螺度导致的湍流压缩性下降及非普适性
(Turbulence Compressibility Reduction and Nonuniversality with Helicity)

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Abstract

The turbulence compressibility reduction with helicity is numerically verified and characterizes in some detail the universality and nonuniversality with respect to helical and nonhelical large-scale forcing. The nonuniversality deep in the dissipation range spectra is identified to be only in the power-law prefactors, with smaller exponents for the helical case.
The classical universal equilibrium theory of turbulence is not directly supported by the systematic results of nonlinear Langevin, Lorentz and Burgers equations by Frisch and Morf [1] (FM81), thus is either wrong or upheld by extra dynamical ingredients beyond those ‘poorer’ systems analyzed by the latter; if correct, it should also be applicable for ‘richer’ systems such as compressible fluids and plasmas under appropriate conditions. Thus, with the corresponding analytical solution of turbulence being not available, the dissipation-range spectra [2] (also Fig. 3 of Ref. [3]) should have been examined with (helical) fluid and plasma flows of different degrees of chirality, particularly for the mutual enlightenment with the crucial helicity effect issue [4]: for example, despite the various advancements (e.g., Refs. [5, 6, 7] and references therein), knowledge about helicity is still insufficient for a good understanding and control of turbulence. Compressible turbulence offers extra physical quantities and channels for dioganization, thus may be a useful analysis tool, besides it own physical relevance [8, 9, 10, 11].

Unlike other investigations on the definite shapes, such as the scaling exponent, of turbulence spectra [10, 11], the recent statistical and mechanical/geometrical analyses [9] indicates that, compared to nonhelical flows, a turbulence with helicity would bear less proportions of the energy of compressibility relevant fluctuations (thus the ‘noise’), such as those of the compressive, density and internal-energy modes, thus the notion of ‘fastening’. The prediction, updating that of Kraichnan [12] (K55) and extending to plasma flows, however is qualitative and should not be considered as the complete systematic result, calling for numerical tests. Actually, direct numerical simulations were soon performed for preliminary verification [13], while at the same time it turned out possible to show that some numerical result of the Gross-Pitaevskii equation modeling the low-temperature superflow or Bose-Einstein condensate [14, 15] was also consistent with the notion of turbulence compressibility reduction with helicity. Besides the multidisciplinary universality, such fundamental research is also closely relevant to practical applications in civil engineering, aeronautics, rapid public transportation and car transportation [17, 18], just to mention the aeroacoustic aspect [16].

FM81’s argument relies on (complex) singularities, which is somehow echoed with the ending remark, that “singularities are often sources of information”, in the recent perspective on turbulence cascade [3]. [Note that regularization of the flow with helicity has recently found mathematical support [19].] But, to our best knowledge, no such work on compressible Navier-Stokes or other plasma models exists, though relevant ideas and techniques have kept evolving and long been extended to various hydrodynamics-type systems, including the familiar Burgers, incompressible Euler, Navier-Stokes and magnetohydrodynamics [20]: it is true that direct numerical tracking and analysis of the complex singularities of three-dimensional compressible
flows would be more nontrivial, which however should not prevent the application of the idea, especially in interpreting the detailed results. This note will be focusing on the numerical tests and spectral analysis of the combination of the two notions, the (non)universality and helicity (fastening) effect.

Let’s consider the adiabatic (actually isothermal) state, with the relation between pressure $p$ and density $\rho$

$$p = c^2 \rho \quad \text{and} \quad \rho = \rho_0 e^\zeta$$

with an equilibrium $\rho_0$, (1)

and the equations of motion

$$\partial_t \zeta + \zeta_\alpha u_\alpha + u_\alpha,\alpha = 0, \quad (2)$$

$$\partial_t u_\lambda + u_\sigma u_{\lambda,\sigma} + c^2 \zeta_\lambda - \nu \theta_{\lambda,\sigma} = 0, \quad (3)$$

where $\theta_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} - \frac{2}{3} \delta_\beta^\gamma u_{\sigma,\sigma}$, $(\bullet)_\gamma = \partial(\bullet)/\partial x^\gamma$ and where $u$, $\nu$ and $c$ denote the velocity, viscosity and speed of sound. [For simplicity of presentation, both $c$ and $\rho_0$ are taken to be unit (= 1) for the time being, with appropriate choice of scales and units, but sometimes we still let them present explicitly to to emphasize the physical reality.] We work in a cyclic box of dimension $2\pi$ with $V = [0, 2\pi]^3$ and applying Fourier representation for all the dynamical variables $v(r) \rightarrow \hat{v}(k)$, say, $u(r) = \sum_k \hat{u}(k) \exp(ik \cdot r)$ with $\hat{k}^2 = -1$. K55 constructs a phase space by the real and imaginary parts of $v$s. Galerkin truncation, imposing all modes with $k = |k|$ greater than some cut-off value $K$ to be zero, is performed, which does not change the (approximate) invariance of energy $E$. K55 expects an ensemble of systems tend to approach an absolute statistical equilibrium state with equipartition of energy among various modes.

