The Center and the Barycenter

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Abstract

In the first part, we deal with the question of which points we have to connect to generate a non self-intersecting polygon. Afterwards, we introduce a *polyhole*, which is a generalization of a polygon. Roughly speaking a polyhole is a big polygon, where we cut out a finite number of small polygons.
In the second part, we present two 'centers', which we call *center* and *barycenter*. In the case that both centers coincide, we call these polygons *nice*. We show that if a polygon has two symmetry axes, it is nice. We yield examples of polygons with a single symmetry axis which are nice and which are not nice. We show that all symmetry axes intersect at one point.
In the third part, we introduce the *Spieker center* and the *Point center* for polygons. We define *beautiful* polygons and *perfect* polygons.

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1 Introduction

We look for a criterion to generate a simple polygon.
Let us assume a set of $k+1$ points called $Points \subset \mathbb{R}^2$, where $Points = \{(x_1, y_1),\ (x_2, y_2),\ \ldots (x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1})\}$. We joint the possible edges. We define the subset $Union$ of $\mathbb{R}^2$, $Union := \bigcup\, [(x_i, y_i), (x_{i+1}, y_{i+1})]$ for $i \in \{1, 2, \ldots k-1, k\}$. With the expression '$[a, b]$' we mean all points on the line segment between a and b and the boundaries a and b. We say that $Points$ is *suitable* if and only if $Union$ is homeomorphic to the circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Definition 1.1. We call $Union$ a *polygon* if and only if it holds $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$ where $i, j \leq k$. We demand that there are no consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$; also $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ and $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ are not collinear. The element (x_i, y_i) is called a *vertex*. We name a polygon such that $Points$ is suitable a *simple polygon*. If we have a simple polygon we include its interior, and it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$ and $k > 2$.

An *r-gon* is a simple polygon with r vertices.

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2 Starlike Polygons

We can start also with a circle $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$. We presume a finite set called A of k different points on the circle, where $k > 2$ and $A = \{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_{k-1}, \vec{a}_k\}$ is in counterclockwise order. We map A bijectively into a set $H = \{\vec{h}_1, \vec{h}_2, \ldots \vec{h}_{k-1}, \vec{h}_k\}$, $\vec{a}_i \mapsto \vec{h}_i$, such that the three points $\vec{h}_i, \vec{a}_i$ and $(0,0)$ are collinear and the norm of $\vec{h}_i$ is positive. We demand that there are no collinear points $\vec{h}_i, \vec{h}_{i+1}, \vec{h}_{i+2}$, and also that $\vec{h}_{k-1}, \vec{h}_k, \vec{h}_1$ and $\vec{h}_k, \vec{h}_1, \vec{h}_2$ are not collinear. We keep the order as in A. We move all points with the same vector $\vec{m}$, i.e. $Z := \{\vec{z}_1, \vec{z}_2, \ldots \vec{z}_i, \ldots \vec{z}_{k-1}, \vec{z}_k\} \subset \mathbb{R}^2$ where $1 \leq i \leq k$ and $\vec{z}_i := \vec{h}_i + \vec{m}$. We keep the order. We call a polygon with a set of vertices $V = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{k-1}, \vec{v}_k\}$ a *starlike polygon* if and only if V provided with an appropriate order can be constructed as it is just described for Z. We add $\vec{v}_{k+1} := \vec{v}_1$ into V. Please see the following Proposition 2.2. Note that V is a suitable set.

Questions 2.1. *Is there an alternative description of starlike polygons? Is every convex simple polygon a starlike polygon?*

Proposition 2.2. *We get a simple polygon P if there is a set of points $V = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{k-1}, \vec{v}_k, \vec{v}_{k+1}\} \subset \mathbb{R}^2$ constructed as above where $k > 2$ and*

$$P := \bigcap \{\ \mathit{Circle} \subset \mathbb{R}^2 \ \mid V \subset \mathit{Circle},\ \text{where } \mathit{Circle} \text{ is homeomorphic to the circle area}$$
$$\{x^2+y^2 \leq 1\},\ \text{and the points between } \vec{v}_i \text{ and } \vec{v}_{i+1},\ 1 \leq i \leq k,\ \text{are a subset of } \mathit{Circle}\}$$

Proof. The claim of the proposition is trivial, since V is a suitable set. □

It follows that a starlike polygon is a compact set, homeomorphic to any circle area, and for all $1 \leq i \leq k$ the vector $\vec{v}_i$ is a boundary point. All starlike polygons are simple polygons.
A triangle is a starlike polygon, too.

