Klein-Gordon Equation
in the theory of deformed tensor products

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This document explains, in a four-dimensional context, how the Klein-Gordon equation can be related to the co-variant version of the Lorentz law. It does it with the help of results inherited from a pure mathematical method explaining how deformed tensor products can be non-trivially decomposed. The exploration is yielding interesting tools for a renewed description of massive particles.

Key words: Klein-Gordon equation, deformed tensor products, co-variant formalism of the Lorentz law.

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English
1 The context of the discussion

1.1 History

The theory of deformed tensor products\(^1\) is revisiting on its own the very old and natural duality linking the scalar and the exterior products. This duality did not catch so much attention until the end of the nineteenth century. There are at least two explanations for this fact: (i) most of, if not all, the research which has been made at this time was only concerning Euclidean three-dimensional spaces; (ii) the Euclidean scalar product and the cross product live in a perfect harmony entirely contained in the simple relation:

\[
\forall a, b \in E(3, R) : ||a \cdot b||^2 + ||a \wedge b||^2 = ||a||^2 \cdot ||b||^2
\]

New developments appear at the beginning of the twentieth century when E. Cartan re-obtains in [01]; 1922, \(^2\] A. Einstein’s field equation again [02]. The revival is due to the fact that E. Cartan demonstrates the famous equation in generalizing the moving triad method to a full four dimensional space, while A. Einstein get it with the concepts of parallel transport and of co-variance.

This situation illustrates the Palatini’s principle [03]; 1919 arguing that metrics and connections follow in general separate destinies. In the same vein, a recent and unfortunately not so well-known work on Clifford’s algebras [04]; 1999 recovers the Einstein’s equation. It should be understood as a supplementary illustration of the thematic developed here.

After that, the scientific community is obliged to state the existence of two equivalent descriptions of the nature. One of them is involving the Levi-Civita connection which is also known to be "metric-compatible" and the other one is related to the spin connection. Hence, the metric is usually related to the scalar product and the spin connection to the exterior product.

In our everyday world, each product exhibits a natural complementary behavior to the other one. Since there should exist a relative independence between these kinds of products (due to the Palatini’s principle), it may be suspected that that beautiful harmony only is an illusion resulting from a statistical coincidence or a door opening a hidden Pandora box.

1.2 Hypothesis

The theory of the (E) question lies in fact on a simple hypothesis which is nothing but the image in a mirror of the hypothesis (i) initially proposed by Riemann, (ii) already accepted and applied within the theory of relativity [02]; namely: mathematical spaces have forms (topological) and they determinate, eventually changing, local metric that must be involved for measurements by a geometer.

By the way, it should be recalled here that the deep links between surfaces and metrics has also been studied by E. Cartan in [05]; 1933 and that, despite of numerous recent experimental successes of Einstein’s theory, [06]; geodetic effect and Thirring-Lense effect, [07]; gravitational waves, the discovery of its solutions represents yet a permanent challenge today. The difficulty is rooted in the Cauchy’s surfaces problem (For a historical reference see [08]).

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1 In my semantic, it is the so-called “Theory of the (E) Question (short: TEQ)”.
2 nota bene: He introduces by side the cosmological constant in Einstein’s theory with a mathematical argumentation and suggests a physical interpretation for it.
Despite of the difficulties, it has been proved that the co-variant version of a force, any one e.g. the Lorentz force in electromagnetism, accepts the relation $g_{\alpha\beta} u^\alpha u^\beta = \text{constant}$ depending on the initial conditions as solutions [Fliessbach; chapter 20, pp. 106-107]. As a matter of facts, a tensor product deformed by the cube containing the Christoffel's symbols of the second kind is the mathematical representation of the co-variance.

I call it the gravitational term and I argue here that, due to the relative independence between connections and metrics, Riemann's hypothesis has an actually forgotten counter-part acting via the deformation of tensor products. More precisely: the deformations of the topology are not only deforming the metrics but the tensor products too. I also argue that the measurements of surfaces ($D = 2$), volumes ($D = 3$) and hyper-volumes ($D = 4$) are a pertinent manner to experimentally verify in which way this hypothesis is effectively realized.

Concretely, I shall consider that a surface ($D = 2$), a volume (when the space has dimension $D = 3$) or a hyper-volume (when the space has dimension $D = 4$) generated by respectively two, three or four vectors are calculated, not with Riemann's integral calculus but, with the definition given by E. Cartan in [10]; §14, p. 15. In my semantic and terminology, the tool which has been introduced in [10] only is what I have called a Pythagorean table acting on the canonical basis of $\mathbb{R}^D$, $\Omega$ ($e_0, ..., e_D - 1$), via the inner scalar product: $<\cdot, \cdot >[G]$:

$$V_{\Omega^2}(\Omega) = |T_2(<\cdot, \cdot >[G])(\Omega, \Omega)|$$


$${}^{3}\text{Deformed tensor products are deformed Lie product when they are built with the help of anti-symmetric cubes.}$$
1.3 Purpose of the document.
In this work, I want to bring the proof for the following propositions:

1. The Klein-Gordon equation describing the propagation of massive waves in four-dimensional spaces can be analyzed with the extrinsic method, a mathematical tool allowing the non-trivial decomposition of deformed tensor products.

2. The non-trivially decomposed deformed tensor product resulting from the analysis which has been made in point 1 is sometimes a representation of the co-variant version of the Lorentz law.

3. The analysis which has been made in point 1 is related to a polynomial which is an indirect representation of the relation of dispersion for the Klein-Gordon wave.