The Fourier coefficient of velocity can be further decomposed into left- and right-handed transversal modes, and, the parallel mode (c.f., Ref. [9] and references therein)

$$\hat{u}(k) = \hat{u}_+(k)\hat{h}_+(k) + \hat{u}_-(k)\hat{h}_-(k) + \hat{u}_i(k)k/k,$$

with the property of the helical basis $\hat{k} \times \hat{h}_s = sk\hat{h}_s$ (for $\hat{k}^2 = -1$ and $s = \pm$), and that,

$$E = \frac{1}{2} \sum_k |\hat{u}_+|^2 + |\hat{u}_-|^2 + |\hat{u}_i|^2 + c^2 |\hat{\zeta}|^2,$$

$$\mathcal{H} = \frac{1}{2} \sum_k k|\hat{u}_+|^2 - k|\hat{u}_-|^2,$$

the latter being the invariant helicity in ideal barotropic flows [4] and, as in the incompressible case [21], should be respected in compressible turbulence.
The canonical distribution \( \sim \exp\{- (\alpha E + \beta H) \} \) for the absolute-equilibrium indicates that the proportion of compressibility-relevant-mode energy may be reduced with helicity (c.f. Ref. [9] for more details and extra arguments and extensions).

For numerical checks, we performed two sets of direct numerical simulations of the compressible Navier-Stokes equations in a cyclic box discretized by 1024\(^3\) grids, stirred by isotropic, white in time, solenoidal forcing at low wavenumbers, with and without helicity injection. In our simulations, the solenoidal forces were added at around wavenumber \( k_f = 1.54 \) with random phase corresponding to each Fourier coefficient \( \hat{f}(k) \) of the force at each time, the same scheme in Ref. [22] for controlling the degree of chirality which is the only genuine difference between the two runs. The helical case is driven purely by the eigenfunctions of the curl operator. [Forcing only on the transversal modes is mainly from the consideration of simplicity and fairness in the comparison, though other forcing schemes with (statistically) ‘equal’ or ‘fair’ drivings on the longitudinal or even the other compressibility relevant modes should also work (but see the discussion of nonuniversality below). We adopted the set up of Ref. [22], with the magnetic field removed, and adjusted Pencil Code for our purposes.] Both cases are of the root mean square turbulent Mach number \( M_t \approx 0.4 \) and of Reynolds numbers \( Re \approx 250 \) in terms of Taylor micro-scales [the ‘\( n \) (on helical)’ case is slightly (6\%) higher than the ‘\( n \) (elical)’ one, \( Re_n = 257 \) versus \( Re_h = 243 \); and, for dissipation wavenumbers, \( k_{dh} = 31.7 \) versus \( k_{dh} = 30 \)]. And, for the analysis, we also introduce the one-dimensional spectra

\[
E_- := \frac{1}{2} \sum_{|k|=k} |\hat{u}_-|^2, \quad E_+ := \frac{1}{2} \sum_{|k|=k} |\hat{u}_+|^2, \\
E_0 := \frac{1}{2} \sum_{|k|=k} |\hat{u}_0|^2 \quad \text{and} \quad E(k) := E_- + E_+ + E_0.
\]  

(7)

[The helicity spectrum is given by \( H(k) = k E_+ - k E_- \). A specific statistical result is that \( E_0 \) is of smaller fraction for a helical, compared to the nonhelical, flow, as specifically realized in Fig. 1 by, respectively,

\[
E_+ \neq E_- \quad \text{and} \quad E_+ = E_-.
\]

(8)

The upper panels of Fig. 1 present various spectra normalized by \( E(k_f) \), including the Kolmogorov spectrum (‘K41’) [23] \( \propto k^{-5/3} \) for reference, while the density spectrum \( R(k) := \sum_{|k|=k} |\hat{\rho}(k)|^2 \) and compressive/parallel spectrum \( E_0(k) \) normalized by \( E(k) \) are plotted in lower panels of Fig. 1. [Dividing the spectra by \( E(k_f) \) or \( E(k) \) effectively cancels \( Re \), and similarly is \( k/k_d \) [23]. We will come back with more remarks on the normalizations.] All plots show smaller (fractions of) \( E_0 \) and \( R \) spectra for the helical case,
Figure 1: Upper panels: Various time-averaged one-dimensional energy/power spectra of velocity (components) for turbulence with and without helicity injection: the nonhelical case shows basically identical spectra of $E_+$ and $E_-$, while the helical case presents $E_+ > E_-$, visibly up to $k \approx 10$ with bare eyes, at small $k$. Due to the numerical noise problem, the results of compressive and density (not shown) modes are not reliable for $k > 6k_d$ or so (depending on the quantities).

