The Complex Square Function

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Abstract

Numbers and their squares show up in many different formulas and theorems, such as the arithmetic sum formulas, the difference of two squares, quadratic functions, and the Pythagorean Theorem. It is beneficial to introduce one universal functional notation that can couple most of these into one coherent form, so that it becomes easier to connect the dots between them and in order to generalize and extend existing formulae and theorems to become more useful. The Complex Square Function introduces this notation and uses its properties to generalize and extend the abilities of the squared numbers.
The Complex Square Function

Choose \( a, b \in \mathbb{C} \). Denote the difference of \( b \) and \( a \) as \( N = b - a \). By the Difference of Two Squares Formula, \( b^2 - a^2 = (b - a)(b + a) \). Now one can rewrite \( b \) in terms of \( N \) and \( a \): \( b = N + a \). By substituting this into the previous formula, the equation becomes

\[(N + a - a)(N + a + a) = N(2a + N)\]

With this, the Complex Square Function can be defined as a multivariable function: \( f(a, N) = N(2a + N) \). This function is defined for all complex \( a, N \) and is based off of the Difference of Two Squares Formula (hence the name of the function: the Complex Square Function). This function has special properties that can make it easier to understand and use other special formulas and theorems and introduces a simpler functional notation for these formulas.

Properties of the Complex Square Function

The Complex Square Function has the following properties (attached are their direct proofs as well):

1. \( f(a, 1) = 2a + 1 \)
   
a. \( f(a, 1) = 1(2a + 1) = 2a + 1 \) by plugging in 1 for \( N \) in the function’s definition.

2. \( f(a, 0) = 0 \)
   
a. \( f(a, 0) = 0(2a + 0) = 0 \) by plugging in 0 for \( N \) in the function’s definition.

3. \( f(x, x) = 3x^2 \)
a. $f(x, x) = x(2x + x) = x(3x) = 3x^2$ by plugging in $x$ for $a$ and $x$ for $N$ in the function’s definition.

4. $\frac{\delta f}{\delta N} = 2a + 2N$

   a. $\frac{\delta}{\delta N}[N(2a + N)] = \frac{\delta}{\delta N}(2aN + N^2) = 2a + 2N$ by taking the partial derivative of $f$ with respect to $N$ using the Power Rule of Differentiation.

5. $\frac{\delta f}{\delta a} = 2N$

   a. $\frac{\delta}{\delta a}[N(2a + N)] = \frac{\delta}{\delta a}(2aN + N^2) = 2a$ by taking the partial derivative of $f$ with respect to $a$ using the Power Rule of Differentiation.

6. $f(a, N) = 2N(a + \frac{N}{2}) \leftrightarrow N \equiv 0 \text{ (mod 2)}$

   a. $N \equiv 0 \text{ (mod 2)}$ implies that $N$ is even. To signify this, the function is rewritten to $f(a, 2N)$.

   b. $f(a, 2N) = 2N(2a + 2N) = 4N(a + N)$ by plugging in $2N$ for $N$ in the function’s definition.

   c. Now one can divide the $N$ expressions by 2 and rescale the function to $f(a, N)$:

      i. $f(a, N) = 2N(a + \frac{N}{2})$

   d. To verify, it is recommended that the reader use their own case for $N$ to create a simple proof by case for this property.
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Application to the Pythagorean Theorem

The Pythagorean Theorem

The Pythagorean Theorem relates the legs \(a\), \(b\) and the hypotenuse \(c\) of a right triangle by the equation \(a^2 + b^2 = c^2\).

