A new proof of positivity criteria of even order derivatives of Riemann Xi function $\xi(s)$

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Abstract

[In this paper a new proof of positivity criteria of even order derivatives of $\xi(s)$ will be given using analytical expression of Riemann Xi function $\xi(s)$]

Key words : Riemann Xi function $\xi(s)$, Positivity criteria of even order derivatives of $\xi(s)$, Riemann Hypothesis

1. Introduction

The positivity criteria of even order derivatives of $\xi(s)$ is a consequence of increasing nature of $|\xi(s)|$ on horizontal lines [1]. Pustyl'nikov [2] showed that

$$\xi^{(2n)}(\tfrac{1}{2}) > 0 \tag{1.1}$$

Pustyl'nikov showed this first assuming Riemann Hypothesis and then without this assumption [1]. We will show this without assuming RH in the following section.

2. Proof of (1.1)

In a recent paper [3] it was shown that analytic expression of Riemann Xi function $\xi(s)$ has the form

$$\xi(s) = \xi(\sigma + it).$$

$$= F_2(l_1) + F_1(l_1)\left[\text{Cos}\, l_1 t\, \text{Cos h}\, l_1\left(\sigma - \tfrac{1}{2}\right) + i\, \text{Sin}\, l_1 t\, \text{Sin h}l_1\left(\sigma - \tfrac{1}{2}\right)\right] \tag{2.1}$$

l_1, $F_2(l_1)$, $F_1(l_1)$ are all positive and can not be determined.

Therefore,

$$\frac{d}{ds}\xi(s)$$

$$= \xi^{(1)}(s)$$

$$= \frac{\partial}{\partial\sigma}\left[F_2(l_1) + F_1(l_1)\, \text{Cos}\, l_1 t\, \text{Cos h}\, l_1\left(\sigma - \tfrac{1}{2}\right)\right] + i\,\frac{\partial}{\partial\sigma}\left[F_1(l_1)\ \text{Sin}\, l_1 t\, \text{Sin h}l_1\left(\sigma - \tfrac{1}{2}\right)\right]$$

$$= l_1\ F_1(l_1)\ \text{Cos}\ l_1 t\ \text{Sin h}\ l_1\left(\sigma - \tfrac{1}{2}\right) + i\ l_1\ F_1(l_1)\ \text{Sin}\ l_1 t\ \ \text{Cos h}\ l_1\left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(2.2)$$

Likewise $\frac{d^2}{ds^2}\xi(s)$

$$= \xi^{(2)}(s)$$

$$= l_1^{\,2}\ F_1(l_1)\ \text{Cos}\ l_1 t\ \text{Cos h}\ l_1\left(\sigma - \tfrac{1}{2}\right) + i l_1^{\,2}\ F_1(l_1)\ \ \text{Sin}\ l_1 t\ \text{Sin h} l_1\left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(2.3)$$

And

$$\frac{d^{2n}}{ds^{2n}}\xi(s)$$

$$= \xi^{(2n)}(s)$$

$$= l_1^{\,2n}\ F_1(l_1)\ \text{Cos}\ l_1 t\ \text{Cos h}\ l_1\left(\sigma - \tfrac{1}{2}\right) + i l_1^{\,2n}\ F_1(l_1)\ \ \text{Sin}\ l_1 t\ \text{Sin h} l_1\left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(2.4)$$

Now taking $t = 0$ in (2.4) we find

$$\xi^{(2n)}(\sigma) = l_1^{\,2n}\ F_1(l_1)\ \text{Cos h}\ l_1\left(\sigma - \tfrac{1}{2}\right) \qquad \ldots(2.5)$$

Therefore

$$\xi^{(2n)}(½) = l_1^{\,2n}\ F_1(l_1) \qquad \ldots(2.6)$$

The R.H.S of (2.6) is clearly positive

Hence $\xi^{(2n)}(½) > 0$ …(2.7)

Thus the positivity criteria of even order derivatives of Riemann Xi function $\xi(s)$ is established.

3. Conclusion .

The above result is a direct consequence from analytic expression of Riemann Xi function [3].

Rererences .

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