A Paradoxical Collection of Sets

By Jim Rock

Abstract: We exhibit a collection of sets, which both have and do not have largest elements.

Introduction. For all real numbers $a$ in the open interval $(0, 1)$
Let the collection of $R_a = \{ y \text{ a real number} \mid 0 \leq y < a \}$

Each set in the collection of $R_a$ has a largest element.
For each $a$ in $(0, 1)$ the group of proper subsets of $R_a$ from the collection of all $R_a$ are nested inside each other in descending order. Each $R_a$ must contain one and only one element $a'$ that is not in a proper subset from among the collection of all $R_a$. Otherwise, since the proper subsets are all nested inside each other, each $R_a$ would be a proper subset of itself.

$a'$ is the largest element of $R_a$.

No Set in the collection of $R_a$ has a largest element.
Suppose there is a largest element $a'$ in $R_a$.

$a' < (a + a')/2 < a$. Let $b = (a + a')/2$. Then $b$ is in $R_a$ and $a' < b$.

Note: the question is not whether $R_a$ actually contains a largest element, but whether or not the two contradictory statements about a largest element are both conclusions of valid logical arguments.

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