Relationship With the Complex Square Function

The Pythagorean Theorem can be rewritten in terms of \(a\), \(c\) and in terms of \(b\), \(c\):

1. \(b^2 = c^2 - a^2\)
2. \(a^2 = c^2 - b^2\)

Using the definition of the Complex Square Function, \(b^2\) and \(a^2\) can be redefined as:

1. \(b^2 = c^2 - a^2 = f(a, c - a)\)
   a. Let \(N = c - a\)
   b. \(c^2 - a^2 = f(a, N) = f(a, c - a)\)
2. \(a^2 = c^2 - b^2 = f(b, c - b)\)
   a. Let \(N = c - b\)
   b. \(c^2 - b^2 = f(a, N) = f(b, c - b)\)

Therefore, the values of \(a\) and \(b\) are \(\sqrt{f(b, c - b)}\) and \(\sqrt{f(a, c - a)}\), respectively. Now, \(c\) can be defined by plugging these values of \(a\) and \(b\) into the Pythagorean Theorem:

1. \(\sqrt{f(b, c - b)}^2 + \sqrt{f(a, c - a)}^2 = c^2\)
2. \(c^2 = f(b, c - b) + f(a, c - a)\)
3. \(c = \sqrt{f(b, c - b)} + f(a, c - a)\)
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From these new values of $a$, $b$, and $c$, the Pythagorean Theorem can be redefined as:

1. $\sqrt{f(b, c-b)^2} + \sqrt{f(a, c-a)^2} = \sqrt{f(b, c-b) + f(a, c-a)^2}$

2. $a = \sqrt{f(b, c-b)}$

3. $b = \sqrt{f(a, c-a)}$

4. $c = \sqrt{f(b, c-b) + f(a, c-a)}$

This portrays the application of the Complex Square Function to the Pythagorean Theorem, by creating notationally simple formulas that can determine the side lengths of a right triangle in terms of the Complex Square Function.

The Difference of Higher Squares

The Difference of Three Squares

The difference of three squares is essentially the difference between three perfect squares. Essentially it is one perfect square subtracted from another perfect square subtracted from a third perfect square as such: $c^2 - b^2 - a^2$.

Formulation of the Difference of Three Squares

The difference of three squares can be defined using the Complex Square Function by the following process:

1. $c^2 - b^2 - a^2 = (c^2 - b^2) - a^2 = f(b, c-b) - a^2 = \sqrt{f(b, c-b)^2} - a^2 = f(a, \sqrt{f(b, c-b)} - a)$

As shown, the Complex Square Function can be used to calculate the difference of three squares by recursively grouping a pair of squares, converting it to the Complex Square Function form, and then doing it again for this new expression and the next term.
The Complex Square Function

The Difference of Higher Squares

The difference of higher squares \((d^2 - c^2 - b^2 - a^2, e^2 - d^2 - c^2 - b^2 - a^2, \text{etc.})\) can be calculated by the same recursive collapsation process, as exhibited in the difference of three squares, until the expression is in the form \(f(a, x)\).

The Difference of Consecutive Squares

Choose \(a, b\) such that \(b = a + 1\). Let \(N = b - a = 1\).

\[ f(a, N) = f(a, 1) = 2a + 1. \]

Note that this is the same as saying that \(b^2 - a^2 = 2a + 1\).

This can also be directly proved:

1. \(b^2 - a^2 = (b - a)(b + a) = (a + 1 - a)(a + 1 + a) = 2a + 1\)

One can also find that \(a + b = 2a + 1\):

1. \(a + b = a + a + 1 = 2a + 1\)

This creates the special formula entitled the Difference of Consecutive Squares:

1. \(b^2 - a^2 = a + b = 2a + 1\) when \(b = a + 1\)

This formula can be used in many different situations, but is especially useful in competitive mathematics, as it says that the difference of two consecutive squares (for example, \(3^2 - 2^2\)) is the sum of the two bases \((2 + 3)\).

Note that this formula is essentially the property of the Complex Square Function \(f(a, 1)\).
Universalization of the Sum Formulas

The Sum Formulas

The sum formulas are the formulas used to find the sum of the first \( n \) positive integers, the first \( n \) even positive integers, and the first \( n \) odd positive integers. The formulas are as follows:

1. \[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \] for the first \( n \) positive integers

2. \[ \sum_{k=1}^{n} 2k = n(n + 1) \] for the first \( n \) even positive integers

3. \[ \sum_{k=1}^{n} (2k - 1) = n^2 \] for the first \( n \) odd positive integers

Relationship With the Complex Square Function

All of the sum formulas can be redefined using the Complex Square Function. This can be directly proved by the following cases:

1. \[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \] (The Sum of the First \( n \) Positive Integers)

   a. \( f\left(\frac{1}{2}, n\right) = n(1 + n) = n(n + 1) \)

   b. Thus, \[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{f\left(\frac{1}{2}, n\right)}{2} \]

   c. Therefore, the sum of the first \( n \) positive integers is \( \frac{f\left(\frac{1}{2}, n\right)}{2} \), when defined using the Complex Square Function.
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2. \[ \sum_{k=1}^{n} 2k = n(n + 1) \] (The Sum of the First \( n \) Even Positive Integers)

   a. \[ f\left(\frac{1}{2}, \; n\right) = n(1 + n) = n(n + 1) \]

   b. Thus, \[ \sum_{k=1}^{n} 2k = n(n + 1) = f\left(\frac{1}{2}, \; n\right) \]

   c. Therefore, the sum of the first \( n \) even positive integers is \( f\left(\frac{1}{2}, \; n\right) \), when defined using the Complex Square Function.

3. \[ \sum_{k=1}^{n} (2k - 1) = n^2 \] (The Sum of the First \( n \) Odd Positive Integers)

   a. \[ f(0, \; n) = n(0 + n) = n(n) = n^2 \]

   b. Thus, \[ \sum_{k=1}^{n} (2k - 1) = n^2 = f(0, \; n) \]

   c. Therefore, the sum of the first \( n \) odd positive integers is \( f(0, \; n) \), when defined using the Complex Square Function.

This effectively universalizes the sum formulas by defining all three of them using the Complex Square Function.

Generalization

The Complex Square Function can also be used to generalize the sum formulas. These generalizations can also be directly proved as follows:

1. \[ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \] (The Sum of the First \( n \) Positive Integers)
a. The generalization is \[ \sum_{k=a}^{n} k, \] with the three cases \( a < 0, \ a > 0, \ n \leq 0. \)

b. \( a < 0 \)

i. \[ \sum_{k=a}^{n} k = [a + ... + (-1)] + 0 + 1 + ... + n = \left( \sum_{k=1}^{n} k \right) - \sum_{k=1}^{\lfloor a \rfloor} k \]

ii. Now by using the previously outlined formulas:

\[
1. \left( \sum_{k=1}^{n} k \right) - \sum_{k=1}^{\lfloor a \rfloor} k = \frac{f\left(\frac{1}{2}, n\right)}{2} - \frac{f\left(\frac{1}{2}, \lfloor a \rfloor\right)}{2} = \frac{f\left(\frac{1}{2}, n\right) - f\left(\frac{1}{2}, \lfloor a \rfloor\right)}{2}
\]

c. \( a > 0 \)

i. \[ \sum_{k=a}^{n} k = a + ... + n = (1 + 2 + ... + n) - (1 + 2 + ... + a - 1) = \]

\[ \left( \sum_{k=1}^{n} k \right) - \sum_{k=1}^{a-1} k \]

ii. Now by using the previously outlined formulas:

\[
1. \left( \sum_{k=1}^{n} k \right) - \sum_{k=1}^{a-1} k = \frac{f\left(\frac{1}{2}, n\right)}{2} - \frac{f\left(\frac{1}{2}, a-1\right)}{2} = \frac{f\left(\frac{1}{2}, n\right) - f\left(\frac{1}{2}, a-1\right)}{2}
\]

d. \( n \leq 0 \)

i. \[ \sum_{k=a}^{n} k = (- \sum_{k=|n|}^{\lfloor a \rfloor} k) \]

ii. Now by applying the case \( a > 0 \):

\[
1. -\left( \sum_{k=|n|}^{\lfloor a \rfloor} k \right) = - \left( \left( \sum_{k=1}^{\lfloor a \rfloor} k \right) - \sum_{k=1}^{\lfloor n \rfloor - 1} k \right) = - \frac{f\left(\frac{1}{2}, \lfloor a \rfloor\right) - f\left(\frac{1}{2}, \lfloor n \rfloor - 1\right)}{2}
\]

e. Thus, \[ \sum_{k=a}^{n} k = \]
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1. \[
\frac{f\left(\frac{1}{2}, n\right) - f\left(\frac{1}{2}, |a|\right)}{2}
\]
   when defined using the Complex Square Function
   and \(a < 0\),

ii. \[
\frac{f\left(\frac{1}{2}, n\right) - f\left(\frac{1}{2}, a - 1\right)}{2}
\]
   when defined using the Complex Square Function
   and \(a > 0\),

iii. \[
-\frac{f\left(\frac{1}{2}, |a|\right) - f\left(\frac{1}{2}, |n| - 1\right)}{2}
\]
   when defined using the Complex Square Function and \(n \leq 0\).