2 A mathematical tool: the extrinsic method for the decomposition of deformed tensor products

2.1 Motivations and basics
The extrinsic method has already been explained in others parts of my work. Readers who are still informed about it can either read a résumé in § 2.5 or directly jump to § 3. Any way, for a better understanding of what follows, I expose it here again.

Remark. Prerequisite.
The first prerequisite is the existence of a non-degenerated bi-linear form acting on elements in $E(4, \mathbb{R})$; its representation is a $(4\times4)$ matrix $[B]$ in $M(4, \mathbb{R})$.

$$(a, b) \in E^2(4, \mathbb{R}) \mapsto b(a, b) = \langle a | \cdot [B] \cdot b \rangle = \langle a, b \rangle |_{B} \in \mathbb{R} ; \ |B| \neq 0$$

Remark. The mathematical context of the physical discussion.
The theory of the (E) question develops mathematical tools; mainly, methods allowing the decomposition of deformed tensor, exterior and Lie products. One of these methods is "intrinsic" because it does not need more ingredients than the ones appearing in the question. It is exposed in the seminal work $[[a]]$ and works perfectly well in a three-dimensional space $E(3, \mathbb{C})$, inclusively at the Euclidean limit, see $[[b]]$. It can also be proposed for investigating some concrete situations (e.g.: $[[c]]$ and $[[d]]$), inclusively the Klein-Gordon Equation (KGE) itself, but until now only in a three plus one approach $[[e]]$.

Unfortunately, that intrinsic method carries two important defaults with it:

- Firstly, it yields no information concerning the remaining/residual part of a non-trivial decomposition.
- Secondly, there is no equivalent and achieved investigation for a space of which the dimension is greater than three.

There are also two arguments justifying the development of another approach:

- A bad one one: basing my opinion on $[[a]]$, I suspect that the investigation of an intrinsic method in a four dimensional context will be very cumbersome and lengthy.
A better one: recalling that mathematics and physics are related to each other and knowing that our measurements are rarely precise (just think about the Heisenberg’s Uncertainty Principle - HUP), I argue that decompositions of deformed products may eventually be realized, but approximately; concretely, there is an error of realization, $\delta E$ such that:

**Definition. Approximative decomposition.**

$$(q_1, q_2) \in E^2(4, R) :$$

$$\exists |\delta E| = |\otimes_A (q_1, q_2) > - \{[P] q_2 > + |z >\}$$

### 2.2 The method.

At this stage, the prerequisite (see the first remark) allows the calculation of two scalars which will prove to have a great importance later; more precisely:

- **Definition. The scalar related to the projectile.**

$$(q_1, q_2) \in E^2(4, R) :$$

$$< q_1, \delta E >_{[B]} =$$

$$< q_1, |\otimes_A (q_1, q_2) > - \{[P] q_2 > + |z >\} >_{[B]}$$

Let suppose that there exists a numerical function $P_1$ depending on the projectile and such that:

$$a, q_1 \in E(4, R), P_1 \in F(E(4, R); R) :$$

$$P_1(a + q_1) - P_1(a) = < q_1, \delta E >_{[B]}$$

This relation can also be written as:

$$P_1(a + q_1) - P_1(a) = < q_1, |\otimes_A (q_1, q_2) > - \{[P] q_2 > + |z >\} >_{[B]}$$

Since there is always at least one trivial decomposition for any given deformed tensor product:

$$P_1(a + q_1) - P_1(a) =$$

$$< q_1, \{[B], \{A\Phi(q_1), q_2 >\}\} - < q_1, \{[B], \{[P] q_2 > + |z >\}\}\}$$

Let suppose that the difference appearing in the l.h.t accepts a Taylor development up to the second order, then:

$$P_1(a + q_1) - P_1(a) =$$

$$\frac{1}{2} < q_1, \{[HessP_1](a), q_1 > + < \text{Grad}_{\{q_1\}}P_1(a), q_1 > + o(3)$$

Let write the three conditions allowing the equivalence between the both expressions:
Terms with degree two:
\[ <q_1| \cdot \{[B] \cdot \{(A)\Phi(q_1)\} \cdot |q_2 >\} > = \frac{1}{2} \cdot <q_1| \cdot \{HessP_1(a)\} \cdot |q_1 >\]

Terms with degree one:
\[ \text{Grad}_{(q_1)}P_1(a) = -[B] \cdot \{[P] \cdot |q_2 > + |z >\}\]

Term with degree zero:
\[ 0(3) = P_1(a + q_1) - P_1(a)\]

Definition. The scalar related to the target.
\[ (q_1, q_2) \in E^2(4, R) : \]
\[ <q_2, \delta E >_{[B]} = <q_2, | \otimes_A (q_1, q_2) > - \{[P] \cdot |q_2 > + |z >\} >_{[B]}\]

Let suppose that there exists a numerical function \( P_2 \) depending on the target and such that:
\[ b, q_2 \in E(4, R), P_2 \in F(E(4, R); R) : \]
\[ P_2(b + q_2) - P_2(b) = <q_2, \delta E >_{[B]}\]

Due to the definition:
\[ P_2(b + q_2) - P_2(b) = \]
\[ <q_2, | \otimes_A (q_1, q_2) > - \{[P] \cdot |q_2 > + |z >\} >_{[B]}\]

Since there is always at least one trivial decomposition for any given deformed tensor product:
\[ P_2(b + q_2) - P_2(b) = \]
\[ <q_2, \{[B] \cdot \{(A)\Phi(q_1) - |P|\} \cdot |q_2 >\} - <q_2, \{[B] \cdot |z >\} >\]

Let suppose it is the beginning of a Taylor development:
\[ P_2(b + q_2) - P_2(b) = \]
\[ \frac{1}{2} \cdot <q_2, \{[HessP_2(b)] \cdot \{q_2 >\} + < \text{Grad}_{(q_1)}P_2(b) \cdot |q_2 >\}\]

That eventuality is only meaningful if three conditions are simultaneously realized:

Terms of degree two:
\[ [B] \cdot \{(A)\Phi(q_1) - |P|\} = \frac{1}{2} \cdot [HessP_2(b)]\]

Terms of degree one:
\[ \text{Grad}_{(q_1)}P_2(b) = -[B] \cdot |z >\]

Term of degree zero:
\[ 0(3) = P_2(b + q_2) - P_2(b)\]
2.3 Results.

