FERMAT'S LAST THEOREM -

A SIMPLE PROOF.

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Abstract.

This paper provides a simple proof of Fermat's Last Theorem via elementary algebraic analysis of a level that would have been extant in Fermat's day, the mid seventeenth century.

1 Introduction.

Fermat's Last Theorem, (or Conjecture), was finally proven in 1995, some 360 years after its proposal in the margin of a book, "Arithmetica" by the Greek mathematician Diophantus, that Pierre de Fermat was reading at the time.
This proof, published by Andrew Wiles, a professor of mathematics at Princeton University, was a proof by association in that, by proving another conjecture, the Taniyama - Shimura Conjecture, Wiles also proved Fermat's Last Theorem via a link between the two, previously established by two other mathematicians, Gerhard Frey and Ken Ribet. The proof was extremely long and complex utilising the most modern 20^{th} Century analytical techniques, the majority of which would not have been available in Fermat's day.
Consequently, although the Fermat Conjecture was at last proven, there remained the tantalising question as to whether it could ever be proven in a direct manner, using only elementary analytical methods. It is the purpose of this paper to provide such a proof.

2 Proof of Fermat's Last Theorem.

2.1 Preamble.

Fermat's equation is

$$x^n + y^n = z^n \tag{2.1}$$

and his Last Theorem states "*There are no integer solutions for x, y and z for n > 2.*"

It is well known, [1], that x and y cannot both be even numbers, and that they must be of different parity and relatively prime. Also, it is well known, [1], [2], that if the Last Theorem can be proved for $n = 4$, then it is also proven for all multiples of $n = 4$. Consequently, because all of the remaining numbers can be reduced to a multiple of the prime numbers, it is therefore only necessary to prove Fermat's Last theorem for all the primes. Indeed this is the approach adopted by many eminent mathematicians in the past, and more recently using computer analysis which, prior to Wiles' proof, established the proof of the Conjecture for all values of n up to 4 million, including the primes therein. However, it is obvious that this eventually becomes a fruitless task because, even if only prime numbers are considered, because there are an infinite number of prime numbers, the Conjecture can never be proven by such number crunching alone.

The method adopted here will however, consider such specific cases of n, because the results so obtained will be shown to exhibit a simple pattern, that can justifiably be extrapolated to the nth order, so proving the general case. This method is initiated as follows.

Because x, y and z are simply numbers, with no physical meaning, (2.1) can be re-written as follows

$$x^n + (x+b)^n = (x+a)^n \tag{2.2}$$

where b is an odd integer and $a > b$. It is important to note that variation of y in (2.1), is effected by variation of b in (2.2) so that with x constant, changes in b directly cause a variation of a in (2.2). Similarly, variation of x in (2.1), keeping y constant means in (2.2), that b varies inversely to x thus also causing a variation in a. Thus, in (2.2) n, x and b are the independent variables and, a the dependent.
Expanding (2.2) binomially and gathering all terms to the left gives

$$x^n - n(a-b)\,x^{(n-1)} - \frac{n(n-1)}{2!}\left(a^2-b^2\right)x^{(n-2)} - \cdots - \frac{n(n-1)(n-2)\cdots(1)}{n!}\left(a^n-b^n\right) = 0 \tag{2.3}$$

Thus Fermat's equation, (2.1) has been transformed into a variable co-efficient nth order polynomial in x. Application of Descartes' Rule of Signs shows that (2.3) contains just one positive root, with all the rest being negative. Also, from the co-efficient of $x^{(n-1)}$ it is clear that all the roots must contain the unique term $(a-b)$. Furthermore, the nature of the other co-efficients also shows that the remaining roots cannot all be real. When n is even, in addition to the one positive root, there can only be one real negative root with all the rest being conjugate complex with negative real parts. And when n is odd, in addition to the one real positive root, all the rest are also conjugate complex with negative real parts.
Fermat's Last Theorem can therefore be proved by determining whether the one generalised positive real root of (2.3) can contain an integer value of the parameter a. The approach here is to accomplish this task for $n =$ 3, 4, 5 and 7, from which extrapolation to the general case will be effected.

2.2 Case n = 3.

When $n = 3$, (2.3) reduces to

$$x^3 - 3(a-b)\,x^2 - 3\left(a^2-b^2\right)x - \left(a^3-b^3\right) = 0 \tag{2.4}$$

From the above discussion, the roots of (2.4) are of the form

$$\{x-(p+k_1)\}\left\{x-\left(p-\frac{k_1}{2}\pm jk_2\right)\right\} = 0 \tag{2.5}$$

where $p = (a-b)$, and the k parameters are secondary variables of the a's and b's that contribute to generating the correct co-efficients of (2.4).
Expanding (2.5) gives

$$x^3 - 3px^2 + \left[\left\{\left(p-\frac{k_1}{2}\right)^2 + k_2^2\right\} + (2p-k_1)(p+k_1)\right]x - \left\{\left(p-\frac{k_1}{2}\right)^2 + k_2^2\right\}(p+k_1) = 0 \tag{2.6}$$

In (2.6) consider the co-efficient of x. Multiplying out this reduces to

$$A_1 = 3p^2 - \frac{3}{4}k_1^2 + k_2^2 \tag{2.7}$$

Comparing (2.7) with the co-efficient of x in (2.4)

$$-3a^2 + 3b^2 = 3(a-b)^2 - \frac{3}{4}k_1^2 + k_2^2 \tag{2.8}$$

which reduces to

$$k_2^2 - \frac{3}{4}k_1^2 = -6a\,(a-b) = -6ap \tag{2.9}$$

Now consider the final term in (2.6). Multiplying out this reduces to

$$A_0 = -p^3 - \left(k_2^2 - \frac{3}{4}k_1^2\right)p + k_1\left(k_2^2 + \frac{k_1}{4}\right) \tag{2.10}$$

