 5]], gridS[[jS, 6]]}];
  j = j + 1]

nGrid = Length[grid];

In[41]=
αGrid = Table[αFROMr[grid[[j, 6]]], {j, Length[grid]}];
δGrid = Table[δFROMr[grid[[j, 6]]], {j, Length[grid]}];
rGrid = Table[grid[[j, 6]], {j, Length[grid]}];

In[44]=
Print["There are ", nGrid, " points on the grid."]
There are 10518 points on the grid.

5. Significance

The problem of "significance" is to determine the likelihood that random polarizations directions would have better alignment or avoidance than the observed polarization directions.

The alignment of the 106 quasars is so remarkable that we take the time to give the statistics special treatment. Instead of relying on the estimates and formulas in Sec. 4 of the Intermediate level notebook in Ref. 3, we perform 10,000 runs with random ψ replacing the observed ψ of the 106 QSOs. Thus the probability distribution and significance formulas are tailored to the task at hand. The main effect is to reduce the uncertainty, plus/minus, values of the results.
To determine the probability distributions and related formulas, we made many runs with random data and fit the results. Each of the 10,000 runs involved the same 106 observed locations of the quasars, the only change to the process is the introduction of random polarization directions $\psi$, a different set of 106 random values $\psi$ for each run. We fit the 10,000 sets of results with approximating functions and determined probability distributions for the best alignment angle $\eta_{\text{min}}$ and the best avoidance angle $\eta_{\text{max}}$. The significance of the observed $\eta_{\text{min}}$ is then the likelihood that the approximating functions that fit the random runs would give a smaller value of $\eta_{\text{min}}$. The statistical formulas found by this method are specific to these 106 quasars.

Many of the significances found for quantities in this notebook are a few parts in tens of millions. Since there were only 10,000 random runs, a comparatively small number compared to tens of millions, an essential assumption is that the exponential tails of the fitting function are suitable approximations. Thus, tiny values of significance may be less accurate than they appear. Yet even if the accuracy is suspect, the extremely small significances found for some quantities in this notebook should be interpreted as showing the likelihood of chance providing the same outcome is essentially nil.

5a. Random Run Generator

Definitions:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r\text{SrcxGrid}$</td>
<td>unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector to a source with the radial vector to a grid point, $S_i \times H_j$</td>
</tr>
<tr>
<td>runRandomData</td>
<td>data generator in the random runs that is needed in the calculations that follow. See listing below for a description of the entries</td>
</tr>
<tr>
<td>$\psi$RandomData</td>
<td>the random PPA $\psi$ values used in the runs</td>
</tr>
<tr>
<td>$r\text{Srcx}$$\psi$Src</td>
<td>unit vector, $S_i \times \psi$, cross product of the radial vector to the source with the vector in the direction of the polarization</td>
</tr>
<tr>
<td>$\eta$BarToGrid</td>
<td>average alignment angle the sources make with each grid point together with the grid point index $j$, i.e. ${j, \eta(H_j)}$</td>
</tr>
<tr>
<td>$\eta$BarMin, $\eta$BarMax</td>
<td>$\eta$BarToGrid, ${j, \eta(H_j)}$, for the smallest and largest $\eta$</td>
</tr>
<tr>
<td>$\eta$BarMinRandomData</td>
<td>the second entry in $\eta$BarMin, i.e. $\eta_{\text{min}}$</td>
</tr>
<tr>
<td>$\eta$BarMaxRandomData</td>
<td>the second entry in $\eta$BarMax, i.e. $\eta_{\text{max}}$</td>
</tr>
</tbody>
</table>

```math
\text{In[45]} = r\text{SrcxGrid1} = \text{Table}[\text{Cross}[\text{rSrc}[[i]], \text{rGrid}[[j]]], \{i, n\text{Src}\}, \{j, n\text{Grid}\}];
(*first step: raw cross product. These are not unit vectors*)
\text{rSrcxGrid} =
\text{Table}[(r\text{SrcxGrid1}[[i, j]]/(r\text{SrcxGrid1}[[i, j]].r\text{SrcxGrid1}[[i, j]] + 0.000001)^{1/2},
\{i, n\text{Src}\}, \{j, n\text{Grid}\}];(*\text{unit vectors}*)
\text{Clear}[r\text{SrcxGrid1}];
(*r\text{SrcxGrid}: table of the unit vectors perpendicular to the plane of the great circle containing the source $S_i$ and the grid point $H_j$*)
```
In[48]=
(*
runRandomData= {}; ψRandomData= {}; nRunPrint= 0;
For[ nRun= 1, nRun≤ nR, nRun++ ,
If[ nRun> nRunPrint, Print[ "At the start of run ", nRun, ", the time is ",
TimeUsed[], " seconds and the memory in use is ", MemoryInUse[], " bytes."];
runRandomData= runRandomData+ 500;
ψSrc= Table[ RandomReal[ {0.00001, π - 0.00001}], {i, nSrc}]; (*
*table of PPA angles ψ for the sources in region j0, in radians*)
rSrcxψSrc = Table[ Sin[ψSrc[[i]]] eNSrc[[i]] - Cos[ψSrc[[i]]] eESrc[[i]], {i, nSrc}]; (*
table of the cross product of rSrc and vector in direction of ψSrc,
a unit vector*) jηBarToGrid = Table[ [j, (1/nSrc) Sum[ ArcCos[ Abs[ rSrcxψSrc[[i]]]. rSrcx rGrid[[i, j]] ] - 0.000001 ], {i, nSrc}]], {j, nGrid}]; (*
* (grid point #, value of the alignment angle ηHj[j] averaged over all sources,
in radians)*) sortηjBarToGrid= Sort[ jηBarToGrid, #1[[2]]<#2[[2]]&]; (*jηBarToGrid, {j, j}, but sorted with the smallest alignment angles first *)
jηBarMin= sortηjBarToGrid[[1]]; (* {j, j}, at the grid point Hj with minimum  η*)
jηBarMax= sortηjBarToGrid[[ -1]]; (* {j, j},
at the grid point Hj with maximum η*) AppendTo[ ψRandomData, { nRun, ψSrc}];
AppendTo[ runRandomData, { nRun, { jηBarMin[[2]], { ηGrid [ [ jηBarMin[[1]] ] ]}, { jηBarMax[[2]], { ηGrid [ [ jηBarMax[[1]] ] ]}] } } (*collect data*) ]
*)

Hint: You can save memory if you do not get the "ψRandomData" in the following cell. The values of ψ in ψRandomData are not needed in any of the following calculations.

In[49]=
SetDirectory[dataDirectory]; (*Save memory space; ψRandomData is not used below.*)
(*Put[ψRandomData, "20210216psiRandomData.dat" ] *) (*Save a new "ψRandomData"*)
(*ψRandomData= Get[ "20210216psiRandomData.dat" ]; *) (*Get an old "ψRandomData"*)

Hint: Saving “runRandomData” avoids the time it takes to complete the “For” statement. Do not forget to make the “For” statement into a remark so that it doesn’t run.

In[50]=
SetDirectory[dataDirectory];
runRandomData= Get[ "20210203runRandomData.dat" ];
(*Get an old "runRandomData".*)

In[51]=
Print[ "The number of runs in the runRandomData table is ", Length[runRandomData], ",." ]

The number of runs in the runRandomData table is 10000.

In[52]=
ηBarMinRandomData = Table[ runRandomData[[ i1, 2, 1]], { i1, Length[runRandomData]}];
ηBarMaxRandomData = Table[ runRandomData[[ i1, 3, 1]], { i1, Length[runRandomData]}];
5b. Analysis of the Random Run Results

For a single source, allowing its polarization direction to be randomly directed means the angle $\psi$ in Fig. 1 can have any value from $0^\circ$ to $180^\circ$, with no particular value favored. For a given grid point $H$, the alignment angle $\eta$ is acute, i.e. $0^\circ \leq \eta \leq 90^\circ$. With random $\psi$, the alignment angle is also random, so no value of $\eta$ is favored over any other. On average, one expects $\eta$ to be $45^\circ$.

For the average alignment angle $\langle \eta(H) \rangle$, Eq. 1, with a sample of $N$ sources with random polarization directions, one expects $45^\circ$ to again be the most likely value for any $H$. However there will a minimum $\eta_{\min}$ at some hub $H_{\min}$ and a maximum $\eta_{\max}$ at some other hub $H_{\max}$. In Ref. 2, we argue that the sum of $N$ now-random numbers is much like a random walk which is well-known to increase with the number of steps $N$ like $N^{3/2}$. Here the “number of steps” is the number of sources $N$. Since we get an average by dividing by $N$, the difference $\frac{\pi}{4} - \eta$ should be proportional to $N^{-1/2}$.

The point is that we expect many random runs to produce a most likely value of $\eta_{\min}$ and a most likely value of $\eta_{\max}$, with nearby values less likely. Thus, we expect some sort of Gaussian-like distributions.

The distributions for minimum alignment angle $\eta_{\min}$ and maximum avoidance angle $\eta_{\max}$ look like slanted Gaussians, each distribution slanted away from $\eta = \pi/4 = 45^\circ$. To accommodate the behavior, the distributions are fit with non-Gaussian functions, which differ from Gaussians by a step curve, an S-curve. Even though the distributions are non-Gaussian, the terms “half-width” $\sigma$ and “mean” $\eta_0$ are used for some parameters. For a discussion see Ref. 2.

Definitions

$\text{sort}\eta_{\text{BarMinRANDOM}}$: collect and sort the best alignment angles $\eta_{\min}$ from the random runs

$\sigma_B$: an estimate of the half-width of the distribution of $\eta_{\min}$

$\eta_0B$: an estimate of the mean value for the distribution of $\eta_{\min}$

$h\text{lMin0}$: histogram list for sorted alignment angles in $\text{sort}\eta_{\text{BarMinRANDOM}}$

$h\text{lMin}$: histogram bar data, i.e. \{$\eta$ at midpoint, height\}

$\text{nlmRandomMin}$: nonlinear model fit to histogram bar data $h\text{lMin}$

$\text{showNLMMin}$: plot of histogram and the fit function

$\text{parametersNLMMin}$: parameters determined for the fit function

$p\text{TableNLMMin}$: parameter table, has standard error and other stats

$\sigma_{\text{RANDOMmin}}$: width parameter, approximately the half-width at 60.7% of peak

$\sigma_{\text{ErrRANDOMmin}}$: standard error in $\sigma$

$\eta_0_{\text{RANDOMmin}}$: mean parameter, approximately the value of $\eta$ at the peak

$\eta_0_{\text{ErrRANDOMmin}}$: standard error in $\eta_0$

$\text{histRandomMin}$: histogram for alignment angles $\eta_{\min}$

$\Delta_R$: number of runs in a bin

$h\text{lMin}$: \{$\eta$ at half-bin width, bin height $\Delta_R$\}
In[55]:= sortηBarMinRANDOM = Sort[ηBarMinRandomData];
σB = nSrc\(^{-1/2}\)/4.;
η0B = π/4. - 1/\(\sqrt{nSrc}\);
hlMin0 = HistogramList[sortηBarMinRANDOM, {η0B - 5 σB, η0B + 4 σB, 0.31 σB}];
hlMin = Table[
    {i, Length[hlMin0[[i]]]},
    {i, Length[hlMin0[[2]]]}];

nlmRandomMin = NonlinearModelFit[hlMin, a \(1 + e^{4 \frac{(x-x0)}{b}}\) \((-1\cdot\(1/2\).((x-x0)/b)^2)),
    {{a, 1200.}, {b, σB}, {x0, η0B}}, x];

(* A Gaussian modified by a Step function
\(1+e^{4 \frac{(x-x0)}{b}}\) \((-1\cdot\(1/2\).((x-x0)/b)^2)),
    
Out[61]=
Figure 4. The histogram is steeper on the \(\pi/4 = 0.785\) side. A Gaussian fit is not approriate. By introducing a step down function, the S-curve \((1+e^{4 \frac{(x-x0)}{b}})^{-1}\), one can reduce the \(\pi/4\) side where the S-curve is small, while leaving the Gaussian untouched on the left, where the S-curve is near unity.