Since the matrix $[B]$ representing the bi-linear form at hand is non-degenerated,

$$|B| \neq 0 \Rightarrow \exists [B]^{-1}$$

the generic deformed tensor product involved in that procedure can be approximately decomposed and:

- Terms of degree two:

$$[P] = A\Phi(q_1) - \frac{1}{2} [B]^{-1} \cdot HessP_2(b)$$

- Terms of degree one:

$$|z > = - [B]^{-1} \cdot \text{Grad}(q_2) P_2(b) >$$

- Term of degree zero:

$$0(3) = P_2(b + q_2) - P_2(b)$$

**Remark. Conclusion.**

The method confronting an approximative non-trivial decomposition and a Taylor development allows the discovery of non-trivial decompositions $([P], z)$ such that:

$$(q_1, q_2) \in E^2(4, R) : |\delta E > = | \otimes_A (q_1, q_2) > - \{[P], |q_2 > + |z >\}$$

The functions $P_1$ and $P_2$, their gradients and their classical Hessians are the main ingredients intervening here. Let me add some complementary comments. For the sake of generality, these functions have been chosen as if they were not related to each other; but there are certainly some situations in physics allowing to make use of the method with a unique function, for example $L(P_1, P_2)$. The method is said to be "extrinsic" because its realization needs the intervention of mathematical objects which are not explicitly contained in the formulation of the question:

$$(q_1, q_2) \in E^2(4, R) : \exists (|P|, z) : | \otimes_A (q_1, q_2) > = \{[P], |q_2 > + |z >\}$$

2.4 Logical test

All mathematical methods have a domain of validity; the extrinsic method too. **Stricto sensu**, the existence of an exact non-trivial decomposition annihilates the scalar related either to the projectile or to the target:

$$\exists (|P|, z) : | \otimes_A (q_1, q_2) > - \{[P], |q_2 > + |z >\} = |0 >$$

$$\delta E = 0 \rightarrow i = 1, 2 : < q_i, \delta E >_{[B]} = 0$$

But conversely, the vanishing of the scalar, for example related to the target, is corresponding to three plausible and different logical configurations:

$$< q_2, \delta E >_{[B]} = 0$$
• **Configuration 1**: The target is null; hence, the deformed tensor product is null too and this situation is obviously meaningless here.

\[ q_2 = 0 \]

• **Configuration 2**: The non-trivial decomposition is realized:

\[ \delta E = 0 \]

• **Configuration 3**: The target and the error are orthogonal but none of them vanishes:

\[ q_2 \neq 0, \delta E \neq 0, \langle q_2, \delta E \rangle = 0 \]

### 2.5 Résumé

Within that theory, tensor products act on elements living in a vector space \( E(D, K) \). They can be deformed via the intervening of cubes containing elements arbitrarily chosen in \( K \). Per convention, for a given product, the first argument is the projectile and the second one is called the target.

The extrinsic method is a mathematical method offering a concrete visage for non-trivial decomposition of these deformed tensor products. It works only on vector spaces equipped with non-degenerated bi-linear forms; i.e. they are represented by non-degenerated \((D-D)\) matrices, i.e.: \([B]\) such that \(|B| \neq 0\).

Per construction, that method involves triples taken in \( K \times E(D, K) \times M(D, K) \) where \( K \) is the "body" for the discussion. \( E(D, K) \) is a vector space (with dimension: \( D \)) built on this body; and \( M(D, K) \) is the set of \((D-D)\) square matrices with elements in \( K \). Each triple represents a scalar associated with an approximative non-trivial decomposition of the deformed tensor product at hand, a vector which is the remaining/residual part of that non-trivial decomposition and a matrix which is its main part: \((s, z, [P])\).

Although the extrinsic method (the label extrinsic is due to the intervening of the scalar \( s \) measuring the error in the realization of the non-trivial decomposition) yields all ingredients of the decomposition, precisely the pair \(([P], z)\), it is handicapped by logical constraints. Precisely, if a non-trivial decomposition is exact, then the scalar \( s \) vanishes \((0, z, [P])\); but the converse is not systematically true. Precisely: the existence of a situation \((0, z, [P])\) is not automatically the signature for an exact non-trivial decomposition.

### 3 The Klein-Gordon equation

#### 3.1 The Klein-Gordon equation and the dispersion relation (recall)

I already have examined the Klein-Gordon equation (KGE) in [[e]]. I did it in involving a \( 3 + 1 \) approach and the intrinsic method exposed in [[a]].