Substituting (2.9) for the co-efficient of p in (2.10) and for k_2^2 in the final term of (2.10) gives

$$A_0 = -p^3 + 6ap^2 - k_1\left(-6ap + k_1^2\right) \tag{2.11}$$

Now compare (2.11) with the final term in (2.4) thus

$$-a^3 + b^3 = -\,(a-b)^3 + 6a\,(a-b)^2 - k_1\left(-6a\,(a-b) + k_1^2\right) \tag{2.12}$$

and this reduces to

$$6a^3 - 9a^2b + 3ab^2 + 6k_1a\,(a-b) - k_1^3 = 0 \tag{2.13}$$

Dividing through by the co-efficient of k_1 now gives

$$a - \frac{b}{2} + k_1\left(1 - \frac{k_1^2}{6a\,(a-b)}\right) = 0 \tag{2.14}$$

Now, k_1 is of unity order in a and b so that k_1^2 in the numerator of the quotient in (2.14) is of the same order in a and b as the denominator. Also because of the independent presence of unity, the term $\left\{1 - \frac{k_1^2}{6a\,(a-b)}\right\}$ must be a pure number. Consequently, the quotient in this term must also be a pure number, i.e. q_3, where the subscript denotes the order of the equation being analysed, i.e. (2.4). The parameter q_3 is not a constant, but a "non-dimensional" function of the a's and b's. Thus (2.14) becomes

$$a - \frac{b}{2} + k_1\,(1 - q_3) = 0 \tag{2.15}$$

So that

$$k_1 = \frac{-\left(a - \frac{b}{2}\right)}{(1 - q_3)} \tag{2.16}$$

Now, substitution of (2.16) into the positive root of (2.5) gives

$$x = a - b - \frac{\left(a - \frac{b}{2}\right)}{(1 - q_3)} \tag{2.17}$$

This must be a positive root so that q_3 must be greater than unity, and thus (2.17) is re-written as

$$x = a - b + \frac{\left(a - \frac{b}{2}\right)}{(q_3 - 1)} \tag{2.18}$$

Re-arranging for a gives

$$a = \left(x + \frac{b}{2}\right)\left(1 - \frac{1}{q_3}\right) + \frac{b}{2} \tag{2.19}$$

Now, because x is an integer, $(x + b/2)$ must be half integer. Therefore, for a to be an integer, the term (1 - 1/q_3) would have to be an odd integer. This is clearly impossible because with $q_3 > 1$, the term (1 - 1/q_3) must be fractional. Thus a in (2.2) and therefore z in (2.1) cannot be integers. Thus, subject to q_3 exhibiting satisfactory characteristics, this proves Fermat's Last Theorem for $n = 3$.

2.3 Case n = 4.

When $n = 4$, (2.3) becomes

$$x^4 - 4(a-b)x^3 - 6\left(a^2 - b^2\right)x^2 - 4\left(a^3 - b^3\right)x - \left(a^4 - b^4\right) = 0 \tag{2.20}$$

The roots of this equation will be of the form

$$\{x - (p + k_1 + k_2)\}\{x - (p - k_1 + k_2)\}\{x - (p - k_2 \pm jk_3)\} = 0 \tag{2.21}$$

When expanded (2.21) becomes

$$x^4 - A_3x^3 + A_2x^2 - A_1x + A_0 = 0 \tag{2.22}$$

Where

$$\begin{aligned} A_3 &= 4p \\ A_2 &= \left[\left\{(p+k_2)^2 - k_1^2\right\} + 4(p+k_2)(p-k_2) + \left\{(p-k_2)^2 + k_3^2\right\}\right] \\ A_1 &= 2\left[\left\{(p+k_2)^2 - k_1^2\right\}(p-k_2) + \left\{(p-k_2)^2 + k_3^2\right\}(p+k_2)\right] \\ A_0 &= \left\{(p+k_2)^2 - k_1^2\right\}\left\{(p-k_2)^2 + k_3^2\right\} \end{aligned} \tag{2.23}$$

Consider the co-efficient of x^2 in (2.22). From (2.23) after multiplying out, this becomes

$$A_2 = 6p^2 - 2k_2^2 - k_1^2 + k_3^2 \tag{2.24}$$

Comparing (2.24) with the co-efficient of x^2 in (2.20) gives

$$-6a^2 + 6b^2 = 6(a-b)^2 - 2k_2^2 - k_1^2 + k_3^2 \tag{2.25}$$

Which reduces to

$$2k_2^2 + k_1^2 - k_3^2 = 12a^2 - 12ab = 12ap \tag{2.26}$$

Now consider the co-efficient of x in (2.22). From (2.23), after multiplying out this becomes

$$A_1 = -4p^3 + 2\left(2k_2^2 + k_1^2 - k_3^2\right)p - 2k_2\left(k_1^2 + k_3^2\right) \tag{2.27}$$

Substituting (2.26) for the co-efficient of p and for k_3^2 in the final term in (2.27) gives

$$A_1 = 20a^3 - 36a^2b + 12ab^2 + 4b^3 + k_2\left\{24a(a-b) - 4k_1^2 - 4k_2^2\right\} \tag{2.28}$$

Comparing (2.28) with the co-efficient of x in (2.20) gives

$$-4a^3 + 4b^3 = 20a^3 - 36a^2b + 12ab^2 + 4b^3 + k_2\left\{24a(a-b) - 4k_1^2 - 4k_2^2\right\} \tag{2.29}$$

and this reduces to

$$24a^3 - 36a^2b + 12ab^2 + k_2\left\{24a(a-b) - 4k_1^2 - 4k_2^2\right\} = 0 \tag{2.30}$$