In[63]:= parametersNLMMin = {a, b, x0}.nlmRandomMin["BestFitParameters"]
pTableNLMMin = nlmRandomMin["ParameterTable"]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1222.99</td>
<td>9.20303</td>
<td>132.89</td>
<td>(2.32353 \times 10^{-38})</td>
</tr>
<tr>
<td>b</td>
<td>0.0297804</td>
<td>0.00025053</td>
<td>118.83</td>
<td>(4.18641 \times 10^{-37})</td>
</tr>
<tr>
<td>x0</td>
<td>0.696699</td>
<td>0.00020957</td>
<td>3324.33</td>
<td>(1.04862 \times 10^{-74})</td>
</tr>
</tbody>
</table>
Repeat for the avoidance angle $\eta_{\text{max}}$. See above for Definitions and adjust the meanings for avoidance, i.e. “min” $\rightarrow$ “max”.

```
In[7]:= sort\$etaBarMaxRandom = Sort[\$etaBarMaxRandomData];
\$sigmaB = nSrc\(^{-1/2}\) / 4.;
\$eta0B = \pi / 4 + 1 / nSrc\(^{1/2}\);
histRangeMax = {\$eta0B - 4 \$sigmaB, \$eta0B + 5 \$sigmaB, 0.31 \$sigmaB};
hlMax0 = HistogramList[sort\$etaBarMaxRandom, histRangeMax];
hlMax = Table[\{1/2 \{hlMax0[[1, i1]] + hlMax0[[1, i1 + 1]]\}, hlMax0[[2, i1]]\},
{\{i1, Length[ hlMax0[[2]] ] \}}];
nlmRandomMax = NonlinearModelFit[hlMax, a \[1 + E^(-4 ((x - x0) / \sigma) / \$sigmaB)] - 1 \ Exp[\(- (1/2.) \ ((x - x0) / \sigma)^2\)],
{\{a, 1200.\}, \{b, \$sigmaB\}, \{x0, \$eta0B\}\}, x];
(*A Gaussian modified by a Step-function \(1 + e^{-4 ((x - x0) / \sigma)^2}\).*)
```

```
In[74]:= showNLMMax = Show[Histogram[sort\$etaBarMaxRandom, histRangeMax, PlotLabel \rightarrow "Avoidance Histogram",
AxesLabel \rightarrow \{"\$eta_{\text{max}}", "\Delta R"\}],
Plot[Normal[nlmRandomMax],
{x, \$eta0B - 5 \$sigmaB, \$eta0B + 5 \$sigmaB}, PlotStyle \rightarrow Purple, ListPlot[hlMax]]]
```

![Avoidance Histogram](image)

```
Out[74]=
```

Figure 5. As in Fig. 4, here the Avoidance Histogram is steeper on the $\pi/4 = 0.785$ side. As before with the Alignment Histogram, the step up function $S$-curve $(1 + e^{-4 ((x - x0) / \sigma)^2})^{-1}$ is combined with the Gaussian, which reduces the $\pi/4$ side on the left, while leaving the Gaussian untouched on the right. “

```
Figure 5. As in Fig. 4, here the Avoidance Histogram is steeper on the $\pi/4 = 0.785$ side. As before with the Alignment Histogram, the step up function $S$-curve $(1 + e^{-4 ((x - x0) / \sigma)^2})^{-1}$ is combined with the Gaussian, which reduces the $\pi/4$ side on the left, while leaving the Gaussian untouched on the right.
\[ n(\theta) = \text{parametersNLMMax} = \{a, b, x0\} \] /\ nlmRandomMax["BestFitParameters"];

\[ \text{parameterErrorsNLMMax} = \text{nlmRandomMax["ParameterErrors"]}; \]

\[ \text{pTableNLMMax} = \text{nlmRandomMax["ParameterTable"]}; \]

\[ \{\sigma \text{RANDOMmax}, \sigma \text{ErrRANDOMmax} \} = \{b, a\} \text{nlmRandomMax["BestFitParameters"]}, \]

\[ \text{nlmRandomMax} \{\eta \text{RANDOMmax}, \eta \text{ERRRANDOMmax} \} = \{x0, a\} \text{nlmRandomMax["BestFitParameters"]}, \]

\[ \text{nlmRandomMax["ParameterErrors"]}[3]; \]

\begin{tabular}{|c|c|c|c|c|}
\hline
Estimate & Standard Error & t-Statistic & P-Value & \\
\hline
a & 1224.8 & 7.2895 & 5.2542 & \times 10^{-4} \\
b & 0.297779 & 0.000198109 & 9.8322 \times 10^{-40} & \\
x0 & 0.874236 & 0.000165742 & 6.42455 \times 10^{-40} & \\
\hline
\end{tabular}

5c. Probability Distributions and Significance Formulas

The histograms in Figs. 4 and 5 are proportional to probability distributions: The height of a bar, \( \Delta R \), is the number of runs in a bin of width \( \Delta \eta \) centered on \( \eta \). Thus, the likelihood of \( \eta \) is roughly \( \Delta R / R \), where \( R \) is the total number of runs \( R = \Sigma \Delta R \). And the likelihood approximates the probability distribution. The histogram and the probability distribution differ by a constant factor, a normalization constant. See Ref. 2 for more detail.

Definitions:

- \( \text{norm} \): a constant used to normalize the distribution so the integral of probability is 1.
- \( \text{probMIN0, probMAX0} \): probability distributions for alignment (MIN) and avoidance (MAX), functions of \( \eta, \eta_0, \sigma \). \( \text{probMIN, probMAX} \): probability distributions assuming the best fit values of \( \eta_0, \sigma \) from the random runs in Sec. 5b above.
- \( \text{signiMIN0, signiMAX0} \): significance as a function of \( \eta, \eta_0, \sigma \).
- \( \text{signiMIN, signiMAX} \): significance for the best fit values of \( \eta_0, \sigma \) from the random runs in Sec. 5b.

\[ \text{norm} = \left( \frac{1}{2\pi} \right)^{1/2} \text{NIntegrate}\left[\left(1 + e^{-4(y-1)^2} \right)^{-1} e^{-y^2/2}, \{y, -\infty, \infty\}\right]^{-1} ; \]

\[ \text{norm} ; (* \text{Constant needed for Eq. (10) and (11) in Ref. 2.}) \]

\[ \text{probMIN0}[\eta, \eta_0, \sigma] := \left( \frac{\text{norm}}{\sigma} \right)^{1/2} \left(1 + e^{\frac{-(\eta - \eta_0)^2}{\sigma}} \right)^{-1} e^{\frac{1}{2} \left(\frac{\eta - \eta_0}{\sigma}\right)^2} ; \]

\[ \text{signiMIN0}[\eta, \eta_0, \sigma] := \text{NIntegrate}[\text{probMIN0}[\eta_1, \eta_0, \sigma], \{\eta_1, -\infty, \eta\}] ; \]

\[ \text{probMAX0}[\eta, \eta_0, \sigma] := \left( \frac{\text{norm}}{\sigma} \right)^{1/2} \left(1 + e^{\frac{-(\eta - \eta_0)^2}{\sigma}} \right)^{-1} e^{\frac{1}{2} \left(\frac{\eta - \eta_0}{\sigma}\right)^2} ; \]

\[ \text{signiMAX0}[\eta, \eta_0, \sigma] := \text{NIntegrate}[\text{probMAX0}[\eta_1, \eta_0, \sigma], \{\eta_1, \eta, \infty\}] ; \]

The significance \( \text{signiMIN0}[\eta, \eta_0, \sigma] \) is the integral of \( \text{probMIN0} \), i.e. \( \text{signiMIN0} = \int_{-\infty}^{\eta} \text{probMIN0}(\eta) \, d\eta \).

The significance \( \text{signiMAX0}[\eta, \eta_0, \sigma] \) is the integral of \( \text{probMAX0} \), i.e. \( \text{signiMAX0} = \int_{\eta}^{\infty} \text{probMAX0}(\eta) \, d\eta \).
The following probability distributions and significances make use of the mean $\eta_0$ and half-width $\sigma$, which are the most likely values according to the random run fitting functions. One expects that the larger the number of random runs, the more accurate the mean $\eta_0$ and half-width $\sigma$ and the more accurate the formulas.

\begin{verbatim}
In[87]:= probMIN[\_] := probMIN0[ \_ , \_ RANDOMmin , \_ RANDOMmin ]
In[88]:= signiMIN[\_] := signiMIN0[ \_ , \_ RANDOMmin , \_ RANDOMmin ]
In[89]:= probMAX[\_] := probMAX0[ \_ , \_ RANDOMmax , \_ RANDOMmax ]
signiMAX[\_] := signiMAX0[ \_ , \_ RANDOMmax , \_ RANDOMmax ]
\end{verbatim}

6. Results and map using the Best Values $\phi n$ of the Polarization Directions

6a. Results Using the Best Values of $\phi n$

“Best” means we used the $\phi n$ that were input in Sec. 3. Later on, in Sec. 7, we allow $\phi n + \delta \phi$, where $\delta \phi$ conforms to the uncertainty $\sigma \phi$ in the measured values.

Definitions:

$v \psi$Src unit vectors along the polarization directions in the tangent planes of the sources
eNSrc unit vectors along local North in the tangent planes of the sources
eESrc unit vectors along local East in the tangent planes of the sources
$j \eta$BarHj \{j,\eta(H)\}, where j is the index for grid point $H_j$ and $\eta(H)$ is the average alignment angle at $H_j$. See Eq. (1) in the Introduction.
sortj$\eta$BarHj \{j,\eta(H)\}, rearranged by value of $\eta(H)$, with smallest angles $\eta(H)$ first.
j$\eta$BarMin \{j,\eta(H)\}, the $j$ and $\eta$ for the smallest value of $\eta(H)$, best alignment $\eta$BarMin the smallest value of $\eta(H)$, measures alignment of the polarization directions
j$\eta$BarMax \{j,\eta(H)\}, the $j$ and $\eta$ for the largest value of $\eta(H)$, most avoided $\eta$BarMax the largest value of $\eta(H)$, measures avoidance

$nSx\psi$ unit vector, $S_i \times \psi_i$, cross product of the radial vector to the source with the vector in the direction of the polarization
$nSxHj$ unit vector, $S_i \times H_j$, cross product of the radial vector to the source with the radial vector to the grid point $H_j$
$\eta$Hj alignment angle between source and grid point $H_j$, see Fig. 1
$\eta$BarHj alignment angle $\eta(H_j)$ between source and grid point $H_j$, averaged over all sources
j$\eta$BarHj \{j,\eta(H_j)\}, the $j$ and $\eta$ for grid point $H_j$

$\eta$BarMin significance of the smallest alignment angle
$\eta$BarMin get the range if sigs using the plus/minus values on the parameters $\eta_0$, $\sigma$ from the random runs, i.e. $\eta$0RANDOMmin, $\sigma$RANDOMmin
$\eta$BarMin the smallest of the values in $\eta$BarMin
$\eta$BarMin the largest of the values in $\eta$BarMin
$\eta$BarMax significance of the largest alignment angle (i.e. avoidance)
$\eta$BarMax get the range if sigs using the plus/minus values on the parameters $\eta_0$, $\sigma$ from the random runs, i.e. $\eta$0RANDOMmax, $\sigma$RANDOMmax
$\eta$BarMax the smallest of the values in $\eta$BarMax
$\eta$BarMax the largest of the values in $\eta$BarMax
αHminDegrees RA of the point $H_{\text{min}}$ where $\eta(H)$ is the smallest
δHminDegrees decl of the point $H_{\text{min}}$ where $\bar{\eta}(H)$ is the smallest
αHmaxDegrees RA of the point $H_{\text{max}}$ where $\eta(H)$ is the largest
δHmaxDegrees decl of the point $H_{\text{max}}$ where $\bar{\eta}(H)$ is the largest

\[ v_\phi, \ e_\eta, \ e_\zeta \text{ unit vectors in the tangent plane of each source } S_i, \]
pointing along the polarization direction, local North, and local East, respectively. See Fig. 1.*
\[ \psi_{\text{Src}} = \text{Table}[\cos(\psi_n[[i]]), \ e_n[\alpha_{\text{Src}}[[i]], \delta_{\text{Src}}[[i]]] + \sin(\psi_n[[i]]) \] \ e_n[\alpha_{\text{Src}}[[i]], \delta_{\text{Src}}[[i]]], \{i, n_{\text{Src}}\}];

\[ \eta_{\text{BarHj}} = \text{Table}[\{j, (1/n_{\text{Src}}) \ \text{Sum}[/\text{ArcCos}\left[ \text{Abs}[r_{\text{Grid}}[[j]].v_{\text{Src}}[[i]]/(\{r_{\text{Grid}}[[j]] - r_{\text{Src}}[[i]]\} . r_{\text{Src}}[[i]]\} . r_{\text{Src}}[[i]]\} . (r_{\text{Grid}}[[j]] - r_{\text{Grid}}[[i]].r_{\text{Src}}[[i]])\} , {i, n_{\text{Src}}}], \{j, n_{\text{Grid}}\}]; \]
\[ \eta_{\text{BarMin}} = \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] < \#2[[2]] \ \&]; \]
\[ \eta_{\text{BarMax}} = \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] > \#2[[2]] \ \&]; \]

\[ (\star \text{ Analysis using Eq } (5) \text{ in Ref. } 2 \text{ to get } \eta_{1\text{H}}, \cos(\eta) = |\vec{V}_H . \vec{V}_\phi|, \text{ then } \{j, \bar{\eta}(H_j)\}. \star) \]
\[ \eta_{\text{BarHj}} = \text{Table}[\{j, (1/n_{\text{Src}}) \ \text{Sum}[\text{ArcCos}\left[ \text{Abs}\left[ r_{\text{Grid}}[[j]].r_{\text{Src}}[[i]]/(\{r_{\text{Grid}}[[j]] - r_{\text{Grid}}[[i]].r_{\text{Src}}[[i]]\} . r_{\text{Src}}[[i]]\} . r_{\text{Src}}[[i]]\} . (r_{\text{Grid}}[[j]] - r_{\text{Grid}}[[i]].r_{\text{Src}}[[i]])\} , {i, n_{\text{Src}}}], \{j, n_{\text{Grid}}\}]; \]
\[ \eta_{\text{BarMin}} = \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] < \#2[[2]] \ \&]; \]
\[ \eta_{\text{BarMax}} = \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] > \#2[[2]] \ \&]; \]

\[ (\star \text{Alternate analysis using Eq } (7) \text{ in Ref. } 2 \text{ to get } \eta_{1\text{H}}, \cos(\eta) = |\vec{H}_{5\text{Sh}} . \vec{H}_{5\text{Sh}}|. \star) \]
\[ \eta_{\text{Hj}}[\_]:=\eta_{\text{Hj}}[\_]=\text{Table[} \text{ArcCos}\left[ \text{Abs}\left[ n_{\text{Src}}[\psi_n[[n]]].n_{\text{Src}}[\psi_n[[n]]]\right]\right] - \sqrt{\left(\left(\text{Cross}\left[ r_{\text{Src}}[[n]], r_{\text{Grid}}[[j]]\right]\right) . \left(\text{Cross}[r_{\text{Src}}[[n]], r_{\text{Grid}}[[j]]]\right)\right)}, \{n, n_{\text{Src}}\}; \]
\[ \eta_{\text{BarHj}}[\_]:=\eta_{\text{BarHj}}[\_]=\text{Sum}[\eta_{\text{Hj}}[\_][[n]], \{n, n_{\text{Src}}\}]/n_{\text{Src}} \]
\[ \eta_{\text{BarHj}}[\_]=\text{Table}[\{j, \eta_{\text{BarHj}}[\_][j], \{j, \text{Length}[\text{grid}]\}]; \]
\[ \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] < \#2[[2]] \ \&]; \]
\[ \eta_{\text{BarMin}} = \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] < \#2[[2]] \ \&]; \]
\[ \eta_{\text{BarMin}} = \eta_{\text{BarMin}}[\_][2]; \]
\[ \eta_{\text{BarMax}} = \text{Sort}[\eta_{\text{BarHj}}, \#1[[2]] < \#2[[2]] \ \&]; \]
\[ \eta_{\text{BarMax}} = \eta_{\text{BarMax}}[\_][2]; \]
(*Significance of the alignment of the polarization directions with hub point H_{min}.*)

\[\text{sig}^\gamma_\text{BarMin} = \text{signiMIN}[\eta_\text{BarMin}]\]

\[\text{sigRangeBaryBarMin} = \]

\[\text{Sort}[\text{Partition}[\text{Flatten}[\text{Table}[\{\text{signiMIN}[\eta_\text{BarMin}, \eta_\text{RANDOMmin} + \gamma_1 \eta_\text{ErrRANDOMmin}, \]
\[\sigma_\text{RANDOMmax} + \gamma_2 \sigma_\text{ErrRANDOMmax}, \gamma_1, \gamma_2, (\gamma_1, -1, 1), (\gamma_2, -1, 1)\}]], 3]]\]

\{\text{sigRangeBaryBarMin}[1], \text{sigRangeBaryBarMin}[-1]\}

\text{sigSmallBaryBarMin} = \text{sigRangeBaryBarMin}[1, 1];

\text{sigBigBaryBarMin} = \text{sigRangeBaryBarMin}[-1, 1];

\text{Print["The best value for the significance of alignment is sig. = ", \text{sig}^\gamma_\text{BarMin}, ".", Using the uncertainties +/- of the \eta_\text{RANDOMmin}, \sigma_\text{RANDOMmin}, we find that the range is from sig. = ", \text{sig}^\gamma_\text{BarMin}, " to ", \text{sig}^\gamma_\text{BarMin}, "]}

The best value for the significance of alignment is sig. = 7.36374 \times 10^{-8}. Using the uncertainties +/- of the \eta_\text{RANDOMmax}, \sigma_\text{RANDOMmax}, we find that the range is from sig. = 5.53373 \times 10^{-8} to 9.72742 \times 10^{-8}.

(*Significance of the polarization directions' avoidance of the hub point H_{max}.*)

\[\text{sig}^\gamma_\text{BarMax} = \text{signiMAX}[\eta_\text{BarMax}]\]

\[\text{sigRangeBaryBarMax} = \]

\[\text{Sort}[\text{Partition}[\text{Flatten}[\text{Table}[\{\text{signiMAX}[\eta_\text{BarMax}, \eta_\text{RANDOMmax} + \gamma_1 \eta_\text{ErrRANDOMmax}, \sigma_\text{RANDOMmax} + \gamma_2 \sigma_\text{ErrRANDOMmax}, \gamma_1, \gamma_2, (\gamma_1, -1, 1), (\gamma_2, -1, 1)\}]], 3]]\]

\{\text{sigRangeBaryBarMax}[1], \text{sigRangeBaryBarMax}[-1]\}

\text{sigSmallBaryBarMax} = \text{sigRangeBaryBarMax}[1, 1];

\text{sigBigBaryBarMax} = \text{sigRangeBaryBarMax}[-1, 1];

\text{Print["The best value for the significance of avoidance is sig. = ", \text{sig}^\gamma_\text{BarMax}, "]}

The best value for the significance of avoidance is sig. = 0.000124075. Using the uncertainties +/- of the \eta_\text{RANDOMmax}, \sigma_\text{RANDOMmax}, we find that the range is from sig. = 0.000109936 to 0.000139688.

\[\{\eta_\text{BarMin}, \eta_\text{BarMax}\}; (* 1. grid, 2. alignment angle \eta \) at Min and Max \eta . *)\]

\[\alpha_\text{HminDegrees0} = \text{grid}[[\{\eta_\text{BarMin}[1]\}][4]) (360 / (2 \pi)); \]

\[\delta_\text{HminDegrees0} = \text{grid}[[\{\eta_\text{BarMin}[1]\}][5]) (360 / (2 \pi)); \]

\text{If}[[180 < \alpha_\text{HminDegrees0} < 361], \alpha_\text{HminDegrees} = \alpha_\text{HminDegrees0} - 180; \]

\[\delta_\text{HminDegrees} = -\delta_\text{HminDegrees0}, \alpha_\text{HminDegrees} = \alpha_\text{HminDegrees0}; \]

\[\delta_\text{HminDegrees} = -\delta_\text{HminDegrees0}; \]

\text{Print["The alignment hubs H_{min} are located at (RA,dec) = ", \{\alpha_\text{HminDegrees}, \delta_\text{HminDegrees}\}, "]}

\text{Print[" and at ", \{\alpha_\text{HminDegrees} - 180, -\delta_\text{HminDegrees}\}, "]}

\text{Print[" in degrees"]}

\text{The alignment hubs H_{min} are located at (RA,dec) =}

\{121.361, -30.\} and at \{-58.6389, 30.\}, in degrees

\text{The avoidance hubs H_{max} are located at (RA,dec) =}

\{96.3107, 58.\} and at \{-83.6893, -58.\}, in degrees
The following Aitoff Plot formulas were be found in, for example, Wikipedia; see Ref. 5.

Definitions

\[ \begin{align*}
  \text{aj\_jBarHjTable} & \quad \text{RA}, \text{dec}, \eta(H) \text{ at each grid point } H = H_j, \text{ in degrees} \\
  \eta\text{BarHjSmooth} & \quad \text{interpolation of \text{aj\_jBarHjTable} yields } \eta(H) \text{ as a smooth function of the (RA,dec) of } H \\
  \text{xy\_BarAitoffTable} & \quad \{x, y, \eta(x,y)\}, \text{ where } x, y \text{ are Aitoff coordinates and } \eta(x,y) \text{ is the alignment angle} \\
  \text{dy\_ContourPlot} & \quad \text{separation of successive contour lines, in degrees} \\
  \text{list\_CP} & \quad \text{list contour plot of } \eta(H), \text{ based on } \{x, y, \eta(x,y)\}, \text{i.e. } \text{xy\_BarAitoffTable} \\
  \text{xy\_AitoffSources} & \quad \{x, y\} \text{ Aitoff coordinates for the sources' locations on the sphere} \\
  \text{map\_OffBar} & \quad \text{contour plot of the alignment angle } \eta(H), \text{ adorned with source locations and labels} \\
\end{align*} \]

\[ aH(a, \delta), xH(a, \delta), yH(a, \delta) \text{ are functions needed when making a 2-D map of the Celestial sphere. The origin xH, yH is centered on } a = \delta = 0. \]

Notice the naming conflict: \( aH(a, \delta) \) is an Aitoff parameter which, in general, differs from the Right Ascension \( a \).

\[ \begin{align*}
  \text{ln[120]} & = \text{(*The names "j\_BarMin", "j\_BarMax" are used again below, so save the current values.*)} \\
  \{\text{j\_BarMinBest, j\_BarMaxBest} \} & = \{\text{j\_BarMin, j\_BarMax}\}; \\
  \text{(* j\_Bar entries: 1. grid at } H_{\text{min}} \text{ or } H_{\text{max}}, \text{ 2. alignment angle } \eta_{\text{min}} \text{ or } \eta_{\text{max}} .\}) \\
\end{align*} \]

6b. Plot of the Alignment Angle Function \( \eta(H) \) Using the Best Values \( \phi \) on

