The Paradoxical Collection of Sets Explained

By Jim Rock

Abstract: We explain why there is a collection of sets, which both have and do not have largest elements.

Introduction. For all real numbers \( a \) in the open interval \((0, 1)\)
let the collection of all \( R_a = \{ y \text{ a real number} \mid 0 \leq y < a \} \)

Each set in the collection of \( R_a \) has a largest element.
Select a single \( R_a \) taken from the collection of all \( R_a \).
Along with itself this selected \( R_a \) has a group of proper subsets taken from the collection of all \( R_a \). This group of proper subsets are nested within each other. When nested within the selected \( R_a \), this group of proper subsets forms a proper subset of the selected \( R_a \).
For any two elements of the selected \( R_a \) the smaller element will be in a proper subset of the selected \( R_a \).
Thus, each \( R_a \) must contain a single largest element not in the group of its nested proper subsets taken from the collection of all \( R_a \).

No Set in the collection of \( R_a \) has a largest element.
Suppose there is a largest element \( a' \) in \( R_a \).
\( a' < (a + a')/2 < a \). Let \( b = (a + a')/2 \). Then \( b \) is in \( R_a \) and \( a' < b \).

Note: the question is not whether \( R_a \) actually contains a largest element, but whether or not the two contradictory statements about a largest element are both conclusions of valid logical arguments.

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