Dividing (2.30) throughout by $24a(a-b)$ then gives

$$a - \frac{b}{2} + k_2\left\{1 - \frac{k_1^2 + k_2^2}{6a(a-b)}\right\} = 0 \tag{2.31}$$

In the quotient in (2.31), the term $\left(k_1^2 + k_2^2\right)$ is of the same order in a and b as the denominator and therefore this quotient must be a pure number, q_4. Eq.(2.31) therefore becomes

$$a - \frac{b}{2} + k_2(1 - q_4) = 0 \tag{2.32}$$

So that

$$k_2 = -\frac{a - b/2}{1 - q_4} \tag{2.33}$$

Substitution of (2.33) into the positive root of (2.21) then gives

$$x_1 = a - b - \frac{a - b/2}{1 - q_4} + k_1 \tag{2.34}$$

Substitution of (2.33) into the negative real root of (2.21) gives

$$x_2 = a - b - \frac{a - b/2}{1 - q_4} - k_1 \tag{2.35}$$

Adding (2.34) and (2.35) then yields

$$\frac{x_1 + x_2}{2} = x' = -\left(a - \frac{b}{2}\right)\left(\frac{q_4}{1 - q_4}\right) - \frac{b}{2} \tag{2.36}$$

and re-arranging for a yields

$$a = -\left(x' + \frac{b}{2}\right)\left(\frac{1 - q_4}{q_4}\right) + \frac{b}{2} \tag{2.37}$$

The parameter a must be positive so that q_4 must be greater than unity and therefore (2.37) is rewritten as

$$a = \left(x' + \frac{b}{2}\right)\left(1 - \frac{1}{q_4}\right) + \frac{b}{2} \tag{2.38}$$

$x^/$ is the average of the real roots and must therefore be either integer or half integer. If $x^/$ is integer, then $(x^/ + b/2)$ must be half integer and a cannot then be integer because (1 - $1/q_4$) cannot be an odd integer.
If $x^/$ is half integer then $(x^/ + b/2)$ is a full integer and for a to be integer, not only would $(x^/ + b/2)$ have to be an odd integer, but (1 - $1/q_4$) would also have to be half integer. This would require q_4 to be exactly equal to 2. In this case from (2.31)

$$\frac{k_1^2 + k_2^2}{6a\,(a - b)} = 2 \tag{2.39}$$

so that

$$k_1^2 = 12a\,(a - b) - k_2^2 \tag{2.40}$$

Then from (2.33) with $q_4 = 2$, (2.40) becomes

$$k_1^2 = 12a\,(a - b) - \left(a - \frac{b}{2}\right)^2 \tag{2.41}$$

Which reduces to

$$k_1^2 = 11a^2 - 11ab - \frac{b^2}{4} \tag{2.42}$$

But under this condition k_1 would have to be an integer for x_1 and x_2 to be integers and this is not possible from (2.42). Therefore q_4 cannot be equal to 2, so that a in (2.20) and therefore z in (2.1) cannot be integers. Thus, subject to q_4 exhibiting satisfactory characteristics, this proves Fermat's Last Theorem for $n = 4$.

2.4 Case n = 5.

When $n = 5$, (2.3) reduces to

$$x^5 - 5\,(a - b)\,x^4 - 10\left(a^2 - b^2\right)x^3 - 10\left(a^3 - b^3\right)x^2 - 5\left(a^4 - b^4\right)x - \left(a^5 - b^5\right) = 0 \tag{2.43}$$

The roots of (2.43) are of the form

$$\{x-(p+k_1+k_2)\}\left\{x-\left(p-\frac{k_1}{2}\pm jk_3\right)\right\}\left\{x-\left(p-\frac{k_2}{2}\pm jk_4\right)\right\}=0 \tag{2.44}$$

Expanding (2.44) gives

$$x^5-A_4x^4+A_3x^3-A_2x^2+A_1x-A_0=0 \tag{2.45}$$

Where

$$A_4=5p$$

$$A_3=\left[\left\{\left(p-\frac{k_2}{2}\right)^2+k_4^2\right\}+(2p-k_2)(2p-k_1)+\left\{\left(p-\frac{k_1}{2}\right)^2+k_3^2\right\}+(4p-k_1-k_2)(p+k_1+k_2)\right]$$

$$A_2=\left[\begin{array}{l}\left\{\left(p-\frac{k_2}{2}\right)^2+k_4^2\right\}(2p-k_1)+\left\{\left(p-\frac{k_1}{2}\right)^2+k_3^2\right\}(2p-k_2)\\+\left\{\left(p-\frac{k_2}{2}\right)^2+k_4^2+(2p-k_2)(2p-k_1)+\left(p-\frac{k_1}{2}\right)^2+k_3^2\right\}(p+k_1+k_2)\end{array}\right]$$

$$A_1=\left[\begin{array}{l}\left\{\left(p-\frac{k_2}{2}\right)^2+k_4^2\right\}\left\{\left(p-\frac{k_1}{2}\right)^2+k_3^2\right\}\\+\left\{\left\{\left(p-\frac{k_2}{2}\right)^2+k_4^2\right\}(2p-k_1)+\left\{\left(p-\frac{k_1}{2}\right)^2+k_3^2\right\}(2p-k_2)\right\}(p+k_1+k_2)\end{array}\right]$$

$$A_0=\left[\left\{\left(p-\frac{k_2}{2}\right)^2+k_4^2\right\}\left\{\left(p-\frac{k_1}{2}\right)^2+k_3^2\right\}(p+k_1+k_2)\right] \tag{2.46}$$