```
ln[121]= \text{(*The following table \text{aj\_jBarHjTable} is interpolated below to yield a smooth function of the alignment angle over the sphere.*)} \\
\text{(\text{* Table Entries: 1. RA at } j\text{th grid point (degrees) 2. dec at } j\text{th grid point (degrees) 3. alignment angle } \eta\text{BarRgnkj at } j\text{th grid point (degrees)*)})} \\
\text{aj\_jBarHjTable} = \{\text{aj\_jBarHjTable\_0} = \{\};} \\
\text{For}[j = 1, j \leq \text{Length}[\text{j\_BarHj}], j++,} \\
\text{AppendTo}[\text{aj\_jBarHjTable\_0}, \text{ \{ \}}; \\
  \text{grid}\left[[j, 4]\right] \times (360. \times (2. \pi)), \text{grid}\left[[j, 5]\right] \times (360. \times (2. \pi)),} \\
  \text{\j\_BarHj}\left[[j, 2]\right] \times (360. \times (2. \pi)) \} \}; \\
\text{If[} 360 \times \text{grid}\left[[j, 4]\right] \times (360. \times (2. \pi)) > 354.3,} \\
\text{AppendTo}[\text{aj\_jBarHjTable\_0}, \text{ \{ \}}; \\
  \text{grid}\left[[j, 4]\right] \times (360. \times (2. \pi)), \text{grid}\left[[j, 5]\right] \times (360. \times (2. \pi)) \} \}; \\
\text{If[} 6. \times \text{grid}\left[[j, 4]\right] \times (360. \times (2. \pi)) \geq 0.,} \\
\text{AppendTo}[\text{aj\_jBarHjTable\_0}, \text{ \{ \}}; \\
  \text{grid}\left[[j, 4]\right] \times (360. \times (2. \pi)), \text{grid}\left[[j, 5]\right] \times (360. \times (2. \pi)) \} \}; \\
\text{aj\_jBarHjTable\_0}]; \\
\text{ln[122]=} \eta\text{BarHjSmooth = Interpolation[\text{aj\_jBarHjTable}]} \\
\text{(*The smooth alignment angle function for the region.*)} \\
\text{... \text{Interpolation: Interpolation on unstructured grids is currently only supported for InterpolationOrder->1 or}} \\
\text{InterpolationOrder->All. Order will be reduced to 1.} \\
\text{Out[122]=} \text{InterpolatingFunction[\{\{\}, \{\}} \}} \\
\text{The following Aitoff Plot formulas were be found in, for example, Wikipedia; see Ref. 5.} \]
In[123]:= \[OH\[\alpha, \delta\] := \alphaH(\alpha^2 / 360^\circ) \delta \cos((2 \pi / 360^\circ) \alpha / 2.)

(* Angles \(\alpha\) and \(\delta\) are in degrees *)

\[xH\[\alpha, \delta\] := xH(\alpha^2 / 360^\circ) \sin((2 \pi / 360^\circ) \alpha / 2.) / \sin(\alphaH(\alpha, \delta))

\[yH\[\alpha, \delta\] := yH(\alpha^2 / 360^\circ) / \sin(\alphaH(\alpha, \delta))

In[126]:= \(\eta BarHjSmooth\) mapped onto a 2D Aitoff projection of the sphere.