2. \[
\sum_{k=1}^{n} 2k = n(n + 1)
\] (The Sum of the First \(n\) Even Positive Integers)

   a. Note that this sum formula is the same as the sum formula for the first \(n\)
      positive integers but multiplied by 2. This can be verified by the fact that
      \[
      \sum_{k=1}^{n} 2k = 2 \sum_{k=1}^{n} k.
      \]

   b. Thus, \[
   \sum_{k=a}^{n} 2k =
   \]
      i. \[
      f\left(\frac{1}{2}, n\right) - f\left(\frac{1}{2}, |a|\right)
      \]
      when defined using the Complex Square Function and \(a < 0\),
      
      ii. \[
      f\left(\frac{1}{2}, n\right) - f\left(\frac{1}{2}, a - 1\right)
      \]
      when defined using the Complex Square Function and \(a > 0\),
      
      iii. \[
      - f\left(\frac{1}{2}, |a|\right) + f\left(\frac{1}{2}, |n| - 1\right)
      \]
      when defined using the Complex Square Function and \(n \leq 0\).
3. \[ \sum_{k=1}^{n} (2k - 1) = n^2 \] (The Sum of the First \( n \) Odd Positive Integers)

a. The generalization is \[ \sum_{k=a}^{n} (2k - 1) \], with the two cases \( a < 0, a > 0 \).

b. \( a < 0 \)

i. \[ \sum_{k=a}^{n} (2k - 1) = \]

\[ \left\{ \left[ (2a - 1) + (2a + 1) + \ldots + 1 \right] + [1 + \ldots + (2n - 1)] \right\} = \]

\[ \sum_{k=1}^{n} (2k - 1) - \sum_{k=1}^{|a|+1} (2k - 1) = f(0, n) - f(0, |a| + 1) \] by the previously outlined formula.

c. \( a > 0 \)

i. \[ \sum_{k=a}^{n} (2k - 1) = \left[ (2a - 1) + (2a + 1) + \ldots + (2n - 1) \right] = \]

\[ [1 + \ldots + (2n - 1)] - [1 + \ldots + (2a - 3)] = \]

\[ \sum_{k=1}^{n} (2k - 1) - \sum_{k=1}^{|a|-1} (2k - 1) = f(0, n) - f(0, |a| - 1) \] by the previously outlined formula.

ii. This case also works for the case \( n \leq 0 \) as \( a < n \) for all \( a, n \).

By these proofs, one can use the Complex Square Function to generalize and extend all of the sum formulas with a simple notation. These formulas will work for all \( a, n \). The reader is recommended to try using these formulas themselves in order to verify their functionality.
**Programming Implementation**

In Python 3, the Complex Square Function can be implemented as:

```
1. def f(a, N):
    return 2*a*N + N*N
```

This function can be copy pasted in order to use the Complex Square Function in Python 3.
Results

The Complex Square Function successfully introduces a simple functional notation that connects formulae and theorems requiring squared numbers (such as the Pythagorean Theorem and the sum formulas).

Outcome 1

The Complex Square Function introduced a functional notation $f(a, N)$.

Outcome 2

The Complex Square Function generalized the Pythagorean Theorem and extended the sum formulas.

Outcome 3

The Complex Square Function introduced two new formulas, the Difference of Consecutive Squares and the Difference of Three Squares, and a new generalization, the Difference of Higher Squares.

Conclusion

These applications may be just a fragment of the full generalization potential of the Complex Square Function. The Complex Square Function may be able to generalize or extend even more theorems, formulas, and even other single-variable and multivariable functions, just by exploiting the power of the squared numbers.