Consider the co-efficient of x^3 in (2.45). In (2.46) after multiplying out this reduces to

$$A_3=10p^2-\frac{3}{4}\left(k_1^2+k_2^2\right)-k_1k_2+k_3^2+k_4^2 \tag{2.47}$$

Comparing (2.47) with the coefficient of x^3 in (2.43) gives

$$-10a^2+10b^2=10(a-b)^2-\frac{3}{4}\left(k_1^2+k_2^2\right)-k_1k_2+k_3^2+k_4^2 \tag{2.48}$$

Which reduces to

$$20a(a-b)-\frac{3}{4}\left(k_1^2+k_2^2\right)-k_1k_2+k_3^2+k_4^2=0 \tag{2.49}$$

Consider the co-efficient of x^2 in (2.45). In (2.46) after multiplying out this reduces to

$$A_2=-10p^3+3\left\{\frac{3}{4}\left(k_1^2+k_2^2\right)+k_1k_2-k_3^2-k_4^2\right\}p-\frac{\left(k_1^3+k_2^3\right)}{4}-k_1k_2(k_1+k_2)-k_1k_3^2-k_2k_4^2 \tag{2.50}$$

Comparing this to the co-efficient of x^2 in (2.43) yields

$$\begin{aligned}-10a^3+10b^3=-10(a-b)^3+3\left\{\frac{3}{4}\left(k_1^2+k_2^2\right)+k_1k_2-k_3^2-k_4^2\right\}p\\-\frac{\left(k_1^3+k_2^3\right)}{4}-k_1k_2(k_1+k_2)-k_1k_3^2-k_2k_4^2\end{aligned} \tag{2.51}$$

and this reduces to

$$30a^2b-30ab^2+3\left\{\frac{3}{4}\left(k_1^2+k_2^2\right)+k_1k_2-k_3^2-k_4^2\right\}p-\frac{\left(k_1^3+k_2^3\right)}{4}-k_1k_2(k_1+k_2)-k_1k_3^2-k_2k_4^2=0 \tag{2.52}$$

Substituting (2.49) for the coefficient of p in (2.52) gives

$$30a^2b-30ab^2+3\{20a(a-b)\}(a-b)-\frac{(k_1^3+k_2^3)}{4}-k_1k_2(k_1+k_2)-k_1k_3^2-k_2k_4^2=0 \tag{2.53}$$

and this reduces to

$$60a^3-90a^2b+30ab^2-\frac{(k_1^3+k_2^3)}{4}-k_1k_2(k_1+k_2)-k_1k_3^2-k_2k_4^2=0 \tag{2.54}$$

Eq.(2.54) may be rewritten as

$$60a^3-90a^2b+30ab^2-(k_1+k_2)\begin{bmatrix}\left\{\frac{3}{4}(k_1^2+k_2^2)+k_1k_2-k_3^2-k_4^2\right\}\\ \\ -\left\{\frac{(k_1^2+k_2^2)}{2}+\frac{k_1k_2}{4}-k_3^2-k_4^2-\frac{(k_1k_3^2+k_2k_4^2)}{(k_1+k_2)}\right\}\end{bmatrix}=0 \tag{2.55}$$

Now substitution from (2.49) gives

$$60a^3-90a^2b+30ab^2-(k_1+k_2)\begin{bmatrix}20a(a-b)\\ -\left\{\frac{(k_1^2+k_2^2)}{2}+\frac{k_1k_2}{4}-k_3^2-k_4^2-\frac{(k_1k_3^2+k_2k_4^2)}{(k_1+k_2)}\right\}\end{bmatrix}=0 \tag{2.56}$$

and dividing through by $20a(a-b)$ gives

$$3\left(a-\frac{b}{2}\right)-(k_1+k_2)\left[1-\frac{\left\{\frac{(k_1^2+k_2^2)}{2}+\frac{k_1k_2}{4}-k_3^2-k_4^2-\frac{(k_1k_3^2+k_2k_4^2)}{(k_1+k_2)}\right\}}{\{20a(a-b)\}}\right]=0 \tag{2.57}$$

The numerator in the quotient of (2.57) is of the same order in a and b as the denominator and is therefore a pure number, q_5. Thus

$$3\left(a-\frac{b}{2}\right)-(k_1+k_2)(1-q_5)=0 \tag{2.58}$$

From which

$$(k_1+k_2)=\frac{3\left(a-\frac{b}{2}\right)}{(1-q_5)} \tag{2.59}$$

Now, substituting (2.59) into the positive root of (2.44) yields

$$x=a-b+\frac{3\left(a-\frac{b}{2}\right)}{(1-q_5)} \tag{2.60}$$

and clearly, in this case for x to be positive, q_5 must be less than unity. Re-arranging (2.60) for a

$$a=\left(x+\frac{b}{2}\right)\left(\frac{1-q_5}{4-q_5}\right)+\frac{b}{2} \tag{2.61}$$

It is clear from (2.61) that a cannot be an integer following the same argument as in Section 2.2. Therefore, z cannot be an integer in (2.1). Thus, subject to q_5 exhibiting satisfactory characteristics, this proves Fermat's Last Theorem for $n=5$.