\(\eta AltoffSources\) = Table\{xH[\alpha Srcc[n]], \delta Srcc[n] \} (360/(2 \pi)),

\(\eta H(\alpha Srcc[n]), \delta Srcc[n] \} (360/(2 \pi)), \eta H[n, Srcc[n]]\}

(* Contour plot of the alignment function \(\eta BarHjSmooth\). *)

listCP = ListContourPlot[Union[\(xyBarHjSmoothTable\)\],

\{\(xH[\alpha MinDegrees, \delta MinDegrees], yH[\alpha MinDegrees, \delta MinDegrees], \eta BarMin (360/(2 \pi)) - 1\}\},

\{\(xH[\alpha MaxDegrees, \delta MaxDegrees], yH[\alpha MaxDegrees, \delta MaxDegrees],

\(\eta BarMax (360/(2 \pi)) + 1\)\}\}, AspectRatio -> 1/2,

Contours -> Table[\(\eta, \eta\), Floor[\(n BarMin[2] \star (360/(2 \pi))\) + 1],

Ceiling[\(\eta BarMax[2] \star (360/(2 \pi))\) - 1, \(\eta\) ContourPlot\}],

ColorFunction -> "TemperatureMap", PlotRange -> {{-7, 7}, {-3, 3}}, Axes -> False, Frame -> False];

In[128]:= (* Construct the map of \(\eta H\). *)

mapOfBar =

Show[\{listCP, Table[ParametricPlot[\{xH[\alpha, \delta], yH[\alpha, \delta]\},

\{\delta, -90, 90\}, PlotStyle -> \{Black, Thickness[0.002]\}], (* Mesh -> \{11, 5, \}

\{\(23, 11, \theta\)\}, MeshStyle -> Thick, \} PlotPoints -> 60, \{\alpha, -180, 180, 30\}, Table[

ParametricPlot[\{xH[\alpha, \delta], yH[\alpha, \delta]\}, \{\alpha, -180, 180\}, PlotStyle -> \{Black, Thickness[0.002]\}],

(* Mesh -> \{11, 5, \}(\(23, 11, \theta\), MeshStyle -> Thick, \} PlotPoints -> 60, \{\delta, -60, 60, 30\},

Graphics[\{PointSize[0.007], Text[StyleForm["N", FontSize -> 10, FontWeight -> "Plain"],

\{0, 1.85\}, (* Sources S -> Purple, Point[xyAltOffSources \]

Black, Text[StyleForm["Max", FontSize -> 8, FontWeight -> "Bold"],

\{xH[-180, 0], yH[0, -60]\}, \{Arrow[BezierCurve[\{(xH[-180, 0], yH[0, -70]), {2.3, -2.0},

(xH[MaxDegrees - 180, -MaxDegrees], yH[MaxDegrees - 180, -MaxDegrees])\}]\}],

Text[StyleForm["Min", FontSize -> 8, FontWeight -> "Bold"], \{xH[180, 0], yH[0, -60]\}],

\{Arrow[BezierCurve[\{(xH[180, 0], yH[0, -70]), {2.3, -2.0},

(xH[MinDegrees, -MinDegrees], yH[MinDegrees, MinDegrees])\}]]\}],

Text[StyleForm["Max", FontSize -> 8, FontWeight -> "Bold"], \{xH[-180, 0], yH[0, 60]\}],

\{Arrow[BezierCurve[\{(xH[-180, 0], yH[0, 70]), {2.3, 2.0},

(xH[MaxDegrees - 180, -MaxDegrees], yH[MaxDegrees - 180, -MaxDegrees])\}]\}],

Text[StyleForm["Min", FontSize -> 8, FontWeight -> "Bold"], \{xH[180, 0], yH[0, 60]\}],

\{Arrow[BezierCurve[\{(xH[180, 0], yH[0, 70]), {2.3, 2.0},

(xH[MinDegrees, -MinDegrees], yH[MinDegrees, MinDegrees])\}]]\}], ImageSize -> 432;
Figure 6: The alignment function $\eta(H)$, Eq. (1). The map is centered on $(RA, dec) = (0^\circ, 0^\circ)$, with contours separated by 4°. Diametrically opposite points $-H$ and $H$ have the same alignment angle.

The sources are located at the dots, shaded Purple.

The best alignment angle (min) is $\eta_{min} = 30.8874^\circ$, located in the most aligned areas shaded Blue.

The best avoidance angle (max) is $\eta_{max} = 56.4281^\circ$, located in the least aligned areas shaded Red.

The alignment hubs $H_{min}$ and $-H_{min}$ are located at $(RA, dec) = \{121.361, -30.\}$ and at $\{-58.6389, 30.\}$, in degrees.

The avoidance hubs $H_{max}$ and $-H_{max}$ are located at $(RA, dec) = \{96.3107, 58.\}$ and at $\{-83.6893, -58.\}$, in degrees.

(*Export the map "mapOfEtaBar" as a pdf. The export location can be reset in Sec. 3.*)

(*To activate, remove the remark brackets "(*) and ")/*. *)

Statistics of the alignment function $\eta(H)$, Eq. 1, determined using the best polarization directions $\psi_n$ as reported in the catalog, Refs. 6 and 7.

The number of sources: $N = 10^6$

The min alignment angle is $\eta_{\text{min}} = 30.8874^\circ$, which has a significance of sig. = $7.36374 \times 10^{-8}$, plus/minus = $2.36368 \times 10^{-8}$ and $1.83 \times 10^{-8}$, giving a range from sig. = $5.53373 \times 10^{-8}$ to $9.72742 \times 10^{-8}$.

The max avoidance angle is $\eta_{\text{max}} = 56.4281^\circ$, which has a significance of sig. = $0.000124075$, plus/minus = $0.0000156127$ and $0.0000141387$, giving a range from sig. = $0.000109936$ to $0.000139688$.

These uncertainties are due to the uncertainties in the constants $\eta_{\text{RANDOM}}$ and $\sigma_{\text{RANDOM}}$ found when fitting functions to the random runs results in Sec. 5b.

7. Uncertainty Runs

To determine the uncertainty in the results due to the uncertainty in the measurements, we make many runs with polarization directions conforming to the normal distribution of values determined by the best value $\psi$ and the uncertainty $\sigma\psi$.

7a. Random Run Generator

The catalog entry for B0051+291 is listed in Sec. 3. The polarization position angle is $\psi = 119^\circ \pm 14^\circ$. We call $\psi_n = 119^\circ$ the best value and $\sigma\psi = 14^\circ$ is the uncertainty. We infer that the likelihood of measured values is normally distributed about the best value with a half-width of $\sigma\psi = 14^\circ$, the width is determined along the curve down by a factor $e^{-1/2}$ from the peak value.

For each run, let the polarization direction $\psi$ for each source be allowed to differ from the best value $\psi_n$ by an amount $\delta\psi$ chosen according to a Gaussian distribution with mean (best) value $\psi_n$ and half-width $\sigma\psi$, $\psi = \psi_n + \delta\psi$. Both values $\psi_n$ and $\sigma\psi$ are taken from the catalog data entered as input in Sec. 3c.

Definitions:

$r_{\text{SrcxGrid}}$ unit vector $S_i \times H_j$ in the direction of the cross product of the radial vector to a source with the radial vector to a grid point, $S_i \times H_j$

$\mu = \psi_n$ by convention, the best value $\psi_n$, input in Sec. 3, is the mean value $\mu$ of a Gaussian of half-width $\sigma_{\psi_n}, \psi \pm \sigma\psi$
σ = σψ

ψ

Data polarization directions ψ = ψn + δψ for each run, δψ consistent with σψ from catalog. Needed to recover results.

runData collection of data from the uncertainty σψ runs, see below for entry list

nRunPrint dummy index controlling when TimeUsed and MemoryInUse data are printed

ψ

Src a polarization direction ψ for the run. This ψ is moved off the best value ψn by an increment determined by the uncertainty σψ

rSrcxrGrid unit vector, S × ψ, cross product of the radial vector to the source with the vector in the direction of the polarization

ηBarToGrid {j, η(Hj)}, where j is the index # for the grid point Hj and η(Hj) is the average of the alignment angles for Hj with the sources.

sortjηBarToGrid {j, η(Hj)}, reordered by the value of η(H), with smallest angles η(H) first.

ηBarHj {j, η(H)}, where j is the index for grid point Hj and η(H) is the average alignment angle at Hj. See Eq. (1) in the Introduction.

sortjηBarHj {j, η(H)}, rearranged by value of η(H), with smallest angles η(H) first.

ηBarMin {j, η(H)}, the j and η for the smallest value of η(H), best alignment

ηBarMax {j, η(H)}, the j and η for the largest value of η(H), most avoided

ηBarMinData values of ηmin from uncertainty runs, alignment

ηBarMaxData values of ηmax from uncertainty runs, avoidance

Hminα Data values of RA = α for hub Hmin from uncertainty runs, alignment

Hminδ Data values of dec = δ for hub Hmin from uncertainty runs, alignment

Hmaxα Data values of RA = α for hub Hmax from uncertainty runs, avoidance

Hmaxδ Data values of dec = δ for hub Hmax from uncertainty runs, avoidance

Tables:

ψ

Data entries: 1. Run # 2. ψSrc, list of polarization position angles ψ These values are not used in any subsequent calculations. Save memory.

runData entries: 1. Run # 2. {ηmin, {α, δ} at Hmin} 3. {ηmax, {α, δ} at Hmax}

In[145] :=

rSrcxrGrid1 = Table[Cross[ rSrc[[i]], rGrid[[j]] ] , {i, nSrc}, {j, nGrid}]; (* first step: raw cross product, not unit vectors*)

rSrcxrGrid = Table[ rSrcxrGrid1[[i, j]] /

       (rSrcxrGrid1[[i, j]].rSrcxrGrid1[[i, j]] + 0.000001)^1/2. , {i, nSrc}, {j, nGrid}];

Clear[rSrcxrGrid1];

(* rSrcxrGrid: table of the unit vectors perpendicular to the plane of the great circle containing the source Si and the grid point Hj*)
In[149]:=
In[148]:=
In[147]:=

(* Get an old "runData". *)
Get["20210207runData.dat"];  
(* Save a new "runData". *)
runData = Table[RandomVariate[NormalDistribution[μ[[i]], σ[[i]]]], {i, nSrc}];  
(* table of PPA angles ψ for the sources in region j0, in radians *)
rSrcxψSrc = Table[Sin[ψSrc[[i]]] eNSrc[[1]] - Cos[ψSrc[[i]]] eESrc[[i]], {i, nSrc}];  
(* table of the cross product of rSrc and vector in direction of ψSrc, 
a unit vector *)
jηBarToGrid = Table[[j, {i, nSrc}]] Sum[ArcCos[  
    Abs[rSrcxψSrc[[i]].rSrcxGrid[[i, j]]] - 0.000001], {i, nSrc}], {j, nGrid}];  
(* grid point η, value of the alignment angle ηHj[j] averaged over all sources, 
in radians.*) sortjηBarToGrid = Sort[jηBarToGrid, #1[[2]] < #2[[2]] &];  
(* ηBarToGrid, {j, η}) 
Sort by the smallest alignment angles first *)
jηBarMin = sortjηBarToGrid[[1]]; (* {j, η}, at the grid point Hj with minimum η*)
jηBarMax = sortjηBarToGrid[[-1]]; (* {j, η}) 
AppendTo[ΨData, {nRun, ψSrc}];  
AppendTo[runData, {nRun, {jηBarMin[[2]], {αGrid [[ jηBarMin[[1]] ]],  
    δGrid[[ jηBarMin[[1]] ]]}}, {jηBarMax[[2]], {αGrid [[ jηBarMax[[1]] ]],  
    δGrid[[ jηBarMax[[1]] ]]}}}] (* collect data *)  
*)

Hint: You can save memory if you do not get the "ΨData". The table ΨData is not needed in any following calculation.

SetDirectory[dataDirectory]; (* Save memory space; ΨData is not used below.*)
(* Put[ΨData,"20210216psiData.dat" ]  *) (* Save a new "ΨData"*)
(*ΨData=Get["20210216psiData.dat"] ;  *) (* Get an old "ΨData"*)

Hint: Saving “runData” to a file avoids the time it takes to complete the “For” statement. Make the “For” statement into a remark so that it doesn’t run.

SetDirectory[dataDirectory];
(* Put[runData,"20210216runData.dat" ]  *) (* Save a new "runData".*)
runData = Get["20210207runData.dat"]; (* Get an old "runData".*)

Print["The number of runs in the runData table is ", Length[runData], "."]
The number of runs in the runData table is 10000.