Here, following a new strategy, I shall confront the KGE and the extrinsic method in a full four-dimensional context. For that purpose, let recall that the dispersion relation [[09]; p. 4, (5)]:

\[ E^2 = m^2 \cdot c^4 + c^2 \cdot p^2 \]

\(^{4}\text{"Körper" in German language and "corps" in French language.}\)
can be related to the KGE with the help of relations [[09]; p. 4, (1)]:

\[ E \rightarrow i.\hbar. \frac{\partial}{\partial t}; \forall a = 1, 2, 3 : p_{a} \rightarrow -i.\hbar. \frac{\partial}{\partial x^{a}} \]

Let write the KGE as in [[09]; chapter 1, p. 4, (6)] where the symbol \( \Delta \) represents a classical (three-dimensional Euclidean space without curvature) Laplace’s operator acting via partial derivations by respect for the respective components of the spatial position:

\[ \forall \psi : \frac{1}{c^{2}} \cdot \frac{\partial^{2} \psi}{\partial t^{2}} - \Delta \psi = -\frac{m^{2}}{\hbar^{2}} \cdot c^{2} \cdot \psi; \quad \Delta \psi = \frac{\partial^{2} \psi}{\partial^{2} x^{1}} + \frac{\partial^{2} \psi}{\partial^{2} x^{2}} + \frac{\partial^{2} \psi}{\partial^{2} x^{3}} \]

Conversely, let then consider the generic solution proposed for it in [[09]; chapter 2, \$2.6, p. 16, (43)]:

\[ \forall (x, t) : \psi(x, t) = u. \exp\left\{ \frac{2.\pi. i}{\hbar} \cdot (p \cdot x - E \cdot t) \right\}; \quad p \cdot x = \sum_{a=1}^{a=3} p^{a} \cdot x^{a} \]

where \( u \) is a constant spinor, \( E \) an energy and \( p \) an impulse. A first partial derivation is yielding:

\[ \frac{\partial \psi(x, t)}{\partial x^{a}} = \frac{2.\pi. i}{\hbar} \cdot p^{a} \cdot \psi(x, t); \quad \frac{\partial \psi(x, t)}{\partial t} = -\frac{2.\pi. i}{\hbar} \cdot E \cdot \psi(x, t) \]

and a second one:

\[ \frac{\partial^{2} \psi(x, t)}{\partial^{2} x^{a}} = -\frac{4.\pi^{2}}{\hbar^{2}} \cdot (p^{a})^{2} \cdot \psi(x, t) \]
\[ \frac{\partial^{2} \psi(x, t)}{\partial^{2} t} = -\frac{4.\pi^{2}}{\hbar^{2}} \cdot E^{2} \cdot \psi(x, t) \]

Injecting these relations into the KGE, one gets -as expected- the dispersion relation again (This can be checked here [[09]; chapter 1, p. 4, (5)]):

\[ \forall \psi : -\frac{E^{2}}{c^{2}} + p^{2} = -m^{2} \cdot c^{2} \]

In supposing now that these classical Euclidean situations occur inside a four-dimensional space related to a Minkowski’s geometry with signature (+ - - -) and adopting the convention [[09]; chapter 2, \$2.9, p. 19, (61)], this can be rewritten in introducing the four-dimensional impulse-energy vector; that relation is nothing else but:

\[ \forall \psi : \langle (4) \ p \rangle_{[\theta]} (4) \ p >_{[\theta]} = (m \cdot c)^{2} \]

This relation induces two sets of situations:

- The particle at hand is massless; \( m = 0 \):

\[ \forall \psi : \langle (4) \ p \rangle_{[\theta]} (4) \ p >_{[\theta]} = 0 \quad (1) \]

The impulse-energy vector is an isotropic vector. The topic can be analyzed with the theory developed by E. Cartan in [[10]].

- The particle at hand is not massless; \( m \neq 0 \). Provided, \( i\ p = m \cdot i\ u \) and \( i\ u/c = i\ V \):

\[ \forall \psi : \langle (4) \ V \rangle_{[\theta]} (4) \ V >_{[\theta]} = 1 \quad (2) \]

This relation can be understood as a peculiar representation of a unit sphere.

There is nothing new, until here.
3.2 Another analysis.

Let rewrite the film back and start the demonstration again. Inspired by the example in [11]: p. 10, (2.1), let consider a generalized formulation of the KGE like:

\[ g^{\mu \nu} \frac{\partial^2 \phi((4) x)}{\partial x^\lambda \partial x^\mu} + \frac{m^2 \cdot c^2}{\hbar^2} \cdot \phi((4) x) = 0 \]

**Proposition:** It can be rewritten as:

\[ P_2((4) x, (4) p, [G], \frac{\partial [G]}{\partial x^\lambda}, \frac{\partial^2 [G]}{\partial x^\lambda \partial x^\mu}) \cdot \phi((4) x) = 0 \]

**Proof.** Let consider the prolongations:

\[ (4) x : (x, t, (3) x) \]
\[ (4) p : (\frac{E}{c}, (3) p) \]

Let equip E(4, R) with the inner scalar product:

\[ \langle (4) p, (4) x \rangle_{[G]} = \sum_{\alpha, \beta = 0, 1, 2} g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta \]

Let examine the generic solution:

\[ \psi((4) x) = u \cdot \exp\{\frac{2 \cdot \pi \cdot i}{\hbar} \cdot \langle (4) p, (4) x \rangle_{[G]}\} \]

It is easy to state that the classical formulation is recovered when \([G] = [\eta]\) with the signature (- + + +). Let now calculate again:

\[ \frac{\partial \psi((4) x)}{\partial x^\lambda} = \frac{2 \cdot \pi \cdot i}{\hbar} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda} \cdot \psi((4) x) \]

and:

\[ \frac{\partial^2 \psi((4) x)}{\partial x^\lambda \partial x^\mu} = \]

\[ \frac{2 \cdot \pi \cdot i}{\hbar} \cdot \left\{ \left( \frac{\partial^2 (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda \partial x^\mu} \right) + \frac{2 \cdot \pi \cdot i}{\hbar} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\mu} \right\} \cdot \psi((4) x) \]