Lower panels: Time-averaged spectra, normalized by $E(k)$, of the helical and non-helical cases, with short black straight lines added to designate the power laws. Also included in the lower-right panel are those with cross normalizations, i.e., nonhelical (resp. helical) $R(k)$ normalized by helical (resp. nonhelical) $E(k)$ denoted by ‘nhR/nhE’ (resp. ‘hR/nhE’) in the legend; two empirical adjustments, $R_n(k+k_{dn}/10)/E_h(k)$ (thin dotted line) and $R_n(k+k_{dn}/5)/E_h(k)$ (thin long-dashed line), together with $R_n(k_{dh}k/k_{dn})/E(k_{dh}k/k_{dn})$ (‘nonhelicalNk’), to account for the slightly different $Re$ and $k_d$ are appended, without legend, to check the robustness of the scaling laws with extra reasonable normalizations. The hooks for largest $k$s are due to numerical noise.

consistent verifying turbulence compressibility (thus ‘noise’) reduction with helicity.

Statistical convergence has been checked by comparing results from averages over different time periods $\Delta t$: actually, the dynamical time scales of turbulence in general vary (monotonically) with the wave lengths, and especially the fluctuations of the benefits at low wavenumbers around the
forced ones are large, leading to slow statistical convergence; but, the forced modes are not of the fundamental consideration in the nonlinear analysis [12, 9], thus not our main concern.

The energy of the longitudinal-velocity ($\hat{u}_l$), transverse-velocity ($\hat{u}_T$) and density ($\rho$) modes are damped by, respectively, $(4/3)\nu k^2$, $\nu k^2$ and $0$, thus, with roughly an average-dissipation wavenumber $k_d \approx 30$ though, in the order of magnitudes of dissipation rates ‘everything’ comes slightly ‘earlier’ (at smaller $k$, due to larger self-dissipation scale) for $E_l(k)$ than for $E(k)$, and even more so than $R(k)$: the last one is not shown in the upper panels of Fig. 1 but can be directly deduced from the comparison of the lower panels. Various scaling laws, ranging from $k^{-5/3}$ to $k^{-11/3}$, for the parallel-mode spectra in the (nearly) inertial range have been indicated [10]. Our data however do not clearly present very clean scale-free spectra for such compressibility relevant modes (neither for $E_l(k)$ in the up-right panel of Fig. 1), and for the time being we shall not address this issue.

Now related to FM81’s concern in the dissipation range, we observe in Fig. 1

$$E_l(k)/E(k) \sim k^{-1} \text{ and } R(k)/E(k) \sim k^{-5/3},$$

a new universality agreed by the helical and nonhelical cases. The above results indicate that, in either the helical or nonhelical case, the $E_l(k)$ and $R(k)$ are different from the $E(k)$ only up to power-law prefactors in the (stretch-)exponentially decaying far-dissipation range, which is consistent with the inspections of other quantities such as $E_l(k)/R(k)$ (not shown). We may then accordingly postulate the (asymptotic) ansatz for the four spectra, helical and nonhelical $R(k)$ and $E(k)$ (the $E_l$ relevant behaviors in Fig. 1 are consistent with but not as clean or as strongly indicative for such a postulation: see also Appendix)

$$\propto k^\alpha F(k).$$

$F(k)$ is some empirical (stretched) exponential function [2], with the argument $k$ usually normalized by $k_d$ in the K41 universality phenomenology, or both further ‘renormalized’ with $\ln Re$ in the multifractal-universality phenomenology [23, 2, 3]. The observed Eq. (9) deep in the dissipation range indicates that, with again ‘h(elical)’ and ‘n(onhelical)’ and the self-evident superscripts,

$$F_h^R(k) = F_h^E(k), \quad F_n^R(k) = F_n^E(k),$$

$$\alpha_h^R - \alpha_h^E = -5/3 = \alpha_n^R - \alpha_n^E. \quad (11)$$

The cross-normalizations added to the lower-right panel of Fig. 1 also present deep in the dissipation range