2.5 Case n =7.

When n =7, (2.3) reduces to

$$
\begin{aligned}
&x^7 - 7(a-b)x^6 - 21\left(a^2-b^2\right)x^5 - 35\left(a^3-b^3\right)x^4 - 35\left(a^4-b^4\right)x^3 - 21\left(a^5-b^5\right)x^2 \\
&-7\left(a^6-b^6\right)x - \left(a^7-b^7\right) = 0
\end{aligned}
\tag{2.62}
$$

The Roots of (2.62) are of the form

$$
\{x-(p+k_1+k_2+k_3)\}\left\{x-\left(p-\frac{k_1}{2}\pm jk_4\right)\right\}\left\{x-\left(p-\frac{k_2}{2}\pm jk_5\right)\right\}\left\{x-\left(p-\frac{k_3}{2}\pm jk_6\right)\right\}=0 \tag{2.63}
$$

Expanding (2.63) gives

$$
x^7 - A_6x^6 + A_5x^5 - A_4x^4 + A_3x^3 - A_2x^2 + A_1x - A_0 = 0 \tag{2.64}
$$

Where

$$
\begin{aligned}
A_6 &= 7p \\
A_5 &= A_4' + A_5'(p+k_1+k_2+k_3) \\
A_4 &= A_3' + A_4'(p+k_1+k_2+k_3) \\
A_3 &= A_2' + A_3'(p+k_1+k_2+k_3) \\
A_2 &= A_1' + A_2'(p+k_1+k_2+k_3) \\
A_1 &= A_0' + A_1'(p+k_1+k_2+k_3) \\
A_0 &= A_0'(p+k_1+k_2+k_3)
\end{aligned}
\tag{2.65}
$$

and where in turn

$$
\begin{aligned}
A_5' &= (6p-k_1-k_2-k_3) \\
A_4' &= \left\{\begin{array}{l}\left(p-\frac{k_3}{2}\right)^2+k_6^2+\left(p-\frac{k_2}{2}\right)^2+k_5^2+(2p-k_2)(2p-k_3) \\ +(4p-k_2-k_3)(2p-k_1)+\left(p-\frac{k_1}{2}\right)^2+k_4^2\end{array}\right\} \\
A_3' &= \left[\begin{array}{l}\left\{\left(p-\frac{k_2}{2}\right)^2+k_5^2\right\}(2p-k_3)+\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2\right\}(2p-k_2) \\ +\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2+\left(p-\frac{k_2}{2}\right)^2+k_5^2+(2p-k_2)(2p-k_3)\right\}(2p-k_1) \\ +(4p-k_2-k_3)\left\{\left(p-\frac{k_1}{2}\right)^2+k_4^2\right\}\end{array}\right] \\
A_2' &= \left[\begin{array}{l}\left\{\left(p-\frac{k_2}{2}\right)^2+k_5^2\right\}\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2\right\} \\ +\left\{\left\{\left(p-\frac{k_2}{2}\right)^2+k_5^2\right\}(2p-k_3)+\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2\right\}(2p-k_2)\right\}(2p-k_1) \\ +\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2+\left(p-\frac{k_2}{2}\right)^2+k_5^2+(2p-k_2)(2p-k_3)\right\}\left\{\left(p-\frac{k_1}{2}\right)^2+k_4^2\right\}\end{array}\right] \\
A_1' &= \left[\begin{array}{l}\left\{\left\{\left(p-\frac{k_2}{2}\right)^2+k_5^2\right\}(2p-k_3)+\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2\right\}(2p-k_2)\right\}\left\{\left(p-\frac{k_1}{2}\right)^2+k_4^2\right\} \\ -\left\{\left(p-\frac{k_2}{2}\right)^2+k_5^2\right\}\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2\right\}(2p-k_1)\end{array}\right] \\
A_0' &= \left[\left\{\left(p-\frac{k_2}{2}\right)^2+k_5^2\right\}\left\{\left(p-\frac{k_3}{2}\right)^2+k_6^2\right\}\left\{\left(p-\frac{k_1}{2}\right)^2+k_4^2\right\}\right]
\end{aligned}
\tag{2.66}
$$

In (2.64) consider the co-efficient of x^5. From (2.65) and (2.66), after multiplying out this reduces to

$$42ap-\frac{3}{4}\left(k_1^2+k_2^2+k_3^2\right)+k_4^2+k_5^2+k_6^2-k_1k_2-k_1k_3-k_2k_3=0 \tag{2.67}$$

Now consider the co-efficient of x^4 in (2.4). From (2.65) and (2.66), after multiplying out this reduces to

$$A_4=-\left[\begin{array}{l} 35p^3-5\left\{\frac{3}{4}\left(k_1^2+k_2^2+k_3^2\right)+k_4^2+k_5^2+k_6^2-k_1k_2-k_1k_3-k_2k_3\right\}p \\ +\left\{\begin{array}{l}\frac{\left(k_1^3+k_2^3+_3^3\right)}{4}+k_1k_2\left(k_1+k_2\right)+k_1k_3\left(k_1+k_3\right)+k_2k_3\left(k_2+k_3\right) \\ +k_1k_4^2+k_2k_5^2+k_3k_6^2+2k_1k_2k_3\end{array}\right\}\end{array}\right] \tag{2.68}$$

Comparing (2.68) with the co-efficient of x^4 in (2.62) gives

$$-35a^3+35b^3=-\left[\begin{array}{l} 35\left(a-b\right)^3-5\left\{\frac{3}{4}\left(k_1^2+k_2^2+k_3^2\right)+k_4^2+k_5^2+k_6^2-k_1k_2-k_1k_3-k_2k_3\right\}\left(a-b\right) \\ +\left\{\begin{array}{l}\frac{\left(k_1^3+k_2^3+_3^3\right)}{4}+k_1k_2\left(k_1+k_2\right)+k_1k_3\left(k_1+k_3\right)+k_2k_3\left(k_2+k_3\right) \\ +k_1k_4^2+k_2k_5^2+k_3k_6^2+2k_1k_2k_3\end{array}\right\}\end{array}\right] \tag{2.69}$$