Let inject these relations into the KGE and get:

\[ \left\{ \frac{2 \cdot \pi \cdot i}{\hbar} \cdot g^{\lambda \mu} \cdot \left\{ \left( \frac{\partial^2 (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda \partial x^\mu} \right) + \frac{2 \cdot \pi \cdot i}{\hbar} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\mu} \right\} \right\} \]

\[ \cdot \phi((4) x) = 0 \]

This is the expected formalism as soon as the following function \(P_2\) is introduced into that theoretical analysis:

\[ P_2((4) x, (4) p, [G], \frac{\partial [G]}{\partial x^\lambda}, \frac{\partial^2 [G]}{\partial x^\lambda \partial x^\mu}) = \]

\[ \frac{2 \cdot \pi \cdot i}{\hbar} \cdot g^{\lambda \mu} \cdot \left\{ \left( \frac{\partial^2 (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda \partial x^\mu} \right) + \frac{2 \cdot \pi \cdot i}{\hbar} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda} \cdot \frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\mu} \right\} + \frac{m^2 \cdot c^2}{\hbar^2} \]

\[ \square \]
Remark: Since experiments are always done locally, let focus the attention on:

$$P_2^{(4)} \mathbf{0}, (4) \mathbf{p}, [G], \frac{\partial[G]}{\partial x^\lambda} \cdot \frac{\partial^2[G]}{\partial x^\lambda \partial x^\mu}$$

Let calculate in the vicinity of the origin:

$$\partial \left( g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta \right) \cdot \frac{\partial x^\mu}{\partial x^\mu} = \frac{\partial g_{\alpha \beta}}{\partial x^\rho} \cdot p^\alpha \cdot x^\beta + g_{\alpha \beta} \cdot \frac{\partial p^\alpha}{\partial x^\mu} \cdot x^\beta + g_{\alpha \beta} \cdot p^\alpha \cdot \delta^\beta_\mu = g_{\alpha \mu} \cdot p^\alpha$$

Remark: Let state by the way that if the metric is symmetric, then:

$$\frac{\partial (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\mu} = g_{\mu \alpha} \cdot p^\alpha = p_\mu$$

Let go a step further and calculate:

$$\frac{\partial^2 (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda \partial x^\mu} = \frac{\partial (g_{\alpha \lambda} \cdot p^\alpha)}{\partial x^\mu} = \frac{\partial g_{\alpha \lambda}}{\partial x^\rho} \cdot p^\alpha + g_{\alpha \lambda} \cdot \frac{\partial p^\alpha}{\partial x^\mu}$$

Remark: Let state by the way that if the metric is symmetric, then:

$$\frac{\partial^2 (g_{\alpha \beta} \cdot p^\alpha \cdot x^\beta)}{\partial x^\lambda \partial x^\mu} = \frac{\partial p_\lambda}{\partial x^\mu}$$

Let inject these relations and get in general:

$$P_2^{(4)} \mathbf{0}, (4) \mathbf{p}, [G], \frac{\partial[G]}{\partial x^\lambda}$$

$$= \frac{2 \pi i}{h} \cdot g^{\lambda \mu} \cdot \left\{ \frac{\partial g_{\alpha \lambda}}{\partial x^\mu} \cdot p^\alpha + g_{\alpha \lambda} \cdot \frac{\partial p^\alpha}{\partial x^\mu} \right\} + \frac{2 \pi i}{h} \cdot (g_{\alpha \lambda} \cdot p^\alpha) \cdot (g_{\beta \mu} \cdot p^\beta) + \frac{m^2 \cdot c^2}{\hbar^2}$$

Remark: Let state by the way that if the metric is symmetric, then:

$$P_2^{(4)} \mathbf{0}, (4) \mathbf{p}, [G], \frac{\partial[G]}{\partial x^\lambda}$$

$$= \frac{2 \pi i}{h} \cdot g^{\lambda \mu} \cdot \left\{ \frac{\partial p_\lambda}{\partial x^\mu} + \frac{2 \pi i}{h} \cdot p_\lambda \cdot p_\mu \right\} + \frac{m^2 \cdot c^2}{\hbar^2}$$

Let now reduce our ambition to the study of a massive wave propagating without interaction with extern actors (there is no force acting on it):

$$\frac{\partial p^\alpha}{\partial x^\mu} = 0$$

↓

$$P_2^{(4)} \mathbf{0}, (4) \mathbf{p}, [G], \frac{\partial[G]}{\partial x^\lambda}$$

$$= \frac{2 \pi i}{h} \cdot g^{\lambda \mu} \cdot \left\{ \frac{\partial g_{\alpha \lambda}}{\partial x^\mu} \cdot p^\alpha + \frac{2 \pi i}{h} \cdot (g_{\alpha \lambda} \cdot p^\alpha) \cdot (g_{\beta \mu} \cdot p^\beta) \right\} + \frac{m^2 \cdot c^2}{\hbar^2}$$
As usual in quantum theory, let write for a single wave:

$$ p = i.\hbar k $$

Let inject that relation in the former:

$$ P_2((4)\mathbf{0}, (4)\mathbf{k}, \{G\}, \frac{\partial \{G\}}{\partial x^\lambda}) = g^{\lambda\mu} \cdot (g_{\alpha\lambda} \cdot k^\alpha) \cdot (g_{\beta\mu} \cdot k^\beta) - g^{\lambda\mu} \cdot \frac{\partial g_{\alpha\lambda}}{\partial x^\mu} \cdot k^\alpha + \frac{m^2 \cdot c^2}{\hbar^2} $$