Proposition 2.3. If we have a set of $k+1$ points called $\mathit{Points} = \{(x_1,y_1),\ (x_2,y_2),\ \ldots (x_k,y_k),\ (x_{k+1},y_{k+1})\} \subset \mathbb{R}^2$ which forms a polygon we get a simple polygon if and only if Points is suitable.

Proof. Trivial. □

3 Polyholes

We define a subset of $\mathbb{R}^2$, which we will call a *polyhole*. This geometric structure consists of a finite number of simple polygons $P, P_1, P_2, P_3, \ldots P_{m-1}, P_m$. From the polygon P we cut out polygons $P_1, P_2, P_3, \ldots P_{m-1}, P_m$.

Definition 3.1. Let $\{P, P_1, P_2, P_3, \ldots P_{m-1}, P_m\}$ be a set of simple polygons. A *polyhole* is defined as P without $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_{m-1} \cup P_m$

A corresponding definition is possible for polytopes. To define them please see [1].

Definition 3.2. Let $\{P, P_1, P_2, P_3, \ldots P_{m-1}, P_m\}$ be a set of polytopes. A *polytopehole* is defined as P without $P_1 \cup P_2 \cup P_3 \cup \ldots \cup P_{m-1} \cup P_m$.

Questions 3.3. *What is the barycenter of a polyhole, if it is realized with homogeneous material of constant thickness? What is the barycenter of a polytopehole of $\mathbb{R}^3$, if it is realized with homogeneous material?*

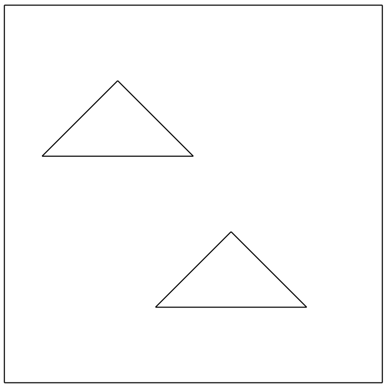

Figure 1:
The polyhole on the left-hand side consists of a square where we cut out two triangles.

4 Nice Polygons

We assume a suitable set of points called $Points = \{(x_i, y_i) \mid i \in \{1, 2, \ldots k-1, k, k+1\}\}$.
We define two 'centers', where the center $Cent$ is just the arithmetic mean of the first and second coordinates of the generating points, respectively.
We got the following formulas for the *barycenter* $B = (B_x, B_y)$ of a simple polygon from [2] or [3]. Please see also [4] and [5]. Area is the area of a simple polygon. Note that $\mathsf{Area} \neq 0$ and that in [2] and [4] the barycenter is called a Centroid, and further that B is the center of gravity of the polygon, if it is realized with homogeneous material of constant thickness. Note that the order in the polygon is counterclockwise. We write

$$D_i = x_i \cdot y_{i+1} - x_{i+1} \cdot y_i, \text{ where } 1 \leq i \leq k \tag{4.1}$$

$$\mathsf{Area} = \frac{1}{2} \cdot \sum_{i=1}^{k} D_i \tag{4.2}$$

$$B_x = \frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=1}^{k} (x_i + x_{i+1}) \cdot D_i, \quad B_y = \frac{1}{6 \cdot \mathsf{Area}} \cdot \sum_{i=1}^{k} (y_i + y_{i+1}) \cdot D_i \tag{4.3}$$

$$Cent = \frac{1}{k} \cdot \left(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i \right) \tag{4.4}$$

Definition 4.1. Let us presume a simple polygon P. We call P *nice* if and only if it holds $B = Cent$.

Proposition 4.2. *Every triangle is nice*

Remark 4.3. When we use the term *symmetry axis* of a polygon P we mean a line segment s in the convex hull of P of maximal length, i.e. it holds for a symmetry axis t in the convex hull of P with more than one common point with s that $t \subset s$.

Proposition 4.4. *If a simple polygon has two different symmetry axes, it is nice*

Proof. The proposition is an easy consequence of the fact that both the center of gravity and $Cent$ must be on a line determined by a symmetry axis. Please see Lemma 4.11.

Lemma 4.5. The property of being nice or being not nice remains with these operations.

- *Revolving P by an arbitrary angle around any point*
- *Shifting P by an arbitrary vector*

Proof. We assume $B \neq Cent$. Let us revolve P by an arbitrary angle around any point. There is a positive distance d between B and $Cent$. It will be kept since a rotation is a distance preserving map. Hence the distance between the image points of B and $Cent$ is also d. After the rotation still P is not nice.
In the case $B = Cent$ we have nothing to show. □

Lemma 4.6. Both operations which we have mentioned above in Lemma 4.5 are distance preserving operations. Therefore the shape of a polygon is kept after these operations.