This reduces to

$$\begin{array}{l}105a^2b-105ab^2+5\left\{\frac{3}{4}\left(k_1^2+k_2^2+k_3^2\right)+k_4^2+k_5^2+k_6^2-k_1k_2-k_1k_3-k_2k_3\right\}\left(a-b\right) \\ -\left\{\frac{\left(k_1^3+k_2^3+_3^3\right)}{4}+k_1k_2\left(k_1+k_2\right)+k_1k_3\left(k_1+k_3\right)+k_2k_3\left(k_2+k_3\right)+k_1k_4^2+k_2k_5^2+k_3k_6^2+2k_1k_2k_3\right\}=0\end{array} \tag{2.70}$$

In (2.70) substitute for the co-efficient of $(a-b)$ from (2.67) and reduce to

$$210a^3-315a^2b+105ab^2-\left\{\begin{array}{l}\frac{\left(k_1^3+k_2^3+_3^3\right)}{4}+k_1k_2\left(k_1+k_2\right)+ \\ k_1k_3\left(k_1+k_3\right)+k_2k_3\left(k_2+k_3\right)+k_1k_4^2+k_2k_5^2+k_3k_6^2+2k_1k_2k_3\end{array}\right\}=0 \tag{2.71}$$

Dividing the term in the k's by $(k_1+k_2+k_3)$ yields

$$210a^3-315a^2b+105ab^2-\left(k_1+k_2+k_3\right)\left\{\begin{array}{l}\frac{1}{4}\left(k_1^2+k_2^2+k_3^2-k_1k_2-k_1k_3-k_2k_3\right) \\ +\frac{1}{\left(k_1+k_2+k_3\right)}\left\{\begin{array}{l}k_1k_2\left(k_1+k_2\right)+k_1k_3\left(k_1+k_3\right)+k_2k_3\left(k_2+k_3\right) \\ +k_1k_4^2+k_2k_5^2+k_3k_6^2+\frac{11}{4}k_1k_2k_3\end{array}\right\}\end{array}\right\}=0 \tag{2.72}$$

and this can be rewritten

$$210a^3-315a^2b+105ab^2-\left(k_1+k_2+k_3\right)\left\{\begin{array}{l}\left\{\frac{3}{4}\left(k_1^2+k_2^2+k_3^2\right)+k_1k_2+k_1k_3+k_2k_3-k_4^2-k_5^2-k_6^2\right\} \\ -\left\{\frac{1}{2}\left(k_1^2+k_2^2+k_3^2\right)-\frac{5}{4}\left(k_1k_2+k_1k_3+k_2k_3\right)-k_4^2-k_5^2-k_6^2\right\} \\ +\frac{1}{\left(k_1+k_2+k_3\right)}\left\{\begin{array}{l}k_1k_2\left(k_1+k_2\right)+k_1k_3\left(k_1+k_3\right)+k_2k_3\left(k_2+k_3\right) \\ +k_1k_4^2+k_2k_5^2+k_3k_6^2+\frac{11}{4}k_1k_2k_3\end{array}\right\}\end{array}\right\}=0 \tag{2.73}$$

Now substitution from (2.67) yields

$$210a^3-315a^2b+105ab^2-(k_1+k_2+k_3)\left\{\begin{array}{l}42a\,(a-b)\\ -\left\{\frac{1}{2}\left(k_1^2+k_2^2+k_3^2\right)-\frac{5}{4}\left(k_1k_2+k_1k_3+k_2k_3\right)-k_4^2-k_5^2-k_6^2\right\}\\ +\frac{1}{(k_1+k_2+k_3)}\left\{\begin{array}{l}k_1k_2\,(k_1+k_2)+k_1k_3\,(k_1+k_3)+k_2k_3\,(k_2+k_3)\\ +k_1k_4^2+k_2k_5^2+k_3k_6^2+\frac{11}{4}k_1k_2k_3\end{array}\right\}\end{array}\right\}=0 \tag{2.74}$$

and dividing through by $42a(a-b)$ then gives

$$5\left(a-\frac{b}{2}\right)-(k_1+k_2+k_3)\left[1-\left\{\frac{\begin{array}{l}\left\{\frac{1}{2}\left(k_1^2+k_2^2+k_3^2\right)-\frac{5}{4}\left(k_1k_2+k_1k_3+k_2k_3\right)-k_4^2-k_5^2-k_6^2\right\}\\ -\frac{1}{(k_1+k_2+k_3)}\left\{\begin{array}{l}k_1k_2\,(k_1+k_2)+k_1k_3\,(k_1+k_3)+k_2k_3\,(k_2+k_3)\\ +k_1k_4^2+k_2k_5^2+k_3k_6^2+\frac{11}{4}k_1k_2k_3\end{array}\right\}\end{array}}{42a\,(a-b)}\right\}\right]=0 \tag{2.75}$$

In (2.75), the numerator in the quotient is of the same order in a and b as the denominator and therefore the quotient will be a pure number, q_7. Therefore, (2.75) becomes

$$5\left(a-\frac{b}{2}\right)-(k_1+k_2+k_3)\,(1-q_7)=0 \tag{2.76}$$

So that

$$(k_1+k_2+k_3)=\frac{5\left(a-\frac{b}{2}\right)}{(1-q_7)} \tag{2.77}$$

Substitution of (2.77) into the positive root of (2.63) then gives

$$x=a-b+\frac{5\left(a-\frac{b}{2}\right)}{(1-q_7)} \tag{2.78}$$

and as in the previous case, q_7 must be less than unity for x to be positive. Re-arranging for a

$$a=\left(x+\frac{b}{2}\right)\left(\frac{1-q_7}{6-q_7}\right)+\frac{b}{2} \tag{2.79}$$

and thus a cannot be an integer following the same argument as in Section 2.2. Therefore z cannot be an integer in (2.1). Thus, subject to q_7 exhibiting satisfactory characteristics, this proves Fermat's Last Theorem for $n=7$.