State that:

$$ P_2((4)\mathbf{0}, (4)\mathbf{0}, \{G\}, \frac{\partial \{G\}}{\partial x^\lambda}) = \frac{m^2 \cdot c^2}{\hbar^2} $$

**Remark:** Let state by the way that if the metric is symmetric, then:

$$ \frac{\partial p_\lambda}{\partial x^\mu} = 0 $$

$$ \downarrow $$

$$ P_2((4)\mathbf{0}, (4)\mathbf{p}, \{G\}, \frac{\partial \{G\}}{\partial x^\lambda}) = -\frac{4 \pi^2}{\hbar^2} \cdot g^{\lambda\mu} \cdot p_\lambda \cdot p_\mu + \frac{m^2 \cdot c^2}{\hbar^2} $$

This is also:

$$ P_2((4)\mathbf{0}, (4)\mathbf{k}, \{G\}, \frac{\partial \{G\}}{\partial x^\lambda}) = g^{\lambda\mu} \cdot k_\lambda \cdot k_\mu + \frac{m^2 \cdot c^2}{\hbar^2} $$

And here again:

$$ P_2((4)\mathbf{0}, (4)\mathbf{0}, \{G\}, \frac{\partial \{G\}}{\partial x^\lambda}) = \frac{m^2 \cdot c^2}{\hbar^2} $$

**Lemma 3.1.** Massive wave propagating without interaction.

A Klein-Gordon (synonym: massive) wave propagating without interaction has an invariant speed and can be described with a polynomial of degree two depending on the components of the wave vector \( \mathbf{k} \): \( P_2(\mathbf{k}) \). This fact allows a renewed analysis involving the extrinsic method; that means more precisely: (i) interpreting \( P_2(\mathbf{k}) \) as a Taylor Mac Laurin development around \( \mathbf{k} = \mathbf{0} \) and (ii) relating that development to a non-trivial decomposition for some generic deformed tensor product \( \otimes_A(\ldots, \mathbf{k}) \).

### 3.3 The Klein-Gordon equation and the extrinsic method.

Let examine what happens when the polynomial \( P_2 \) is a Taylor Mac Laurin development:

$$ \frac{1}{2} \cdot < \mathbf{k}, \{ [Hess_{(\mathbf{k}, \mathbf{0})} P_2(\mathbf{0})] \cdot |\mathbf{k} > \} = g^{\lambda\mu} \cdot (g_{\alpha\lambda} \cdot k^\alpha) \cdot (g_{\beta\mu} \cdot k^\beta) $$

$$ | \text{Grad}_{(\mathbf{k})} P_2(\mathbf{0}) > = - | g^{\lambda\mu} \cdot \frac{\partial g_{\alpha\lambda}}{\partial x^\mu} > $$

Let suppose that development is involved into a procedure attempting to discover the non-trivial decomposition of a generic deformed tensor product \( \otimes_A(\ldots, \mathbf{k}) \):

$$ \exists ((4)\mathbf{O}, (4)\mathbf{y}) \in M(4, \mathbb{R}) \times E(4, \mathbb{R}) : | \otimes_A((4)\ldots, (4)\mathbf{k}) > = (4)\mathbf{O} | ((4)\mathbf{k}) > + | (4)\mathbf{y} > $$

It amounts to the same to say that the polynomial \( P_2 \) is resulting from an approximative non-trivial decomposition; in that case, there is a set of non-degenerated elements \( [B] \) in \( M(4, \mathbb{R}) \) such that:
Terms of degree two:
\[ [O] = A\Phi(...) - \frac{1}{2} [B]^{-1} \cdot \text{Hess}_k P_2(0) \]

Terms of degree one:
\[ |y > = [B]^{-1} \cdot g^{\lambda \mu} \frac{\partial g_{\lambda \mu}}{\partial x^\alpha} > \]

Term of degree zero:
\[ 0(3) = P_2(k) - \frac{m^2 c^2}{\hbar^2} \]

When \( P_2 \) is a smooth function, its Hessian is symmetric and each trivial decomposition respects the relation:
\[ [B] \cdot [O] - [O]^t \cdot [B]^t = [B] \cdot A\Phi(...) - A\Phi^t(...).[B]^t \]

Remark: The Hessian takes a peculiar visage when the metric is symmetric; precisely:
\[ [G] = [G]^t \Rightarrow \frac{1}{2} \cdot \text{Hess}_k P_2(0) = [G]^{-1} \]

The metric is Cartan; read [[05]].

When \( P_2 \) is a smooth function, each trivial decomposition respects the relation:
\[ [B] \cdot [O] - [O]^t \cdot [B]^t = [B] \cdot A\Phi(...) - A\Phi^t(...).[B]^t \]

More precisely:
\[ [B] \cdot \{ A\Phi(u) - [B]^{-1} \cdot [G]^{-1}, [G]^{-1} \} - \{ A\Phi(u) - [B]^{-1} \cdot [G]^{-1} \}, [B]^t \]
\[ = [B] \cdot A\Phi(u) - A\Phi^t(u) \cdot [B]^t \]
\[ \downarrow \]
\[ [G]^{-1} + \{(B)^{-1}, (G)^{-1}\}, [B]^t \] = [0]
\[ \downarrow \]
\[ [G]^{-1} + \{(G)^{-1}t, (B)^{-1}t\}, [B]^t \] = [0]

Since \( \text{M}(4, R) \) is equipped with an associative multiplication:
\[ [G]^{-1} + \{(G)^{-1}t, (B)^{-1}t\} \cdot [B]^t \] = [0]