ηBarMinData = Table[runData[[i1, 2, 1]], {i1, Length[runData]}];
ηBarMaxData = Table[runData[[i1, 3, 1]], {i1, Length[runData]}];
HminData = Table[ runData[[i1, 2, 2, 1]] , {i1, Length[runData]}];
Hmin0Data = Table[runData[[i1, 2, 2]], {i1, Length[runData]}];
HmaxData = Table[runData[[i1, 3, 2, 1]], {i1, Length[runData]}];
Hmax0Data = Table[runData[[i1, 3, 2]], {i1, Length[runData]}];

?b. Uncertainty in the Best Alignment Angle η_{min}

This section fits a Gaussian distribution to the η_{min} from the uncertainty runs.
Definitions

\[
\text{sort}\eta\text{BarMin} \quad \text{list of } \eta_{\text{min}} \text{ from the data files, sorted small to large}
\]

\[
\eta_{0B} \quad \text{estimated mean of the Gaussian fit}
\]

\[
\sigma_B \quad \text{estimated half-width of the Gaussian fit}
\]

\[
\text{histogramRANGE} \quad \{ \text{min } \eta, \text{ max } \eta, \Delta \eta \} \text{ for the histogram}
\]

\[
hl0, hl \quad \text{histogram } \{ \eta, \text{ bin height} \} \text{ tables needed to set up the NonlinearModelFit}
\]

\[
lphl \quad \text{list plot of the histogram table } hl
\]

\[
nlmB \quad \text{non-linear model fit of a Gaussian to the } \eta_{\text{min}} \text{ histogram}
\]

\[
\text{showNLMB} \quad \text{plot of Gaussian and histogram}
\]

\[
\text{parametersNLMB} \quad \text{amplitude, half-width, and mean of the Gaussian fit}
\]

\[
pTableNLMB \quad \text{table of parameter attributes, including standard error}
\]

\[
\text{bestVersusMeanMin} \quad \text{ratio of the } \eta_{\text{min}} \text{ from best values } \psi \text{ to the mean } \eta_{0} = \eta_{\text{min}} \text{ at the peak of the fit}
\]

\[
\text{In[158]= sort}\eta\text{BarMin} = \text{Sort}[\eta\text{BarMinData}];
\]
\[
\eta_{0B} = \text{sort}\eta\text{BarMin}[[\text{Floor}[\{1/2\} \text{ Length}[\text{sort}\eta\text{BarMin}]]]]; (*\text{Guess the mean. }*)
\]
\[
\sigma_B = \text{sort}\eta\text{BarMin}[[\text{Floor}[\{4/5\} \text{ Length}[\text{sort}\eta\text{BarMin}]]]] - \eta_{0B}; (*\text{Guess the width.}*)
\]
\[
\text{histogramRANGE} = \{ \eta_{0B} - 5 \sigma_B, \eta_{0B} + 5 \sigma_B, 0.4 \sigma_B \};
\]
\[
hl0 = \text{HistogramList}[\text{sort}\eta\text{BarMin}, \text{histogramRANGE}];
\]
\[
hl = \text{Table}[[\{1/2\} (hl0[[1, i1]] + hl0[[1, i1 + 1]]), hl0[[2, i1]]], \{i1, \text{Length}[ hl0[[2]] ]\}];
\]
\[
nlmB = \text{NonlinearModelFit}[hl, a \text{ Exp}\left[-\left(\frac{1}{2}\right) \left(\frac{x - x0}{b}\right)^2\right],
\]
\[
\{\{a, \text{Length}[\text{sort}\eta\text{BarMin}/6], \{b, \sigma_B\}, \{x0, \eta0B\}\}, x\}; (*x \text{ is } \eta\text{BarMin}*)
\]

\[
\text{In[164]= showNLMB = Show}[\{\text{Histogram}[\text{sort}\eta\text{BarMin}, \text{histogramRANGE},
\]
\[
\text{PlotLabel } \rightarrow "\eta_{\text{min}}", \text{AxesLabel } \rightarrow \{"\eta_{\text{min}}, \text{ radians}, \Delta R\}\},
\]
\[
\text{Plot}[\text{Normal}[nlmB], \{x, \eta0B - 5 \sigma_B, \eta0B + 5 \sigma_B\}, \text{PlotLabel } \rightarrow "\eta_{\text{min}}"],
\]
\[
\text{ListPlot}[hl, \text{PlotLabel } \rightarrow "\eta_{\text{min}}" ]\}
\]
\[
\text{Print}[\"Figure 7: The Gaussian fit to the alignment angle } \eta_{\text{min}} \text{ histogram, where the height is the number of runs } \Delta R \text{ in each bin of width } \Delta \eta_{\text{min}} = \text{, 0.4 } \sigma_B, \text{ radians, centered on } \eta_{\text{min}}.\"
\]

Figure 7: The Gaussian fit to the alignment angle \( \eta_{\text{min}} \) histogram, where the height is the number of runs \( \Delta R \) in each bin of width \( \Delta \eta_{\text{min}} = 0.0043482 \) radians, centered on \( \eta_{\text{min}} \).
In[166]:= parametersNLMB = {a, b, x0} /. nlmB["BestFitParameters"];
   pTableNLMB = nlmB["ParameterTable"]
   {σηBarMinFit, ηBarMinFit} = {parametersNLMB[[2]], parametersNLMB[[3]]};(*radians*)

Out[167]=

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1325.51</td>
<td>13.1489</td>
<td>100.808</td>
</tr>
<tr>
<td>b</td>
<td>0.0130625</td>
<td>0.000149625</td>
<td>87.3019</td>
</tr>
<tr>
<td>x0</td>
<td>0.551343</td>
<td>0.000149625</td>
<td>3684.85</td>
</tr>
</tbody>
</table>

In[169]:= Print[
"Results of the Gaussian fit to the uncertainty runs for the alignment angle ηₘᵢₙ:"
]
   Print[""
   Print["For the value of the alignment angle ηₘᵢₙ, we find ηₘᵢₙ = ",
   ηBarMinFit (360./(2.π)), "° ± ", σηBarMinFit (360./(2.π)), "°."]
   Print["The mean ηₘᵢₙ = ", ηBarMinFit (360./(2.π)),
   "° has a significance of ", signiMIN[ηBarMinFit], "."]
   Print["The value ηₘᵢₙ + σηₘᵢₙ = ", (ηBarMinFit + σηBarMinFit) (360./(2.π)),
   "° has a significance of ", signiMIN[ηBarMinFit + σηBarMinFit], "."]
   Print["The value ηₘᵢₙ - σηₘᵢₙ = ", (ηBarMinFit - σηBarMinFit) (360./(2.π)),
   "° has a significance of ", signiMIN[ηBarMinFit - σηBarMinFit], "."]

Results of the Gaussian fit to the uncertainty runs for the alignment angle ηₘᵢₙ:

For the value of the alignment angle ηₘᵢₙ, we find ηₘᵢₙ = 31.5897° ± 0.748427°.
The mean ηₘᵢₙ = 31.5897° has a significance of 6.44344 × 10⁻⁷.
The value ηₘᵢₙ + σηₘᵢₙ = 32.3381° has a significance of 5.43124 × 10⁻⁶.
The value ηₘᵢₙ - σηₘᵢₙ = 30.8412° has a significance of 6.34791 × 10⁻⁸.

7c. Uncertainty in the Largest Avoidance Angle ηₘₐₓ

This section fits a Gaussian distribution to the ηₘₐₓ from the uncertainty runs.

Definitions:
The quantities for avoidance (Max) here have similar definitions as the quantities for alignment (Min). See the list of Definitions in Sec. 7b.

In[175]:= sortηBarMax = Sort[ηBarMaxData];
   η₀MaxB = sortηBarMax [[Floor[(1/2) Length[sortηBarMax]]]];
   σMaxB = sortηBarMax [[Floor[(4/5) Length[sortηBarMax]]]] - η₀MaxB;
   histogramRANGEMAX = {η₀MaxB - 5 σMaxB, η₀MaxB + 5 σMaxB, 0.4 σMaxB};
   h10Max = HistogramList[sortηBarMax, histogramRANGEMAX];
   h1Max = Table[{(1/2) (h10Max[[1, i1]] + h10Max[[2, i1]]), h10Max[[2, i1]]},
                  {i1, Length[ h10Max[[2]] ]}];
   nlmMaxB = NonlinearModelFit[h1Max, a Exp[-(1/2.) (x - η₀)/b]²],
              {{a, 300.}, {b, σMaxB}, {x0, η₀MaxB}}, x;(*x is ηBarMax *)
showNLMMaxB = Show[
  Histogram[sortηBarMax, histogramRANGEMAX, PlotLabel -> "\(\bar{\eta}_{\text{max}}\)", AxesLabel -> {"\(\bar{\eta}_{\text{max}}\), radians", "\(\Delta R\)"},
  Plot[Normal[nlmMaxB], {x, η0MaxB - 5 σMaxB, η0MaxB + 5 σMaxB}, PlotLabel -> "\(\bar{\eta}_{\text{max}}\)"],
  ListPlot[hlMax, PlotLabel -> "\(\eta_{\text{max}}\)"]
]
Print["Figure 8: The Gaussian fit to the avoidance angle \(\bar{\eta}_{\text{max}}\) histogram. The bins have a width \(\Delta \bar{\eta}_{\text{max}} = \)
0.00443473 radians, are centered on \(\bar{\eta}_{\text{max}}\), and have a height equal to the number of runs \(\Delta R\) in the bin."
]

Figure 8: The Gaussian fit to the avoidance angle \(\bar{\eta}_{\text{max}}\) histogram. The bins have a width \(\Delta \bar{\eta}_{\text{max}} = \)
0.00443473 radians, are centered on \(\bar{\eta}_{\text{max}}\), and have a height equal to the number of runs \(\Delta R\) in the bin.

parametersNLMMaxB = {a, b, x0} /. nlmMaxB["BestFitParameters"];
pTableNLMMaxB = nlmMaxB["ParameterTable"]
{σηBarMaxFit, ηBarMaxFit} = {parametersNLMMaxB[[2]], parametersNLMMaxB[[3]]}; (*radians*)

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1344.95</td>
<td>6.86385</td>
<td>195.947</td>
</tr>
<tr>
<td>b</td>
<td>0.0131474</td>
<td>0.000077477</td>
<td>169.695</td>
</tr>
<tr>
<td>x0</td>
<td>0.975124</td>
<td>0.000077477</td>
<td>12586</td>
</tr>
</tbody>
</table>

Print["Results of the Gaussian fit to the uncertainty runs for the avoidance angle \(\bar{\eta}_{\text{max}}\):"
]
Print[""
Print["For the value of the avoidance angle \(\bar{\eta}_{\text{max}}\), we find \(\bar{\eta}_{\text{max}} = \),
\(\eta_{\text{BarMaxFit}}(360. \div (2.\pi))\), \(" \pm \), \(\sigma_{\eta_{\text{BarMaxFit}}}(360. \div (2.\pi))\), \(" \)."
]
Print["The mean \(\bar{\eta}_{\text{max}} = \), \(\eta_{\text{BarMaxFit}}(360. \div (2.\pi))\),
\(" \) has a significance of \("\), signiMAX[\(\eta_{\text{BarMaxFit}}\), \("\)."
]
Print["The value \(\bar{\eta}_{\text{max}} + \sigma_{\bar{\eta}_{\text{max}}} = \), \(\eta_{\text{BarMaxFit}} + \sigma_{\eta_{\text{BarMaxFit}}}(360. \div (2.\pi))\),
\(" \) has a significance of \("\), signiMAX[\(\eta_{\text{BarMaxFit}} + \sigma_{\eta_{\text{BarMaxFit}}\), \("\)."
]
Print["The value \(\bar{\eta}_{\text{max}} - \sigma_{\bar{\eta}_{\text{max}}} = \), \(\eta_{\text{BarMaxFit}} - \sigma_{\eta_{\text{BarMaxFit}}}(360. \div (2.\pi))\),
\(" \) has a significance of \("\), signiMAX[\(\eta_{\text{BarMaxFit}} - \sigma_{\eta_{\text{BarMaxFit}}\), \("\) ."
]
Results of the Gaussian fit to the uncertainty runs for the avoidance angle $\eta_{\text{max}}$:

For the value of the avoidance angle $\eta_{\text{max}}$, we find $\eta_{\text{max}} = 55.8705^\circ \pm 0.753293^\circ$.

The mean $\eta_{\text{max}} = 55.8705^\circ$ has a significance of 0.000429587.

The value $\eta_{\text{max}} + \sigma \eta_{\text{max}} = 56.6238^\circ$ has a significance of 0.0000783435.

The value $\eta_{\text{max}} - \sigma \eta_{\text{max}} = 55.1172^\circ$ has a significance of 0.00196114.

7d. Location $(\alpha, \delta)$ of the Alignment Hubs $H_{\text{min}}$

Each uncertainty run returns an alignment hub $H_{\text{min}}$. In this section, we attempt to find functions that fit the distribution of those hubs.

Issues:

(a) In any one run, the analysis produces an alignment angle $\eta$ at each grid point. There can be just one minimum alignment angle $\eta_{\text{min}}$, and, therefore, just one grid point $H_{\text{min}}$ determined. However, by the symmetry across a diameter, the diametrically opposite location $-H_{\text{min}}$ should have the same minimum alignment angle, within the accuracy of the computed values. Note that $-H_{\text{min}}$ may not be a grid point. So we need to collect the hubs by moving some of them across a diameter.