2.6 Extrapolation to n = Any Odd Number.

From the results for $n=5$ and $n=7$, although only two cases are involved, the analytical process being completely rigorous, permits extrapolation from these two cases to the general one of $n=$ any odd number.

Thus for the general case

$$a=\left(x+\frac{b}{2}\right)\left(\frac{1-q_n}{n-1-q_n}\right)+\frac{b}{2} \tag{2.80}$$

and clearly a in (2.80) and z in (2.1) cannot be integer because the term $\left(\frac{1-q_n}{n-1-q_n}\right)$ cannot be integer. Thus, subject to q_n exhibiting satisfactory characteristics, this, in conjunction with the result for $n=4$, proves Fermat's Last Theorem for all n.

2.7 The q Numbers.

To consolidate the proof, it is necessary to show that the q numbers exhibit satisfactory characteristics as x, b and n independently explore their theoretical maxima and minima, and therefore do not cause anomalies in the relationships developed for the a's.

2.7.1 *The q Numbers in Cases n = 3 and 4.*

These cases are unique in that as indicated in Sections 2.2 and 2.3, these q numbers must always be greater than unity.
From (2.19) or (2.39), re-arranging for q

$$q=\frac{x+{}^{b}\!/_{2}}{x-a+b} \tag{2.81}$$

(a) Minimum Values of x and b.
The minimum values of x and b are each unity. Substitution into (2.81) then gives

$$q=\frac{1{}^{1}\!/_{2}}{2-a} \tag{2.82}$$

As $b<a<2$, q must be greater than unity. Note that, because all three independent variables have been specified here, it is possible to calculate a from (2.2) for the two cases involved. Thus for $n=3$, $a=1.08008$ and for $n=4$, $a=1.03054$.

(b) Maximum x, (b must be unity).
The maximum value of x is $x\to\infty$. Dividing (2.81) throughout by x gives

$$q=\frac{1+{}^{1}\!/_{2x}}{1-\dfrac{(a-1)}{x}} \tag{2.83}$$

and as $x\to\infty$, $q\to 1$ from a value >1.

(c) Maximum b, (x = any value $<\infty$).
The maximum value of b is $b\to\infty$ which incurs $a\to\infty$ and therefore as $b\to\infty$ (2.81) becomes

$$q=\frac{x+{}^{b}\!/_{2}}{x} \tag{2.84}$$

so that as $b\to\infty$, $q\to\infty$. Thus in the case of $n=3$ and $n=4$, the range of variation of q numbers do not cause an anomaly in the relationships developed for the parameter a.

2.7.2 *The q Numbers for n > 4.*

In this case the q numbers must always be less than unity as shown in Sections 2.4 and 2.5. From (2.80), re-arranging for q_n

$$q_n=\frac{x-(n-1)\ a+n{}^{b}\!/_{2}}{x-a+b} \tag{2.85}$$

(a) Minimum Values.
The minimum values of x and b are each unity. Substitution into (2.85) then gives

$$q_n = \frac{1 + {}^{n}\!/_{2} - (n-1)\,a}{2-a} \tag{2.86}$$

The minimum value of n in this case is 5, which when substituted into (2.86) gives

$$q_n = \frac{3^1\!/_2 - 4a}{2-a} \tag{2.87}$$

Because all three independent variables have been specified here, it is possible to determine the value of a from (2.2). The value is 1.0123. Clearly, when substituted into (2.87), $q_n < +1$.
Also it is obvious from (2.2) that as n is increased, a reduces and as $n \to \infty$, $a \to 1$ so that in (2.86), $q_n \to -\infty$. Thus in this case, q_n does not therefore incur anomalies in the relationship developed for a for all $n > 4$.

(b) Maximum x, (b must be unity).
The maximum value of x is $x \to \infty$. Dividing (2.85) throughout by x gives

$$q_n = \frac{1 - (n-1)\,{}^{a}\!/_{x} + {}^{n}\!/_{2x}}{1 - \dfrac{(a-1)}{x}} \tag{2.88}$$

so that as $x \to \infty$, $q_n \to 1$ from a value < 1.
In this case as $n \to \infty$, ${}^{n}\!/_{x} \to 1$ so that (2.88) becomes

$$q_n = \frac{{}^{3}\!/_{2} - a}{1} \tag{2.89}$$

and because $a \to 1$, $q_n < 1$. Thus in this case q_n does not incur anomalies in the relationship developed for a for all $n > 4$.

(c) Maximum b, (x = any value $< \infty$).
The maximum value of b is $b \to \infty$. Dividing (2.85) throughout by b gives

$$q_n = \frac{{}^{x}\!/_{b} - \dfrac{(n-1)\,a}{b} + {}^{n}\!/_{2}}{1 + {}^{x}\!/_{b} - {}^{a}\!/_{b}} \tag{2.90}$$

As $b \to \infty$, so $a \to \infty$ and (2.90) then becomes

$$q_n \to \frac{-{}^{n}\!/_{2} + 1}{0} \to -\infty \tag{2.91}$$

Clearly this result is unaffected by changes in n and so q_n does not cause anomalies in this case in the relationships developed for a for all $n > 4$.

(d) Maximum n, (x and b = any value $< \infty$).
The maximum value of n is $n \to \infty$. Dividing (2.85) throughout by n gives

$$q_n = \frac{{}^{x}\!/_{n} - \left(1 - {}^{1}\!/_{n}\right)\, a + {}^{b}\!/_{2}}{(x-a+b)\!/_{n}} \tag{2.92}$$

As $n \to \infty$, this becomes

$$q_n \to \frac{-a + {}^{b}\!/_{2}}{0} \to -\infty \tag{2.93}$$

Thus in this case also q_n does not cause anomalies in the relationships developed for a for all $n > 4$.