Proof. Trivial. □

In a polygon we fix four real numbers.

Definition 4.7. We define
min_x := minimum of the set of the first coordinates of the set of the vertices $Points$ of P.
min_y := minimum of the second coordinates of $Points$,
max_x := maximum of the first coordinates of $Points$,
max_y := maximum of the second coordinates of $Points$.
Let $min_x < max_x$ and $min_y < max_y$. Note that in a simple polygon P these conditions are fulfilled. We define a simple polygon called $Rectangle(P)$ by four vertices (max_x, max_y), (min_x, max_y), (min_x, min_y), (max_x, min_y).

Remark 4.8. In a simple polygon P it holds that both P and the convex hull of $Points$ are in $Rectangle(P)$.

Definition 4.9. Let $s = \{\vec{a} + r \cdot \vec{d} \mid r \in [v, w]$ for fixed real numbers $v, w\}$ be a symmetry axis of a polygon. We define $l(s)$ as the line $\{\vec{a} + r \cdot \vec{d} \mid r \in \mathbb{R}\}$.

Remark 4.10. It holds that s is a subset of $l(s)$.

Lemma 4.11. Let s be a symmetry axis of a simple polygon P. Both B and $Cent$ are on the line segment $l(s) \cap Rectangle(P)$.

Proof. We assume a simple polygon P with a symmetry axis s and centers B and $Cent$. Note that B is the center of gravity of P. Hence B must be on $l(s)$, since s is a symmetry axis of P. For the same reason, B is also in $Rectangle(P)$.
We use Lemma 4.6. We map P by a rotation and a shift parallel the vertical y axis into a second polygon P' with a symmetry axis s' and centers B' and $Cent'$ such that s' is on the x axis. Assume a vertex (x', y') of P'. Since s' is a symmetry axis and it is on the x axis either $y' = 0$ or there is a second vertex $(x', -y')$ of P'. If we add all vertices together we get $Cent' = (c', 0)$ with any real number c', i.e. $Cent'$ is on the x axis. This means that $Cent'$ is on $l(s')$. Since P' has the same shape as P we get that $Cent$ is on $l(s)$.
It is easy to show that $Cent$ is a point in $Rectangle(P)$: It holds

$$min_x = \frac{1}{k} \cdot \sum_{i=1}^{k} min_x < \frac{1}{k} \cdot \sum_{i=1}^{k} x_i < \frac{1}{k} \cdot \sum_{i=1}^{k} max_x = max_x \qquad (4.5)$$

If we consider correspondingly the second coordinate of $Cent$ we get $min_y < \frac{1}{k} \cdot \sum_{i=1}^{k} y_i < max_y$, and it follows that $Cent$ is in $Rectangle(P)$.
We get that $Cent$ is in $l(s) \cap Rectangle(P)$. Lemma 4.11 has been proved. □

Two symmetry axes intersect in one point. It is both B and $Cent$. The proof of Proposition 4.4 is finished. □

Corollary 4.12. In a simple polygon with multiple symmetry axes all symmetry axes intersect in one point. It is both B and $Cent$. It follows that a simple polygon with more than a single symmetry axis is nice.

Note that one symmetry axis is not sufficient, as the kite defined by $(0,0), (1,-1), (3,0), (1,1)$ shows, since $\frac{5}{4} \neq \frac{4}{3}$. It is not nice.

We give an example of a polygon with a single symmetry axis which is nice, but not a triangle.
We present a 5-gon. It has centers $B = Cent$, it is about $(0.50, 0.85)$. The vertices are

$$(0,0), (1,0), (1,1), \left(\frac{1}{2}, 1 + \frac{1}{2} \cdot \sqrt{6}\right), (0,1). \text{ We get } B = Cent = \left(\frac{1}{2}, \frac{1}{10} \cdot \left(6 + \sqrt{6}\right)\right). \quad (4.6)$$

This example proves the conjecture that besides triangles only polygons with two or more symmetry axes are nice is wrong.