(b) The spread of near-minimum $\eta$ may extend over a large area of the Celestial sphere. See the blue area in Fig. 6. Thus the small variations with the uncertainty runs may produce more than one local minimum of the alignment angle function $\eta(H)$. There may be several disparate places where hubs $H_{\text{min}}$ appear to collect. The plan is to respond reasonably to whatever appears.

(c) Since the hubs are finitely spaced grid points, the cluster of hubs may be so tightly determined that just a handful of grid points are populated. In such cases, the Gaussian fit is not appropriate and estimating the most likely location and the range of likely RAs and decs can be done by inspection.

Definitions

- $H_{\text{min}} \alpha$ RA = $\alpha$ in radians for $H_{\text{min}}$, “0” is raw data, “1” has been worked on, ...
- $H_{\text{min}} \delta$ dec = $\delta$ in radians for $H_{\text{min}}$, “0” is raw data, “1” has been worked on, ...
- $H_{\text{min}} \alpha \text{AVE}$ arithmetic average of the RAs, in radians
- sort$H_{\alpha} \text{Min}$ list of RA = $\alpha$ for $H_{\text{min}}$ from the data files, sorted small to large
- $\mu \alpha_{\text{MinB}}$ estimate of the mean value for the RA = $\alpha$ of $H_{\text{min}}$
- $\sigma \alpha_{\text{MinB}}$ estimate of the half-width of the RA = $\alpha$ of $H_{\text{min}}$
- histogramRangeMin parameter range for several histograms
- hl$0\text{Min}, hl\text{Min}$ tables of histogram data needed for plot and fit
- nlm$\alpha_{\text{MinB2}}$ two Gaussian fit to the histogram of $\alpha$ for $H_{\text{min}}$
- showNLMzMinB2plot histogram and the function that fits it
- parametersNLM$\alpha_{\text{MinB2}}$ values of the two Gaussians’ parameters
- pTableNLM$\alpha_{\text{MinB2}}$ table with values and standard errors of the parameters
- $\sigma \alpha_{\text{MinFit1}}$ half-width of the larger peak
- $\alpha_{\text{MinFit1}}$ value of $\alpha$ at the top of the larger peak
- $\sigma \alpha_{\text{MinFit2}}$ half-width of the smaller peak
- $\alpha_{\text{MinFit2}}$ value of $\alpha$ at the top of the smaller peak
- $H_{\text{min}} \alpha \text{AVE}$ average over all uncertainty runs of $\alpha$ for $H_{\text{min}}$
Many of the following sections have similarly named quantities with similar definitions.

(i) Replace “α” by “δ” for the sections dealing with the uncertainty in dec = δ.

(ii) Replace “min” with “max” in the context of the avoidance hubs $H_{max}$.

In[193]:= (* Move hubs, if necessary, so that $-180^\circ \leq \alpha < 180^\circ$ *)
Hmina0 = HminaData;
Hminδ0 = HminδData;
HminaBy180n = Round[Hmina0/Pi];
Hmina1 = Table[Hmina0[[i1]] - HminaBy180n[[i1]] Pi, {i1, Length[Hmina0]}];
Hminδ1 = Table[(-1)HminaBy180n[[i1]] Hminδ0[[i1]], {i1, Length[Hminδ0]}];

In[197]= ListPlot[Hmina1, PlotRange \to \{-\Pi, \Pi\}]
Print[
"Figure 9: Most of the RA occur at negative values near RA = -1. By the symmetry\nacross a diameter, we can move all the hubs to positive values, $\alpha > 0$ band. The move across a diameter changes the sign of the dec = δs."\n]

Out[197]=

Figure 9: Most of the RA occur at negative values near RA = -1. By the symmetry across a diameter, we can move all the hubs to positive values, $\alpha > 0$ band. The move across a diameter changes the sign of the dec = δs.

In[199]=

Hmina = Table[
    If[Hmina1[[i1]] < 0, Hmina1[[i1]] + Pi, Hmina1[[i1]], "huh?"] , {i1, Length[Hmina1]}];
Hminδ = Table[If[Hmina1[[i1]] < 0, -Hminδ1[[i1]], Hminδ1[[i1]], "huh?"] , {i1, Length[Hminδ1]}];
In[201]:= ListPlot[{Sort[Hminα], Sort[Hminδ]},
PlotLabel -> "RA = α and dec = δ for H_min, radians", AxesLabel -> {"Run #", "α, δ"}]
Print["Figure 10: It looks like we can fit Gaussians to the RA
= α values, since the values are spread out over many times the
grid spacing. But the values of dec. = δ occupy only about 6 grid
points, it may be better to judge the uncertainty for δ another way."]

Figure 10: It looks like we can fit Gaussians to the RA = α values, since the values are
spread out over many times the grid spacing. But the values of dec. = δ occupy only about 6 grid
points, it may be better to judge the uncertainty for δ another way.

Fit a Gaussian, or two, to the histogram for RA = α of H_min.

In[203]:= sortHαMin = Sort[Table[Hminα[[i2]], {i2, Length[Hminα]}]];
μαMinB = sortHαMin[[Floor[(4/5) Length[sortHαMin]]]] - μαMinB;
histogramRangeMin = {μαMinB - 5 σαMinB, μαMinB + 5 σαMinB, 0.4 μαMinB};
hlMin = HistogramList[sortHαMin, histogramRangeMin];
{Length[hlMin[[1]]], Length[hlMin[[2]]]};
hlMin = Table[{(1/2) hlMin[[1, i1]] + hlMin[[1, i1 + 1]], hlMin[[2, i1]]},
{i1, Length[hlMin[[2]]]}];
nlmαMinB2 = NonlinearModelFit[hlMin, a1 Exp[-(1/2.) ((x - x01)/b1)^2] +
a2 Exp[-(1/2.) ((x - x02)/b2)^2], {(a1, 1600.), (b1, 0.08),
{x01, 2.2}, (a2, 800.), (b2, 0.04), (x02, 2.35)}, x];(*x is α*)
In[211]:= showNLMαMinB2 = Show[
{Plot[Normal[nlmαMinB2], {x, μ0αMinB - 5 σαMinB, μ0αMinB + 5 σαMinB},
  PlotLabel -> "α for H\textsubscript{min}", AxesLabel -> {"α, radians", "ΔR"},
  PlotRange -> {0, 1.1 a1/ nlmαMinB2["BestFitParameters"]}],
 Histogram[sortHαMin, histogramRangeMin, PlotLabel -> "α for H\textsubscript{min}"
  PlotRange -> {0, 700}],
 ListPlot[hlMin, PlotLabel -> "α for H\textsubscript{min}"
  PlotRange -> {0, 1.1 a1/ nlmαMinB2["BestFitParameters"]}],
  PlotRange -> {0, 1.1 a1/ nlmαMinB2["BestFitParameters"]}]
]

Print["Figure 11: The two-peak Gaussian fit to the H\textsubscript{min} hub’s RA = α histogram. The bin width Δα is Δα = 0.4 σαMinB, radians."]

Figure 11: The two-peak Gaussian fit to the H\textsubscript{min} hub’s RA = α histogram. The bin width Δα is Δα = 0.0328535 radians.

In[213]:= parametersNLMαMinB2 = {a1, b1, x01, a2, b2, x02} /.
 nlmαMinB2["BestFitParameters"];
 pTableNLMαMinB2 = nlmαMinB2["ParameterTable"]
{σαMinFit1, αMinFit1} = {parametersNLMαMinB2[[2]], parametersNLMαMinB2[[3]]} (*radians*)
{σαMinFit2, αMinFit2} = {parametersNLMαMinB2[[5]], parametersNLMαMinB2[[6]]} (*radians*)

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Standard Error</th>
<th>t-Statistic</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a1 1805.08</td>
<td>70.3942</td>
<td>25.6425</td>
<td>3.33296 × 10^{-16}</td>
</tr>
<tr>
<td>b1 0.0584131</td>
<td>0.00319401</td>
<td>18.2883</td>
<td>1.60906 × 10^{-13}</td>
</tr>
<tr>
<td>x01 2.17341</td>
<td>0.00287388</td>
<td>756.265</td>
<td>5.12954 × 10^{-44}</td>
</tr>
<tr>
<td>a2 734.703</td>
<td>98.6679</td>
<td>7.44622</td>
<td>4.78239 × 10^{-7}</td>
</tr>
<tr>
<td>b2 0.0277968</td>
<td>0.00459073</td>
<td>60.05</td>
<td>7.99152 × 10^{-6}</td>
</tr>
<tr>
<td>x02 2.31896</td>
<td>0.00492096</td>
<td>471.242</td>
<td>4.10247 × 10^{-40}</td>
</tr>
</tbody>
</table>

Out[214]=

In[217]:= Print["The uncertainty runs produce two peaks for the RA of H\textsubscript{min}, one peak more likely than the other."]
 Print["The more likely peak has α = \"360.\/(2.\π)\" ° ± \", σαMinFit1(360.\/(2.\π)) ° and the less likely peak has α = \", αMinFit2(360.\/(2.\π)) °. "]

The uncertainty runs produce two peaks for the RA of H\textsubscript{min}, one peak more likely than the other. The more likely peak has α = 124.527° ± 3.34682 ° and the less likely peak has α = 132.867° ± 1.59264 °.
\[ H_{\text{minAVE}} = \left( \frac{1}{\text{Length}[H_{\text{min}}]} \right) \text{Sum}[H_{\text{min}}[[\{i4\}]], \{i4, \text{Length}[H_{\text{min}}]\}] \]

(* average α for H_{\text{max}} in radians *)

Print["Also note that, averaging over all runs, the average α for H_{\text{min}} in degrees is \(125.637^\circ\).""]

Next, attempt to fit a Gaussian to the δ for H_{\text{min}}. (Spoiler alert: The attempt fails.)

Definitions: For most quantities below for δ, replace “α” with “δ” in the quantities defined above for RA = α.

- \(H_{\text{minδ}}\): the locations (α,δ) of the hubs H_{\text{min}}
- lpHmin: list plot of the locations (α,δ) of the hubs H_{\text{min}}
- αMinj, δMinj: values needed to draw the uncertainty boxes