$$R_h/E_n \sim k^{-12/5} \text{ and } R_n/E_h \sim k^{-14/15}. \quad (12)$$

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There is no ‘classical’ phenomenology for normalizing $ks$ in our cross-normalizations, thus they are left unnormalized (or ‘naturally’ normalized by $k_0 = 1$) in this plot; but, since the two sets of data have very close $Res$ (even closer for $\ln Re$ in the multifractal-universality normalization [23]), reasonable extra normalizations will not essentially change the results: a K41 relative shift of $\log(k_{dn}/k_{dh}) \approx \log 1.06$, which is already seen in the other three panels to be tiny, for the line of $R_n(k)/E_n(k)$ to the left presents in this panel barely visible, thus negligible, difference. We have also checked in the graph that the scaling laws are robust with respect to yet two extra ‘empirical normalizations’. Thus, we deduce

$$F_h^R(k) = F_h^E(k) = F_n^R(k) = F_n^E(k)$$

and

$$\Delta \alpha = \alpha_n^E - \alpha_h^E = 11/15 = \alpha_n^R - \alpha_h^R,$$

with all exponents determined up to a constant $\gamma$. Evaluation of the common part, $k^\gamma F(k)$, by fitting the spectra however can be subtle due to unknown ansatz for $F(k)$ (but see Ref. [2]), and is not of our current interest.

Eq. (14) can be verified, a posteriori, by $E_h(k)/E_n(k)$ and $R_h(k)/R_n(k)$ and even extended to characterize $E_i$, with however much less obvious scaling behaviors to be used for derivation from the beginning, due to (numerical) errors (Appendix).

In general, the asymptotic spectrum is determined by the strengths, locations and distributions of the singularities [1], thus it appears on the first sight impossible to get any precise insights from our result of the steepened power-law prefactors for the helical case quantified by $\Delta \alpha$ in Eq. (14). However, associating the power-law prefactor to the strength/singular exponent and the exponential to the distance $\delta(t)$ away from the real domain for a single complex singularity [1, 20], it may be said that, effectively or over all, helicity reduces the short-wavelength “noise” in the real world with weakened/less singular, but not more distant, singularities. Such a ‘mean field’ statement is of course not precise and could be misleading, but may still be of value for further analytical studies and applications.

Now in the (nearly) inertial range, a scaling law $k^{\alpha_i}$, with inertial-range exponent $\alpha_i$ estimated to be around $-4/3$ as designated by the solid line in that range, appears roughly universal to all cases. This, if indeed, is consistent FM81 and part of the universal equilibrium theory [23].

Finally, relevant to (aero)acoustic noise measured by the pressure fluctuation $p'$, it is typical to gain a benefit

$$\Delta SPL_{rms} := 20 \log_{10} \frac{p'_{\text{nonhelical}}}{p'_{\text{helical}}} \approx 3.0 \text{ dB},$$

evaluated here with the root mean squares of the pressure fluctuation of the helical and nonhelical cases computed from the normalized time-averaged
power spectra in our simulations. The gains fluctuating with time and
depending on the scales, the value 3.0 dB should however mainly be considered
as a more definite ‘proof of the concept’ rather than a solid number for prac-
tical guidance, though it is not impossible that our fundamental results may
also further be applied to various situations, from understanding to control,
and, from nature to laboratory and to industry (c.f., e.g., Refs. [17, 18]).

In conclusion, we have verified and characterized the helicity (large-scale
injection) effect of turbulence compressibility reduction conjectured in Ref.
[9] and the nonuniversality (in the dissipation range spectra) in accordance
with FM81. All the results are so far only with respect to the differences in
the helicity injection or not of the specific forcing scheme [22], with other
characteristics — isotropic, white in time, stochastic and solenoidal — iden-
tical [22], but we hope to motivate more systematic studies, say, on classi-
ifying the (non)universalities and in turn to help practice such as better
noise control.

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on δ(t) (private communications, 2005–2006).

References

[2] Extrapolation of the systematic results of FM81 would indicate that ex-
ponential, stretched exponential and even power law are all possible, and
since ‘everything’ is limited to be finite in any experimental or numerical
measurements, the interpretation of the latter can also become subtle.
Relevant information of fluids and plasmas can be found in recent publica-
tions by, e.g., D. Buaria and K. R. Sreenivasan, [Physical Review Fluids
5, 092601(R) (2020)], S. Khurshid, D. Donzis and K. R. Sreenivasan,
[Physical Review Fluids 3, 082601 (2018)], and, J. E. Maggs and G. J.