Here, the metric has been supposed to be symmetric:
\[ [G]^{-1} + \{G^{-1}t, [B]^t\} = [0] \]

This relation is true when:
\[ ([B]^{-1})t \cdot [B]^t = -\text{Id}_4 \]

It amounts to the same to write this condition as:
\[ ([B]^{-1})^t + ([B]^t)^{-1} = [0] \]

**Lemma 3.2.** The necessary condition allowing the use of the extrinsic method.

When the metric is symmetric, the polynomial \( P_2(k) \) describing the propagation of the massive wave can be related to the non-trivial decomposition of a generic deformed tensor product \( \otimes_A(...) \cdot k \) provided that decomposition is obtained with the extrinsic method in involving elements \([B]\) taken in \( \text{M}(4, R) \) and respecting what will be called the anti-golden rule. There is absolutely no obligation for these matrices to be related to the metric.
3.4 A link with the Lorentz law.

Let go a step further and consider the second proposition (subsection: purpose of this document) in exploring the correspondence:

\[ |m_\square \frac{du}{ds} + \otimes_{\Gamma(2)}(u, p) \rangle = q \cdot |F\rangle \cdot |u \rangle \iff \otimes_A(..., k) = |O\rangle \cdot |k \rangle + |y \rangle \]

Due to (recall):

\[ p = i.\hbar k \]

Let look for the circumstances allowing the equivalence:

\[ |m_\square \frac{du}{ds} + \otimes_{\Gamma(2)}(u, p) \rangle = q \cdot |F\rangle \cdot |u \rangle \iff \otimes_A(..., p) = |O\rangle \cdot |p \rangle + i.\hbar \cdot |y \rangle \]

Let try with the relation:

\[ |p \rangle = [T]. |u \rangle + |a \rangle \iff p^\beta = T^\beta_\epsilon \cdot u^\epsilon + a^\beta \]

And let start for the pedagogy with:

\[ A^\chi_{\alpha\beta} \cdot u^\alpha \cdot p^\beta = a_{\chi\beta} \cdot p^\beta + h \cdot i.\gamma^\lambda \]

draw

\[ A^\chi_{\alpha\beta} \cdot u^\alpha \cdot (T^\beta_\epsilon \cdot u^\epsilon + a^\beta) = \alpha_{\chi\beta} \cdot (T^\beta_\epsilon \cdot u^\epsilon + a^\beta) + i.\gamma \cdot y^\lambda \]

draw

\[ A^\chi_{\alpha\beta} \cdot T^\beta_\epsilon \cdot u^\alpha \cdot u^\epsilon + A^\chi_{\alpha\beta} \cdot u^\alpha \cdot a^\beta = \alpha_{\chi\beta} \cdot T^\beta_\epsilon \cdot u^\epsilon + (\alpha_{\chi\beta} \cdot a^\beta + i.\gamma \cdot y^\lambda) \]

down

\[ (A^\chi_{\alpha\beta} \cdot T^\beta_\epsilon)_\alpha \cdot u^\alpha \cdot u^\epsilon = \alpha_{\chi\beta} \cdot T^\beta_\epsilon \cdot u^\epsilon - A^\chi_{\alpha\beta} \cdot a^\beta \cdot u^\alpha + (\alpha_{\chi\beta} \cdot a^\beta + i.\gamma \cdot y^\lambda) \]

down

\[ \forall \chi : (A^\chi_{\alpha\beta} \cdot T^\beta_\epsilon)_\alpha \cdot u^\alpha \cdot u^\epsilon = (\alpha_{\chi\beta} \cdot T^\beta_\epsilon - A^\chi_{\alpha\beta} \cdot a^\beta) \cdot u^\epsilon + (\alpha_{\chi\beta} \cdot a^\beta + i.\gamma \cdot y^\lambda) \]

The co-variant version of the Lorentz law is recovered when, simultaneously:

\[ |p \rangle = [T]. |u \rangle + |a \rangle \]

\[ \Gamma^\chi_{\alpha\epsilon} = \Gamma^\chi_{\epsilon\alpha} \]

\[ A^\chi_{\alpha\beta} \cdot T^\beta_\epsilon = m \cdot \Gamma^\chi_{\alpha\epsilon} \]

\[ \alpha_{\chi\beta} \cdot T^\beta_\epsilon = A^\chi_{\alpha\beta} \cdot a^\beta = q \cdot F^\chi_\epsilon \]

\[ -m \cdot \frac{du^\chi}{ds} = \alpha_{\chi\beta} \cdot a^\beta + i.\gamma \cdot y^\lambda \]

**Lemma 3.3.** A link with the covariant version of the Lorentz law.

When the deformed tensor product that can be related with the polynomial \( P_2(k) \) describing a massive wave propagating at invariant speed (i) is non-trivially decomposed with the help of the extrinsic method and (ii) can be written as:

\[ \otimes_A(u, p) = |O\rangle \cdot |p \rangle + i.\hbar \cdot |y \rangle \]

Then, it can be interpreted as a representation of the covariant version of the Lorentz law if, simultaneously:

\[ |p \rangle = [T]. |u \rangle + |a \rangle \]

\[ \Gamma^\chi_{\alpha\epsilon} = \Gamma^\chi_{\epsilon\alpha} \]

\[ A^\chi_{\alpha\beta} \cdot T^\beta_\epsilon = m \cdot \Gamma^\chi_{\alpha\epsilon} \]

\[ \alpha_{\chi\beta} \cdot T^\beta_\epsilon = A^\chi_{\alpha\beta} \cdot a^\beta = q \cdot F^\chi_\epsilon \]

\[ -m \cdot \frac{du^\chi}{ds} = \alpha_{\chi\beta} \cdot a^\beta + i.\gamma \cdot y^\lambda = 0 \]
3.5 Recovering the relation of dispersion.