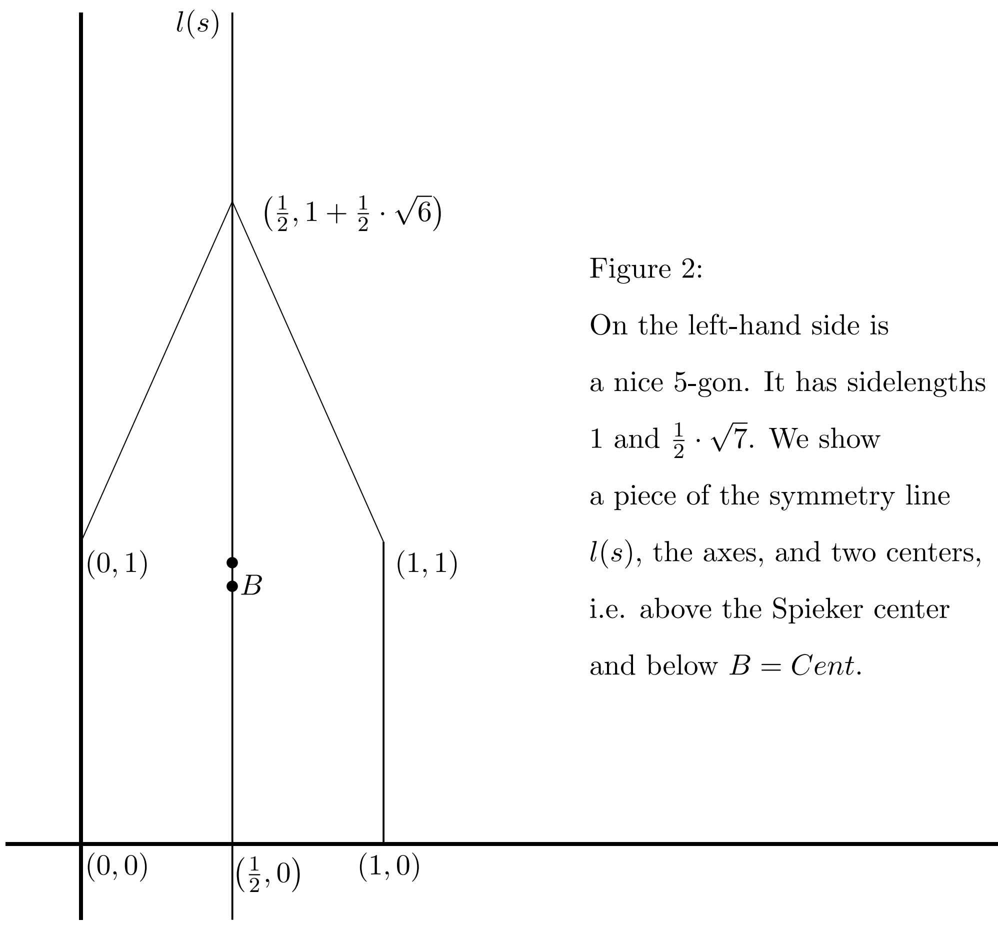

Figure 2: On the left-hand side is a nice 5-gon. It has sidelengths 1 and $\frac{1}{2} \cdot \sqrt{7}$. We show a piece of the symmetry line $l(s)$, the axes, and two centers, i.e. above the Spieker center and below $B = Cent$.

5 Spieker Center and Point Center

In a triangle the Spieker center is well-known. We have gotten the formulas of the Spieker center from [5]. Please see also [6]. The Spieker center of a triangle $A = (x_A, y_A), B = (x_B, y_B), C = (x_C, y_C)$ is its barycenter, at which the triangle is formed by a wire of constant diameter. The interior of the triangle is not regarded. The Spieker center is outside the wire. The sidelengths of the triangle are l_1, l_2 and l_3, where sides with lengths l_1 and l_3 intersect in A, while sides with lengths l_1 and l_2 intersect in B. The coordinates of the Spieker center $(spieker_x, spieker_y)$ are

$$spieker_x = \frac{(l_1 + l_3) \cdot x_A + (l_1 + l_2) \cdot x_B + (l_2 + l_3) \cdot x_C}{2 \cdot (l_1 + l_2 + l_3)} \quad \text{and} \tag{5.1}$$

$$spieker_y = \frac{(l_1 + l_3) \cdot y_A + (l_1 + l_2) \cdot y_B + (l_2 + l_3) \cdot y_C}{2 \cdot (l_1 + l_2 + l_3)} \tag{5.2}$$