Figure 12: For H_{\text{min}}, the dec = δ has only 6 distinct values separated by the grid spacing \(2^\circ\). By inspection of this plot, let the "δ = -29° ± 1.5° = -0.506 ± 0.026 radians."
\text{\texttt{In[225]}:} = \texttt{Hmin\delta AVE = (1/\text{Length}[Hmin\delta]) \text{Sum}[Hmin\delta[[i4]], \{i4, \text{Length}[Hmin\delta]\}];}
\text{(* average \(\delta\) for \(H_{\text{max}}\) in radians *)}
\text{Print["Note that, averaging over all runs, the average \(\delta\) for \(H_{\text{min}}\) in degrees is", \(H_{\text{min}}\delta \text{AVE (360}/(2. \pi))\), "\(\circ\), averaging over all runs."]}
\text{Note that, averaging over all runs, the average \(\delta\) for \(H_{\text{min}}\) in degrees is} 
\text{29.0036\(\circ\), averaging over all runs.}
\text{\texttt{In[227]}:} = \texttt{Print["Allowing the measured PPA \(\psi = \psi n + \delta \psi\), with \(\delta \psi\) consistent with reported \(\sigma \psi\) uncertainties, produces a value of \((\text{RA,dec}) = (\alpha, \delta)\) for \(H_{\text{min}}\) of \((\alpha, \delta) = "\}, \alpha_{\text{MinFit1}}(360./ (2. \pi)), \delta_{\text{MinFit}}(360./ (2. \pi))\}, " + ", \alpha_{\text{MinFit1}}(360./ (2. \pi)), \sigma_{\text{MinFit}}\{360./ (2. \pi)\})], ", in degrees, \(\alpha\) according to the Gaussian fit and \(\delta\) by eye."]
\text{Allowing the measured PPA \(\psi = \psi n + \delta \psi\), with \(\delta \psi\) consistent with reported \(\sigma \psi\) uncertainties, produces a value of \((\text{RA,dec}) = (\alpha, \delta)\) for \(H_{\text{min}}\) of \((\alpha, \delta) = \{124.527, -29\}) \pm \{3.34682, 1.5\}, in degrees, \(\alpha\) according to the Gaussian fit and \(\delta\) by eye.}
\text{\texttt{In[228]}:} = \texttt{Print["The best values \(\psi n\) produce an alignment hub \(H_{\text{min}}\) with \((\text{RA,dec}) = \"\), \{\alpha_{\text{MinDegrees}}, \delta_{\text{MinDegrees}}\}\]}
\text{The best values \(\psi n\) produce an alignment hub \(H_{\text{min}}\) with \((\text{RA,dec}) = \{121.361, -30\}).}
\text{\texttt{In[229]}:} = \texttt{\{\alpha_{\text{MinFit1}}, \alpha_{\text{MinFit1}}\} = \{\text{parametersNLMaxMinB2[[2]], parametersNLMaxMinB2[[3]]}`; \(\{\text{radians}\}\); \{\alpha_{\text{MinFit2}}, \alpha_{\text{MinFit2}}\} = \{\text{parametersNLMaxMinB2[[5]], parametersNLMaxMinB2[[6]]}`; \(\{\text{radians}\}\);}
\text{\texttt{In[231]}:} = \texttt{\{\text{Plot the values for \(H_{\text{min}}\). }\} \text{\texttt{Hmin\alpha = Sort[Table[Hmin\alpha[[i5]], Hmin\alpha[[i5]]], \{i5, \text{Length}[Hmin\alpha]\}];}}
\text{\{Hmin\alpha[[1]], Hmin\alpha[[[-1]]\} \{360./ (2. \pi)\}; \(\{\text{\text{radians}\} \} \text{\texttt{Hmin\alpha[[1]], Hmin\alpha[[[-1]]\} \{360./ (2. \pi)\}; \(\{\text{\text{degrees}\} \} \text{\texttt{lpHmin = ListPlot[Hmin\alpha\{360./ (2. \pi)\}],}} \text{\texttt{PlotRange} \rightarrow \{(0, 180), (-90, 90)\}, \text{\texttt{PlotMarkers} \rightarrow \text{Automatic},}} \text{\texttt{AxesLabel} \rightarrow \{(\text{RA, degrees\)), \"(dec, degrees\)), \text{\texttt{PlotLabel} \rightarrow \text{\"Locations of the H_{\text{min}} hubs\"}}\};}
\text{\texttt{a1Min1} = \{\alpha_{\text{MinFit1}} - \alpha_{\text{MinFit1}}, 360./ (2. \pi)\}; \texttt{a2Min1} = \{\alpha_{\text{MinFit1}} + \alpha_{\text{MinFit1}}, 360./ (2. \pi)\}; \texttt{\delta1Min1} = \{\delta_{\text{MinFit1}} - \delta_{\text{MinFit1}}, 360./ (2. \pi)\}; \texttt{\delta2Min1} = \{\delta_{\text{MinFit1}} + \delta_{\text{MinFit1}}, 360./ (2. \pi)\};}
\text{\texttt{a1Min2} = \{\alpha_{\text{MinFit2}} - \alpha_{\text{MinFit2}}, 360./ (2. \pi)\}; \texttt{a2Min2} = \{\alpha_{\text{MinFit2}} + \alpha_{\text{MinFit2}}, 360./ (2. \pi)\}; \texttt{\delta1Min2} = \{\delta_{\text{MinFit2}} - \delta_{\text{MinFit2}}, 360./ (2. \pi)\}; \texttt{\delta2Min2} = \{\delta_{\text{MinFit2}} + \delta_{\text{MinFit2}}, 360./ (2. \pi)\};}
Show[lpHmin,
      Graphics[{Line[{{α1Min1, δ1Min1}, {α1Min1, δ2Min1}, {α2Min1, δ1Min1},
         {α1Min1, δ1Min1}}], Line[{{α1Min2, δ1Min2}, {α1Min2, δ2Min2},
         {α2Min2, δ1Min2}, {α1Min2, δ1Min2}}]}
      ]],
      Print["Figure 13: The locations of the alignment hubs \( H_{\text{min}} \) from the uncertainty runs. The boxes outline the most likely locations on the left and a less likely cluster on the right."
      ]

Figure 13: The locations of the alignment hubs \( H_{\text{min}} \) from the uncertainty runs. The boxes outline the most likely locations on the left and a less likely cluster on the right.

7e. Location of the Avoidance Hubs \( H_{\text{max}} \)

Find the likelihood of the location \((α, δ)\) of \( H_{\text{max}} \) in the uncertainty runs. See the comments at the start of Sec. 7d.

Definitions: Explore the definitions for \( H_{\text{min}} \) at the start of Sec. 7d. Find the similarly named quantity by interchanging Max for Min.

Adjust the definition to the present context.

(* Move hubs, if necessary, so that \(-180° ≤ α < 180°\)*)

Hmaxα0 = HmaxαData;
Hmaxδ0 = HmaxδData;
HmaxαBy180n = Round[Hmaxα0 / π];
Hmaxα1 = Table[Hmaxα0[[i1]] - HmaxαBy180n[[i1]] π, {i1, Length[Hmaxα0]}];
Hmaxδ1 = Table[(-1)HmaxαBy180n[[i1]] Hmaxδ0[[i1]], {i1, Length[Hmaxδ0]}];
Figure 14: By the symmetry across a diameter, the band at positive \( \alpha \) can be moved to negative values by deducting \( \pi \) radians from each positive value of \( \alpha \). The move changes the sign of \( \delta \).

Figure 15: The avoidance hubs found in the uncertainty runs occupy a wide range for both \( \alpha \) and \( \delta \). Both ranges span intervals much larger than the grid spacing. So fitting \( \alpha \) and \( \delta \) with Gaussians is appropriate.
Fit a Gaussian to the $\alpha$ for $H_{\text{max}}$.

```math
\text{sortH}\alpha\text{Max} = \text{Sort}\left[\text{Table}[H\text{max}[[i]], \{i, \text{Length}[H\text{max}]\}]\right];
\mu\alpha\text{MaxB} = \text{sortH}\alpha\text{Max}[\lfloor \text{Length}[\text{sortH}\alpha\text{Max}] / 2 \rfloor];
\sigma\alpha\text{MaxB} = \text{sortH}\alpha\text{Max} - \mu\alpha\text{MaxB};
\text{histogramRangeMax} = (\mu\alpha\text{MaxB} - 10 \sigma\alpha\text{MaxB}, \mu\alpha\text{MaxB} + 5 \sigma\alpha\text{MaxB}, 0.4 \sigma\alpha\text{MaxB});
\text{hl}\alpha\text{Max} = \text{Table}\left[\left\{(1/2) \left(\text{hl}\alpha\text{Max}[[i]] + \text{hl}\alpha\text{Max}[[i+1]]\right), \text{hl}\alpha\text{Max}[[i]], \text{hl}\alpha\text{Max}[[i+1]]\right\}\right];
\text{nln}\alpha\text{MaxB} = \text{NonlinearModelFit}\left[\text{hl}\alpha\text{Max}, a \exp\left(-\frac{1}{2} \left(\frac{x-x0}{b}\right)^2\right),\left\{a, \text{Length}[H\text{max}] / 6\right\}, \{x0, \mu\alpha\text{MaxB}\}\right];
\text{normalNln}\alpha\text{MaxB} = \text{Normal}[\text{nln}\alpha\text{MaxB}];
```

```math
\text{showNln}\alpha\text{MaxB} = \text{Show}\left[\left\{\text{Plot}[\text{Normal}[\text{normalNln}\alpha\text{MaxB}],\{x, \mu\alpha\text{MaxB} - 10 \sigma\alpha\text{MaxB}, \mu\alpha\text{MaxB} + 5 \sigma\alpha\text{MaxB}\}, \text{AxesLabel} \rightarrow \{"\alpha", \text{radians}", \"\Delta R\"\},\text{PlotLabel} \rightarrow \"\alpha \text{ for } H_{\text{max}}\", \text{PlotRange} \rightarrow \{0, 1.1 \text{ a} / \text{nln}\alpha\text{MaxB["BestFitParameters"]}\}]\right\},\right\}\right]
```

Print["Figure 16: The Gaussian fit to the $H_{\text{max}}$ hub's RA = $\alpha$ histogram. One could argue that as many as four Gaussians could be used to fit the data better, two near the peak, one down the left slope and one just before the tail on the left side. See Fig. 11, for the similar situation with $H_{\text{min}}$. For the RAs of $H_{\text{max}}$, however, we prefer to stay with the single Gaussian fit."]

Figure 16: The Gaussian fit to the $H_{\text{max}}$ hub’s RA = $\alpha$ histogram. One could argue that as many as four Gaussians could be used to fit the data better, two near the peak, one down the left slope and one just before the tail on the left side. See Fig. 11, for the similar situation with $H_{\text{min}}$. For the RAs of $H_{\text{max}}$, however, we prefer to stay with the single Gaussian fit.
In[265]:= parametersNLMMaxB = {a, b, x0} /. nlmMaxB["BestFitParameters"];
parametersNLMMaxB = nlmMaxB["ParameterTable"];
{aMaxFit, bMaxFit} = {parametersNLMMaxB[[2]], parametersNLMMaxB[[3]]};
(* Gaussian values, radians *)

\[\begin{array}{|c|c|c|c|c|}
\hline
\text{Estimate} & \text{Standard Error} & \text{t-Statistic} & \text{P-Value} \\
\hline
a & 1124.19 & 39.1823 & 28.6914 & 2.06109 \times 10^{-25}
\hline
b & 0.197621 & 0.00795386 & 24.846 & 2.21108 \times 10^{-23}
\hline
x0 & -1.45215 & 0.00795326 & -182.586 & 1.86614 \times 10^{-52}
\hline
\end{array}\]

\textbf{Out[266]=}

In[268]:= Print["The uncertainty runs produce a value of RA = \(\alpha\) for H\textsubscript{max} of \(\alpha = \), 
\(\alpha\text{MaxFit} (360. \div (2. \pi))\), "\(° \pm \), "\(\sigma\text{MaxFit} (360. \div (2. \pi))\), "\(° ." \]
The uncertainty runs produce a value of RA = \(\alpha\) for H\textsubscript{max} of \(\alpha = -83.2022 ° \pm 11.3229 °\).

In[269]:= H\text{max AVE} = \(1 \div \text{Length[Hmax]}\) \text{Sum[Hmax[[i4]], {i4, Length[Hmax]}]};
(* average \(\alpha\) for H\textsubscript{max} in radians *)
Print["Note that, averaging over all runs, the average \(\alpha\) for H\textsubscript{max} in degrees is " , 
\(\text{Hmax AVE} (360 \div (2. \pi))\), "° ."
Note that, averaging over all runs, the average \(\alpha\) for H\textsubscript{max} in degrees is \(-85.5611° .

Fit a Gaussian to the \(\delta\) for H\textsubscript{max}.

In[271]:= sortH\delta Max = Sort[Table[Hmax\delta [[i2]], {i2, Length[Hmax\delta ]}]]; 
\(\mu\delta MaxB = \text{sortH\delta Max}[[2]]\)/2];
\(\sigma\delta MaxB = 0.09;\)
histogramRangeMax = \{\(\mu\delta MaxB - 5 \sigma\delta MaxB\), \(\mu\delta MaxB + 3 \sigma\delta MaxB\), 0.4 \sigma\delta MaxB\};
hlMax = HistogramList[sortH\delta Max, histogramRangeMax];
{Length[ hlMax[[1]]], Length[ hlMax[[2]]]});
h\delta Max = Table[\{1/2 \(hlMax[[1, i1]] + hlMax[[1, i1 + 1]]\), hlMax[[2, i1]]\}, 
\{i1, Length[ hlMax[[2]]]\}];
nlm\delta MaxB = NonlinearModelFit[h\delta Max, \(\epsilon \div (1/2.) \times (x - x0) / b\)^2, 
\{a, Length[Hmax\delta ]/6], \{b, \sigma\delta MaxB\}, \{x0, \(\mu\delta MaxB\}\}]\);(*x is \(\delta\), y is \(\Delta R\) *)
normal\delta MaxB = Normal[nlm\delta MaxB];
showNLM\deltaMaxB = Show[{Plot[Normal[nlm\deltaMaxB], {x, \mu_0 \deltaMaxB - 5 \sigma\deltaMaxB, \mu_0 \deltaMaxB + 5 \sigma\deltaMaxB}, PlotLabel -> "\delta for H max", PlotRange -> All, AxesLabel -> {"\delta, radians", "\Delta R"}], Histogram[sortHMax, histogramRangeMax, PlotLabel -> "\delta for H max", PlotRange -> {0, 700}], ListPlot[hl\deltaMax, PlotLabel -> "\delta for H max"], Plot[Normal[nlm\deltaMaxB], {x, \mu_0 \deltaMaxB - 5 \sigma\deltaMaxB, \mu_0 \deltaMaxB + 5 \sigma\deltaMaxB}, PlotLabel -> "\delta for H max", PlotRange -> {0, 700}]}]

Print["Figure 17: The Gaussian fit to the H max hub's dec = \delta histogram. The bin width \Delta \delta is \Delta \delta = 0.036 radians, which is just a little wider than the grid spacing, d\theta = d\theta, radians."

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17.png}
\caption{The Gaussian fit to the H max hub's dec = \delta histogram. The bin width \Delta \delta is \Delta \delta = 0.036 radians, which is just a little wider than the grid spacing, d\theta = d\theta, radians.}
\end{figure}

The uncertainty runs produce a value of declination \delta for H max of \delta = -55.2793 \pm 3.56496 \degree.