Within Einstein’s theory, the metric is related to the energy-impulse tensor via the famous relation \([11];\ p.\ 154,\ (6.7)\):

\[
[R_{\alpha\beta}] + (\Lambda - \frac{R}{2}) \cdot [G_{\alpha\beta}] = \frac{8\pi G}{c^4} \cdot [T_{\alpha\beta}]
\]

When the geometry is invariant, this equation is reduced to:

\[
\Lambda \cdot [G] = \frac{8\pi G}{c^4} \cdot [T]
\]

Remark: For invariant symmetric metrics, the function \(P_2\) is reduced to:

\[
P_2^{(4)}(0, (4)k, [G], \frac{\partial [G]}{\partial x^\lambda}) = g^{\lambda\mu} \cdot k_\lambda \cdot k_\mu + \frac{m^2 \cdot c^2}{\hbar^2}
\]

\[
P_2^{(4)}(0, (4)k, [G], \frac{\partial [G]}{\partial x^\lambda}) = <k| [G]^{-1} |k> + \frac{m^2 \cdot c^2}{\hbar^2}
\]

Since (recall):

\[
p = i \cdot h \cdot k
\]

The function \(P_2\) is also:

\[
P_2^{(4)}(0, (4)k, [G], \frac{\partial [G]}{\partial x^\lambda}) = \frac{\Lambda \cdot c^4}{8\pi G} \cdot <k| [T]^{-1} |k> + \frac{m^2 \cdot c^2}{\hbar^2}
\]

This relation can be identified with the usual relation of dispersion for the physical conditions at hand when two relations are true:

1. The left hand term is the square of the total energy of the wave:

\[
E^2 = \hbar^2 \cdot P_2^{(4)}(0, (4)k, [G], \frac{\partial [G]}{\partial x^\lambda}) \cdot c^2
\]

2. And:

\[
-\frac{\Lambda \cdot c^4}{8\pi G} \cdot <p| [T]^{-1} |p> = p^2
\]

\[
\forall p: [T]^{-1} \cdot \frac{8\pi G}{\Lambda \cdot c^2} \cdot I_{d_4} = [0]
\]

\[
\forall p: [T] \cdot \frac{\Lambda \cdot c^4}{8\pi G} \cdot I_{d_4} = [0]
\]

From which it may be induced that:

\[
\rho_{\text{vacuum}} \sim -\frac{\Lambda \cdot c^2}{8\pi G}; \ \Lambda < 0
\]
**Lemma 3.4.** The dispersion relation for symmetric and invariant metrics.

An analyze of the Klein-Gordon equation in a context involving symmetric invariant metrics with the help of the intrinsic method:

1. recovers the dispersion relation characterizing this context (recall):
   \[ E^2 = m^2 \cdot c^4 + c^2 \cdot p^2 \]

2. relates the function \( P_2 \) to the square of the total energy of the propagating wave and indirectly to its frequency:
   \[ \nu^2 = P_2(0, k; [G], \frac{\partial[G]}{\partial x^\lambda}) \cdot c^2 \]

3. gives the volumetric density of matter in vacuum again with a negative cosmological constant.

### 4 Conclusion

The purposes which have been described in § 2.3 are reached. Exactly as in the introducing sections of the famous quantum field theory [[09]], a circular and coherent demonstration has been exposed. It is able to connect the dispersion relation and the Klein-Gordon equation too. But, instead of doing it with the relations [[09]; p. 4, (1)]:

\[ E \rightarrow i.\hbar \frac{\partial}{\partial t}; \forall a = 1, 2, 3 : p_a \rightarrow -i.\hbar \frac{\partial}{\partial x^a} \]

it involves a tiny more sophisticated mathematical analysis.

This analysis injects presumed generic solutions into the KGE. This manoeuvre is transforming it into a polynomial \( P_2 \) of degree two depending on the components of the wave vector \( k : P_2(k) \).

This polynomial is interpreted as the signature of the existence of a generic deformed tensor product \( \otimes_A(\ldots, k) \) which has been approximately and non-trivially decomposed. The interpretation is done with the help of the extrinsic method. Here, \( A \) is some cube of elements arbitrarily chosen in \( \mathbb{R} \) and \( \ldots \) is a priori any vector in \( E(4, \mathbb{R}) \).

The end of the document focuses on symmetric metrics. The mathematical conditions authorizing the use of the extrinsic method are given; there must exist a set of non-degenerated elements \([B]\) in \( M(4, \mathbb{R}) \) for which the anti-golden rule is true:

\[ ([B]^{-1})^t + ([B]^t)^{-1} = [0] \]

A specialization on the decomposition of \( \otimes_A(u, k) \) where \( u \) is the speed of the wave allows to build a link with the covariant version of the Lorentz law. This formal junction suggests the important intellectual idea that massive waves can behave like a particle.

The polynomial can be related to the Einstein’s field equation. This link is particularly simple and obvious for invariant symmetric metrics. In these circumstances, the usual dispersion relation is re-obtained without technical complication. This approach has diverse advantages, among them: it gives (i) a physical interpretation for the polynomial \( P_2 \) (which is no more only a mathematical tool), (ii) the volumetric density of energy in vacuum and (iii) the sign of the cosmological constant.

This approach should be tested in more complicated geometric situations.
5 Bibliography

References

5.1 My contributions


5.2 Articles, Books and Courses


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