The concept of the *Spieker center* can easily be generalized to polygons. We assume the polygon is made from a wire of constant diameter. The shape of the polygon is formed by the wire. We look for its center of gravity; it is generally outside the wire. We consider a new polygon, constructed by k mass centers. Therefore it also has k vertices. We compute the *Point center* of the new polygon. The Point center of an r-gon is defined by the imagination that the masses are in the vertices of the polygon. Let $m_1, m_2, \dots m_{r-1}, m_r$ be r masses. The polygon has the Point center $Point = (point_x, point_y)$, where

$$point_x = \frac{1}{M} \cdot \sum_{i=1}^{r} m_i \cdot a_i \tag{5.3}$$

$$point_y = \frac{1}{M} \cdot \sum_{i=1}^{r} m_i \cdot b_i, \text{ and} \tag{5.4}$$

$$M = \sum_{i=1}^{r} m_i \text{ is the sum of the masses and} \tag{5.5}$$

$$(a_1, b_1), (a_2, b_2), \dots (a_{r-1}, b_{r-1}), (a_r, b_r) \text{ are the vertices of the polygon.} \tag{5.6}$$

To calculate the Spieker center of a given polygon we have to consider a new polygon, constructed by k mass centers of the k edges. Therefore it also has k vertices. We assume that in the new polygon the masses are on these k vertices. The Spieker center of the given polygon is the Point center of the new polygon. The formulas are

$$spieker_x = \frac{1}{U} \cdot \sum_{i=1}^{k} l_i \cdot \left(\frac{1}{2} \cdot (x_i + x_{i+1}) \right) = \frac{1}{2 \cdot U} \cdot \sum_{i=2}^{k+1} (l_i + l_{i-1}) \cdot x_i \tag{5.7}$$

$$spieker_y = \frac{1}{U} \cdot \sum_{i=1}^{k} l_i \cdot \left(\frac{1}{2} \cdot (y_i + y_{i+1}) \right) = \frac{1}{2 \cdot U} \cdot \sum_{i=2}^{k+1} (l_i + l_{i-1}) \cdot y_i \tag{5.8}$$

$$\text{where } l_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \quad \text{and} \quad U = \sum_{i=1}^{k} l_i \tag{5.9}$$

We define $l_{k+1} := l_1$. Note the indices in the formulas! Note that it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$. The variable '$l_i$' means the length of one edge of the polygon. Every edge $l_i :=$

$[(x_i, y_i), (x_{i+1}, y_{i+1})]$ has a center of gravity $\frac{1}{2} \cdot ((x_i, y_i) + (x_{i+1}, y_{i+1}))$. U is the perimeter of the polygon.
As an example we take the 5-gon of above. Its Spieker center is about $(0.50, 0.93)$. In exact coordinates it is

$$\left(\frac{1}{2}, \frac{1}{6 + 2 \cdot \sqrt{7}} \cdot \left(2 + 2 \cdot \sqrt{7} + \frac{1}{2} \cdot \sqrt{42}\right)\right) = \left(\frac{1}{2}, \sqrt{7} - 2 + \frac{1}{8} \cdot \left(3 \cdot \sqrt{42} - \sqrt{294}\right)\right). \quad (5.10)$$

Definition 5.1. Let us presume a simple polygon P. We call P *beautiful* if and only if it holds that B equals the Spieker center. We call P *perfect* if and only if all three centers are the same, i.e. it holds that B equals both $Cent$ and the Spieker center.

Lemma 5.2. Let s be a symmetry axis of a simple polygon P. The Spieker center is on $s \cap Rectangle(P)$.

Proof. The line segment s is a symmetry axis both for the entire polygon and for its contour. Because the Spieker center is the barycenter of the contour it must be on s. The statement that the Spieker center is in $Rectangle(P)$ is trivial. □

Proposition 5.3. *Let a simple polygon has two or more symmetry axes. Then it is perfect.*

Proof. The three points $B, Cent$ and the Spieker center all are on the line determined by a symmetry axis. There is only a single possibility that all points are on every line. □

Conjecture 5.4. *A triangle is perfect if and only if it is an equilateral triangle.*

The next conjecture contains the previous one.

Conjecture 5.5. *An r-gon is perfect if and only if it is an regular r-gon.*

Conjecture 5.6. *A simple polygon is beautiful if and only if it is perfect. In other words it holds $B =$ Spieker center if and only if $B =$ Spieker center and $B = Cent$.*

Conjecture 5.7. *A simple polygon is beautiful if and only if it has more than one symmetry axis.*

We end the paper with an infinite number of questions. We define that a *connection* between two points A and B is a continuous map $f : [0, 1] \to \mathbb{R}^2$ such that $f(0) = A$ and $f(1) = B$.

Questions 5.8. *We assume a set of n different fixed points in $\mathbb{R}^2$. The first set of questions is: Are there connections between all points such that the union of the connections is homeomorphic to a circle? The second is the same, except that only line segments are allowed as connections. In the next questions, we may use a subset of the set of points. We ask whether there are connections of all points such that these connections are homeomorphic to m circles. In the last set of questions, we ask the same, except that only line segments are allowed.*

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