Hmax\deltaAVE = (1/Length[Hmax\delta]) Sum[Hmax\delta[[i4]], {i4, Length[Hmax\delta]}];
(* average \delta for H max in radians *)

Print["Note that, averaging over all runs, the average \delta for H max is \delta, Hmax\deltaAVE \{360 \div (2 \pi)\}, \"\degree \.."

Note that, averaging over all runs, the average \delta for H max is -55.2616 \degree.
The uncertainty runs give the location of $H_{\text{max}}$ as $(\alpha, \delta) =$ 

\[
\left\{ \alpha_{\text{MaxFit}} - 360. \pi, \delta_{\text{MaxFit}} - 360. \pi \right\}, \left\{ \alpha_{\text{MaxFit}} + 360. \pi, \delta_{\text{MaxFit}} + 360. \pi \right\},
\]

in degrees.

The uncertainty runs give the location of $H_{\text{max}}$ as $(\alpha, \delta) =$ 

\[-83.2022, -55.2793 \pm 11.3229, 3.56496\], in degrees.

(* Plot the values for $H_{\text{max}}$. *)

\[
H_{\alpha} = \text{Table}[[H_{\alpha}[i8]], \{i8, \text{Length}[H_{\alpha}])\};
\]

\[
H_{\alpha} = \{H_{\alpha}[1], H_{\alpha}[-1]\}; (*\text{radians}*)
\]

\[
lpH_{\alpha} = \text{ListPlot}[H_{\alpha} \left\{ \right\}, \text{PlotRange} \rightarrow \left\{ \right\}, \text{PlotMarkers} \rightarrow \text{Automatic}, \text{AxesLabel} \rightarrow \left\{ \right\}, \text{PlotLabel} \rightarrow \left\{ \right\};
\]

\[
\alpha_{1\text{Max}} = (\alpha_{\text{MaxFit}} - \sigma_{\alpha_{\text{MaxFit}}} - 360. \pi); \alpha_{2\text{Max}} = (\alpha_{\text{MaxFit}} + \sigma_{\alpha_{\text{MaxFit}}} - 360. \pi);
\]

\[
\delta_{1\text{Max}} = (\delta_{\text{MaxFit}} - \sigma_{\delta_{\text{MaxFit}}} - 360. \pi); \delta_{2\text{Max}} = (\delta_{\text{MaxFit}} + \sigma_{\delta_{\text{MaxFit}}} - 360. \pi);
\]

(* Plot of $H_{\alpha}$ hubs with the most likely region indicated *)

\[
\text{Show}[[lpH_{\alpha}, \text{Graphics}[\text{Line}[\left\{\{\alpha_{1\text{Max}}, \delta_{1\text{Max}}\}, \{\alpha_{1\text{Max}}, \delta_{2\text{Max}}\}, \{\alpha_{2\text{Max}}, \delta_{1\text{Max}}\}, \{\alpha_{2\text{Max}}, \delta_{2\text{Max}}\}\right\}]])
\]

Print["Figure 18: The locations of the alignment hubs $H_{\alpha}$ from the uncertainty runs. The 
box outlines the most likely locations according to the Gaussian fits for $\alpha$ and $\delta.$"]

Figure 18: The locations of the alignment hubs $H_{\alpha}$ from the uncertainty runs. The 
box outlines the most likely locations according to the Gaussian fits for $\alpha$ and $\delta.$

7f. Map of the Hubs for the Uncertainty Runs

In this subsection, we map the locations of the alignment hubs $H_{\alpha}$ and the locations of the avoidance hubs $H_{\text{max}}$, one set for each uncertainty run.

Definitions:

\[
\text{xyAitoffH_{\alpha}} \quad \text{Aitoff coordinates for the alignment hubs $H_{\alpha}$ from the uncertainty runs}
\]

\[
\text{xyAitoffH_{\text{max}}} \quad \text{Aitoff coordinates for the avoidance hubs $H_{\text{max}}$ from the uncertainty runs}
\]
The Aitoff coordinates for the hubs $H_{\text{min}}$ locations.

\[
\text{xyAitoffHmin} = \text{Table}[[xH[Hmin][n](360/(2\pi)), \text{Hmin}[n](360/(2\pi))], \{n, \text{Length[Hmin]}\}];
\]

The Aitoff coordinates for the hubs $H_{\text{max}}$ locations.

\[
\text{xyAitoffHmax} = \text{Table}[[xH[Hmax][n](360/(2\pi)), \text{Hmax}[n](360/(2\pi))], \{n, \text{Length[Hmax]}\}];
\]

The Aitoff coordinates for the hubs $-H_{\text{min}}$ locations.

\[
\text{xyAitoffOppositeHmin} = \text{Table}[[xH[If[0 \leq \text{Hmin}[n](360/(2\pi)) < 180, \text{Hmin}[n](360/(2\pi)) - 180, If[0 > \text{Hmin}[n](360/(2\pi)) > -180, \text{Hmin}[n](360/(2\pi)) + 180]], \text{Hmin}[n](360/(2\pi))]], \{n, \text{Length[Hmin]}\}];
\]

The Aitoff coordinates for the hubs $-H_{\text{max}}$ locations.

\[
\text{xyAitoffOppositeHmax} = \text{Table}[[xH[If[0 \leq \text{Hmax}[n](360/(2\pi)) < 180, \text{Hmax}[n](360/(2\pi)) - 180, If[0 > \text{Hmax}[n](360/(2\pi)) > -180, \text{Hmax}[n](360/(2\pi)) + 180]], \text{Hmax}[n](360/(2\pi))]], \{n, \text{Length[Hmax]}\}];
\]
(*Construct the map of \(H_{\text{min}}\) and \(H_{\text{max}}\) hubs with ± regions indicated.*)

\[\text{mapOfgHminHmax = }
\text{Show}[[\text{Table}[
\text{ParametricPlot}[[xH[\alpha, \delta], yH[\alpha, \delta]], \{\delta, -90, 90\}, \text{PlotStyle} \rightarrow \{\text{Black, Thickness[0.002]}\},
\text{PlotPoints} \rightarrow 60, \text{PlotRange} \rightarrow \{(-7, 7), (-3, 3)\}, \text{Axes} \rightarrow \text{False}]], \{\alpha, -180, 180, 30\}]]
\text{, Table}[\text{ParametricPlot}[[xH[\alpha, \delta], yH[\alpha, \delta]], \{\alpha, -180, 180\},
\text{PlotStyle} \rightarrow \{\text{Black, Thickness[0.002]}\}, \text{PlotPoints} \rightarrow 60, \text{PointSize[0.007]}, \text{Text}[[\text{StyleForm}["N", \text{FontSize} \rightarrow 10, \text{FontWeight} \rightarrow \text{"Plain"}], \{0, 1.85\}],
\text{LightBlue}, \{\alpha\text{Hmin} \rightarrow \text{Point}[[xH\text{AlitoffHmin}], \{\alpha\text{Hmax} \rightarrow \text{Point}[[xH\text{AlitoffOppositeHmin}],
\text{LightRed}, \text{Arrow}[[\text{BezierCurve}[[xH[180, 0], yH[0, -70]], \{-2.3, -2.0\}],
\text{Text}[[\text{StyleForm}["Max", \text{FontSize} \rightarrow 8, \text{FontWeight} \rightarrow \text{"Bold"}], \{xH[180, 0], yH[0, -70]\}],
\text{Arrow}[[\text{BezierCurve}[[xH[180, 0], yH[0, -70]], \{2.3, 2.0\}],
\text{Text}[[\text{StyleForm}["Min", \text{FontSize} \rightarrow 8, \text{FontWeight} \rightarrow \text{"Bold"}], \{xH[-180, 0], yH[0, -70]\}],
\text{Arrow}[[\text{BezierCurve}[[xH[-180, 0], yH[0, -70]], \{2.3, -2.0\}],
\text{Text}[[\text{StyleForm}["Min", \text{FontSize} \rightarrow 8, \text{FontWeight} \rightarrow \text{"Bold"}], \{xH[-180, 0], yH[0, 70]\}],
\text{Arrow}[[\text{BezierCurve}[[xH[-180, 0], yH[0, 70]], \{-2.3, 2.0\}],
\text{Text}[[\text{StyleForm}["Max", \text{FontSize} \rightarrow 8, \text{FontWeight} \rightarrow \text{"Bold"}], \{xH[180, 0], yH[0, 70]\}],
\text{Arrow}[[\text{BezierCurve}[[xH[180, 0], yH[0, 70]], \{2.3, 2.0\}],
\text{Text}[[\text{StyleForm}["Min", \text{FontSize} \rightarrow 8, \text{FontWeight} \rightarrow \text{"Bold"}], \{xH[180, 0], yH[0, 70]\}],
\text{Arrow}[[\text{BezierCurve}[[xH[180, 0], yH[0, 70]], \{2.3, -2.0\}],
\text{Text}[[\text{StyleForm}["Max", \text{FontSize} \rightarrow 8, \text{FontWeight} \rightarrow \text{"Bold"}], \{xH[-180, 0], yH[0, 70]\}],
\text{Arrow}[[\text{BezierCurve}[[xH[-180, 0], yH[0, 70]], \{-2.3, -2.0\}]}}]
\]
Figure 19: The hubs found for the uncertainty runs. The hubs are represented as lightly shaded dots, light blue for alignment and pink for avoidance. The arrows point to the hubs found with the best values of the polarization directions $\psi$. Note that the clumps of light blue alignment hubs closely follow the areas of convergence (blue) in Fig. 6 and the clumps of pink avoidance hubs follow the areas of divergence (red) in Fig. 6. By shifting the polarization directions slightly due to experimental uncertainty, the locations of the extreme alignment angles shift. But the shift is small, favoring those areas that are near the extremes for the best values $\psi_n$ of the polarization directions.

Notes:

(i) The map is centered on (RA,dec) = (0°,0°).

(ii) The alignment hubs $H_{\text{min}}$ and $-H_{\text{min}}$ are plotted as light blue dots.

(iii) The avoidance hubs $H_{\text{max}}$ and $-H_{\text{max}}$ are plotted as pink dots.

(iv) The regions $(\alpha,\delta) \pm (\sigma\alpha,\sigma\delta)$ where the hubs $H$ and $-H$ are most likely found are enclosed in purple lines.

8. Concluding Remarks

The polarization directions of the 106 QSOs are very well aligned. The minimum alignment angle $\eta_{\text{min}}$, with a best value $\eta_{\text{min}} = 30.9^\circ$, is extremely low for this many QSOs. The likelihood that random polarization directions, i.e. the significance, would be better aligned is extremely small. We find in Sec. 6a, that the significance of the best value for $\eta_{\text{min}}$ is $7.4^{+2.4}_{-1.8} \times 10^{-8}$, or about one in ten million. The plus/minus values are due to the finite number of random runs and not due to experimental uncertainties. In
general, the more random runs the smaller these plus/minus values. Random runs do not use any measured polarization directions. The accuracy of the statistics improves with increasing the number of runs, not with more accurate data.

In Sec. 7b, by taking into account the uncertainty in the measured polarization directions, the uncertainty runs produce a range of minimum alignment angles $\eta_{\text{min}}$. The histogram of the results, Fig. 7, fits a normal distribution quite well. The mean value of $\eta_{\text{min}}$ at the peak of the distribution is $\eta_{\text{min}} = 31.6^\circ \pm 0.7^\circ$. One finds the significance of the alignment angle $\eta_{\text{min}}$ is $6.4 \times 47.9 \times 10^{-7}$, on the order of one in a few hundred thousand or more.

In contrast with random runs, making more uncertainty runs should not change the $+47.9$ and $-5.8$ displayed in the previous paragraph, except perhaps returning somewhat more accurate values. The plus/minus range reflects experimental data listed in the catalog.

By the uncertainty runs, the significance of the alignment most likely has a value of $6.4 \times 10^{-7}$, with $1/6.4 \times 10^{-7}$ or one in 1.6 million random runs having a better alignment. And, the significance is highly likely to be somewhere in the range from $(6.4 - 5.8) \times 10^{-7} = 0.6 \times 10^{-7}$ to $(6.4 + 47.9) \times 10^{-7} = 54.3 \times 10^{-7}$. Equivalently, the significance range implies that from one in 180 thousand to one in 16 million random runs have better alignment.

The analysis shows that alignment of the polarization directions of these 106 optical QSOs is extremely unlikely to be due to chance.

References

1. R. Shurtleff, the ready-to-run Mathematica version of this notebook is available, for a limited time, at the following URL: https://www.dropbox.com/s/dhuq6uo5k1kq98i/20210205IntermediateKitFor193BestOpticalQSORegions2.nb?dl=0 (2021).
7. The catalog in Ref. 6 is also available online at http://cdsarc.u-strasbg.fr/viz-bin/qcat?J/A+A/441/915