The overall result from this review of the q numbers, is that they incur no anomalies in the relationships developed for the a's as the three independent variables, x, b and n, independently explore their maxima and minima. Therefore, the proof of Fermat's Last Theorem, presented herein, is sound for all values of x, y and n.

3 Conclusions.

Although the analytical process used here includes some very long and involved algebraic derivations, the description of a "simple" proof is justified, because the technical level of the analysis is very elementary, being far below graduate level, and utilising only the very minimum amount of interpretative logic. It is easily of a level available to Fermat and his peers in their day.
In the same book margin that Fermat penned his famous conjecture, he also stated, [2],

"I have discovered a remarkable proof of this theorem which the margins of this book are too small to contain."

It is now thought, [2], that Fermat's stated proof of the general case was flawed, and that he subsequently realised this, which may explain why he apparently did not again mention the conjecture, after challenging two of his peers to prove the cases $n = 3$ and $n = 4$, proofs for which he obviously must have known in order to issue the challenges.
Just before his death in 1665, Fermat penned a communication stating that all of his proofs used a technique that he himself devised known as "infinite descent". If this was the stated "remarkable proof" of his conjecture, then he would almost certainly have realised that it was flawed, because the method of infinite descent can only be successful when starting from a finite position. When starting from an infinite position, as proof of the Last Theorem would require, it must fail because even after an infinite number of descents, there would still be an infinite number to go, and it would therefore never reach a final result.
Also, in view of his communication above, it is also considered unlikely that Fermat's remarkable proof was similar to that presented here.

Finally, as is shown in the Appendix, Fermat's equation is but one version of a more general equation, some variations of which do exhibit integer solutions. It is not yet known whether there are others which, like Fermat's, only partially do so, i.e. as for $n = 2$.

Appendix A

Derivation of All Solutions for $n = 2$.

When $n = 2$, (2.1) exhibits an infinite number of solutions. All of these can, theoretically, be derived as follows.
When $n = 2$, (2.3) reduces to

$$x^2 - 2(a-b)x - \left(a^2 - b^2\right) = 0 \tag{A.1}$$

Solving for x using the standard formula,

$$x = (a-b) \pm \{2a(a-b)\}^{1/2} \tag{A.2}$$

For simplicity, as before, put $p = (a-b)$, so that (A.2) becomes

$$x = p \pm (2ap)^{1/2} \tag{A.3}$$

Thus with a and b, and therefore p, as integers, for x to be an integer, only even squares can appear under the root sign in (A.3). Accordingly, there is a solution to Fermat's equation for $n = 2$ whenever $2pa$ is an even square. When $2pa > 4$, there are multiple solutions for each even square. The following table lists all the solutions for the first six even squares.

#	**2pa**	**p**	**a**	**b**	**x**	**y**	**z**	x^2	y^2	z^2
1	4	1	2	1	3	4	5	9	16	25
2	16	1	8	7	5	12	13	25	144	169
3		2	4	2	6	8	10	36	64	100
4	36	1	18	17	7	24	25	49	576	625
5		2	9	7	8	15	17	64	225	289
6		3	6	3	9	12	15	81	144	225
7	64	1	32	31	9	40	41	81	1600	1681
8		2	16	14	10	24	26	100	576	676
9		4	8	4	12	16	20	144	256	400
10	100	1	50	49	11	60	61	121	3600	3721
11		2	25	23	12	35	37	144	1225	1369
12		5	10	5	15	20	25	225	400	625
13	144	1	72	71	13	84	85	169	7056	7225
14		2	36	34	14	48	50	196	2304	2500
15		3	24	21	15	36	39	225	1296	1521
16		4	18	14	16	30	34	256	900	1156
17		6	12	6	18	24	30	324	576	900
18		8	9	1	20	21	29	400	441	841

Table A.1 - Pythagorean Triples for the First Six Even Squares of 2*pa*.

As shown by the caption to Table A.1, these solutions are, for obvious reasons, known as Pythagorean Triples. It is also obvious that not all such triples are unique. In the above Table for instance, entries 3, 6, 8, 9, 12, 14, 15, 16 and 17 are multiples of lower triples. Those that are unique are known as Principle Triples, and are distinguished by having prime numbers for x and/or z.
The second point to note in the Table is that, both entries 4 and 12 can also be represented in the form

$$x^2 + y^2 = z^4 \tag{A.4}$$

which raises the question as to how many other Pythagorean triples can be so represented. It also raises the question as to whether Fermat's equation can be further generalised to

$$x^l + y^m = z^n \tag{A.5}$$

and whether integer solutions exist for this equation, or any of its possible other variants.
Of interest is a version of (A.5) thus

$$x^2 + y^2 = z^3 \tag{A.6}$$

which, with $z = x + a$, converts to the variable co-efficient elliptic curve

$$y^2 = x^3 + (3a - 1)\, x^2 + 3a^2 x + a^3 \tag{A.7}$$

for which integer solutions would, in the light of Wiles' proof of the Taniyama - Shimura Conjecture incur associated Modular Forms.
In the method used here, (A.6) also converts to the variable co-efficient 3^{rd} order polynomial in x

$$x^3 + (3a - 2)\, x^2 + \left(3a^2 - 2b\right) x + \left(a^3 - b^2\right) = 0 \tag{A.8}$$

The search for integer solutions would not be as simple because identification of the form of the roots is not straightforward.

REFERENCES.

[1] Harold M. Edwards, *Fermat's Last Theorem - A Generic introduction to Algebraic Number Theory.* Springer - Verlag, 1977.

[2] Simon Singh, *Fermat's Last Theorem,* Harper Perennial, 1997.