[16] In the nonlinear, especially turbulent, regime, the notion of ‘aeracoustics’ and ‘noise’ becomes subtle [c.f., the spatio-temporal spectra in, e.g., recent work of J. Cerretani and P. Dmitruk, Phys. Fluids 31, 045102 (2019)]. We will not make specific identification of ‘turbulence noise’, but only refer to the latter by associating it to the compressibility of the flow.


Appendix

Figure 2: Plots of $E_h(k)/E_n(k)$ (‘hE/nE’) and $R_h(k)/R_n(k)$ (‘hR/nR’): the short straight line denotes the power law $\propto k^{-11/15}$, verifying a posteriori the result obtained in the main body of the paper and indicating the extension to the parallel-mode spectra $5E_h(k)/E_n(k)$ (‘5hE/nE’) in the short range circled out (above which the parallel-mode spectrum of the helical case, $E_{ih}$ shown in the upper-right panel of Fig. 1 in the main body of the paper is dominated by numerical errors).

The scaling behaviors and the $\Delta \alpha$ quantifying the steepened prefactors
of the helical, compared to the nonhelical, turbulence spectra can be summarized to be

\[ \frac{c_h k^{\alpha_h} F(k) + e_h(k)}{c_n k^{\alpha_n} F(k) + e_n(k)} = \frac{C_h(\hat{k})}{C_n(\hat{k})} \hat{k} - \Delta \alpha, \]

(16)

where \( e_h(k) \) and \( e_n(k) \) are respectively the (numerical) errors, systematic or random, in the data of helical and nonhelical cases. \( c_h \) and \( c_n \) are constants independent of \( k \). \( k \) in the plot is in the range around 100 while \( \hat{k} := k/K \) for, say, \( K = 100 \), thus of \( O(1) \), and \( C_h(\hat{k}) \) and \( C_n(\hat{k}) \) are respectively the numerical coefficients summarizing up the precise values and the errors: in the precise case \( C_h \) and \( C_n \) are independent of \( \hat{k} \).

Now, since the objective spectra are deep in the dissipation range in which (numerical) errors may not be so small. Assuming for simplicity that the maximum relative errors at different \( k \) in the interested range is the same, 10\%, say. Then, if the spectra of the helical and nonhelical cases are of the same order of magnitudes with \( C_h = C_n \) independent of \( \hat{k} \) in the precise case, then, for each \( \hat{k} \), both \( C_h(\hat{k}) \) and \( C_n(\hat{k}) \) may acquire extreme values, say, 10±1, resulting in the extreme possibility of values 11/9 and 9/11, i.e., an error \( \approx 20\% \). However, Then if the spectra of the two cases are of quite different orders of magnitudes, say, with a ratio of 1000 with \( C_n = 1000C_h \) independent of \( \hat{k} \) in the precise case, then, for each \( \hat{k} \), \( C_h(\hat{k}) \) and \( C_n(\hat{k}) \) with errors may acquire extreme values, say, 10±1 and 10000±1000 respectively, resulting in the extreme possibility of values 11/9000 and 9/11000, i.e., an error \( \approx 10\% \).

According to the above numerical analysis, \( E_h(k)/E_n(k) \) is closer to 1 than \( E_{lh}/E_{ln} \) and \( R_h/R_n \) in Fig. 2 (left), thus is easier to be affected by the (numerical) errors. \( R(k)/E(k) \) in the main body of the paper is even much smaller than those in this figure, so the results there look more obvious and should be more reliable. That is, the error in this respect from \( E_h(k)/E_n(k) \) of our data can be much larger than those measurements in the main body of the paper: Fig. 2 also shows that the systematic errors deviating from \( F_h^E = F_n^E \) is more obvious. [Extra reasonable normalizations with about 6\% of differences in \( R \) and \( k \) for the two cases do not essentially change the results, as said in the main body of the paper.] For incompressible turbulence, it is advised to simulate helical and nonhelical flows with a big difference of energy levels for quantifying \( \Delta \alpha \) in the dissipation range, but on the other hand, in order to detect the possible (minute) difference between \( F_h^E \) and \( F_n^E \), the two simulations should have close energy levels.

In summary, the error analysis together with the energy levels of various spectra of our data explains why the corresponding results from the plots of \( F_h(k)/E_{hl}(k) \) and \( F_n(k)/E_{ln}(k) \) in Fig. 2, a more straightforward practice though, are not as accurate and then not as empirically illuminating: \( \Delta \alpha = 11/15 \) is precisely for and indeed verified by such spectra [and even extendable to the parallel-mode spectra \( E_{lh}(k)/E_{ln}(k) \) in the short range.
$100 < k < 150$, but the latter is definitely much less indicative to derive such a $\Delta \alpha$. 